S. TANNO AND K, MASUDA
KODAI MATH. J.
8 (1985), 420—429

A CHARACTERIZATION OF THE PRODUCT OF TWO 3-SPHERES BY THE SPECTRUM

BY SHUKICHI TANNO AND KAZUO MASUDA

§1. Introduction.

Let (M, g) be a compact Riemannian manifold. By Spec(M, g) we denote the spectrum of the Laplacian acting on functions on (M, g). Let $S^m(c)$ be the m-sphere of constant curvature c.

For $m \leqq 6$, $S^m(c)$ is characterized by the spectrum (Berger [1], Tanno [5]); that is, Spec$(M, g)=$Spec$\,S^m(c)$ implies that (M, g) is isometric to $S^m(c)$.

For $m \geqq 7$, it is an open question if $S^m(c)$ is characterized by the spectrum. As for partial answers see [6].

In this paper we obtain the following theorem on product Riemannian manifolds.

THEOREM A. *Let (M, g) and (M', g') be 3-dimensional compact Riemannian manifolds. Assume that*

$$\mathrm{Spec}\,[(M, g)\times(M', g')]=\mathrm{Spec}\,[S^3(c)\times S^3(c')].$$

Then, (M, g) and (M', g') are of constant curvature K and K', respectively, and $K+K'=c+c'$.

Furthermore, if the sectional curvatures K and K' are positive, then (M, g) is isometric to $S^3(c)$ (or $S^3(c')$) and (M', g') is isometric to $S^3(c')$ (or $S^3(c)$, resp.).

Let $CP^n(H)$ be the n-dimensional complex projective space of constant holomorphic sectional curvature H. Corresponding to Theorem A we get

THEOREM B. *Let (M, g, J) and (M', g', J') be (complex) 3-dimensional compact Kählerian manifolds. Assume that*

$$\mathrm{Spec}\,[(M, g, J)\times(M', g', J')]=\mathrm{Spec}\,[CP^3(H)\times CP^3(H')].$$

Then, (M, g, J) is holomorphically isometric to $CP^3(H)$ (or $CP^3(H')$) and (M', g', J') is holomorphically isometric to $CP^3(H')$ (or $CP^3(H)$, resp.).

Received March 19, 1985

§ 2. Preliminaries.

Let (M, g) be a compact Riemannian manifolds of dimension m and let $\mathrm{Spec}(M, g)=\{0=\lambda_0<\lambda_1\leqq\lambda_2\leqq\cdots\}$ be the spectrum of the Laplacian acting on functions on (M, g). By $R=(R^i{}_{jkl})$, $\rho=(R_{jl})=(R^i{}_{jil})$ and S we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature of (M, g), respectively. For a tensor field T on (M, g), $|T|^2$ denotes the square of the norm of T with respect to g. Then, a formula of Minakshisundaram-Pleijel is

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} \underset{t\downarrow 0}{\sim} \left(\frac{1}{4\pi t}\right)^{m/2}[a_0+a_1t+a_2t^2+\cdots],$$

where a_0, a_1, a_2 and a_3 are given by the following (Berger [1], Mckean-Singer [2], Sakai [4])

$$a_0=\mathrm{Vol}(M, g),$$

$$a_1=(1/6)\int_M S,$$

$$a_2=(1/360)\int_M [2|R|^2-2|\rho|^2+5S^2], \tag{2.1}$$

$$a_3=(1/6!)\int_M [-(1/9)|\nabla R|^2-(26/63)|\nabla\rho|^2-(142/63)|\nabla S|^2$$
$$+(2/3)S|R|^2-(2/3)S|\rho|^2+(5/9)S^3+A],$$

where

$$A=(8/21)(R, R, R)-(8/63)(\rho\,;R, R)+(20/63)(\rho\,;\rho\,;R)-(4/7)(\rho\rho\rho),$$

$$(R, R, R)=R^{ij}{}_{kl}R^{kl}{}_{ab}R^{ab}{}_{ij},$$

$$(\rho\,;R, R)=R_{ij}R^i{}_{abc}R^{jabc},$$

$$(\rho\,;\rho\,;R)=R^{ij}R^{kl}R_{ikjl},$$

$$(\rho\rho\rho)=R^i{}_jR^j{}_kR^k{}_i.$$

We denote the Weyl conformal curvature tensor by C and put

$$G=\rho-(1/m)Sg.$$

Then we get (cf. Tanno [5])

$$a_2=\frac{1}{360}\int_M\left[2|C|^2+\frac{2(6-m)}{m-2}|G|^2+\left(\frac{2(6-m)}{m(m-2)}+\frac{5m(m-3)+6}{(m-1)(m-2)}\right)S^2\right]. \tag{2.2}$$

Let (M', g') be another compact Riemannian manifold. The Riemannian product $(M, g)\times(M', g')$ is denoted by M^*. We denote the geometric object of

(M', g') or M^* corresponding to T on (M, g) by T' or T^*.

Since

$$\sum e^{-\lambda_r^* t}=(\sum e^{-\lambda_k t})(\sum e^{-\lambda_l' t}),$$

we obtain

$$a_0(M^*)=a_0(M)a_0(M'),$$

$$a_1(M^*)=a_1(M)a_0(M')+a_0(M)a_1(M'),$$

$$a_2(M^*)=a_2(M)a_0(M')+a_1(M)a_1(M')+a_0(M)a_2(M').$$

For a function f on $M(M'$, resp.), we denote its extended function on M^* by the same letter f. The following is evident.

$$a_1(M^*)=(1/6)\int_{M^*}[S+S'] \tag{2.3}$$

LEMMA 2.1. *$a_2(M^*)$ is given by*

$$\begin{aligned}a_2(M^*)=\frac{1}{360}\int_{M^*}\Big[&2(|C|^2+|C'|^2)+\frac{2(6-m)}{m-2}|G|^2+\frac{2(6-m')}{m'-2}|G'|^2\\&+\Big(\frac{2(6-m)}{m(m-2)}+\frac{5m(m-3)+6}{(m-1)(m-2)}\Big)S^2+10SS'\\&+\Big(\frac{2(6-m')}{m'(m'-2)}+\frac{5m'(m'-3)+6}{(m'-1)(m'-2)}\Big)S'^2\Big].\end{aligned} \tag{2.4}$$

Proof. Since

$$a_1(M)a_1(M')=(1/36)\int_M S\int_{M'} S'=(10/360)\int_{M^*}SS',$$

we get (2.4) by (2.2). q. e. d.

Now, let (M, g, J) and (M', g', J') be compact Kählerian manifolds. We denote the Bochner curvature tensor of (M, g, J) by B. Then, putting $\dim_C M=n$, we get (cf. Tanno [5])

$$a_2(M)=\frac{1}{360}\int_M\Big[2|B|^2+\frac{2(6-n)}{n+2}|G|^2+\frac{5n^2+4n+3}{n(n+1)}S^2\Big].$$

Corresponding to Lemma 2.1, we get

LEMMA 2.2. *$a_2(M^*)$ for $M^*=(M, g, J)\times(M', g', J')$ is given by*

$$\begin{aligned}a_2(M^*)=\frac{1}{360}\int_{M^*}\Big[&2(|B|^2+|B'|^2)+\frac{2(6-n)}{n+2}|G|^2+\frac{2(6-n')}{n'+2}|G'|^2\\&+\frac{5n^2+4n+3}{n(n+1)}S^2+10SS'+\frac{5n'^2+4n'+3}{n'(n'+1)}S'^2\Big].\end{aligned} \tag{2.5}$$

§ 3. Proof of Theorem A.

First we prove the following.

PROPOSITION 3.1. *Let* (M, g) *and* (M', g') *be* 3*-dimensional compact Riemannian manifolds. Let* $N(c)$ *and* $N'(c')$ *be* 3*-dimensional compact Riemannian manifolds of constant curvature* c *and* c'. *For* $i=0$, 1, *and* 2, *assume*

$$a_i[(M, g)\times(M', g')]=a_i[N(c)\times N'(c')].$$

Then, (M, g) *and* (M', g') *are of constant curvature* K *and* K', *and* $K+K'=c+c'$.

Proof. We denote the Riemannian products by

$$M^*=(M, g)\times(M', g'), \qquad M_o^*=N(c)\times N'(c').$$

$a_0(M^*)=a_0(M_o^*)$ implies $\mathrm{Vol}(M^*)=\mathrm{Vol}(M_o^*)$, and $a_1(M^*)=a_1(M_o^*)$ implies

$$\int_{M^*}(S+S')=\int_{M_o^*}(S_o+S_o'),$$

where S_o and S_o' denote the scalar curvature of $N(c)$ and $N'(c')$. By Schwarz inequality we get

$$\int_{M^*}(S+S')^2\geqq\int_{M_o^*}(S_o+S_o')^2, \tag{3.1}$$

where equality holds if and only if $S+S'=S_o+S_o'$.

Since $C=C'=0$ for $m=m'=3$, by (2.4) we see that $a_2(M^*)=a_2(M_o^*)$ is equivalent to

$$\int_{M^*}[6(|G|^2+|G'|^2)+5(S+S')^2]=\int_{M_o^*}5(S_o+S_o')^2. \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$G=G'=0, \qquad S+S'=S_o+S_o'.$$

Thus, (M, g) and (M', g') are of constant curvature K and K'. Since $S=6K$, we get $K+K'=c+c'$. q. e. d.

LEMMA 3.2. *Let* $M(K)$ *and* $M'(K')$ *be* 3*-dimensional compact Riemannian manifolds of constant curvature* K *and* K'. *Assume that*

$$\mathrm{Spec}\,[M(K)\times M'(K')]=\mathrm{Spec}\,[S^3(1)\times S^3(c')].$$

If $K\leqq 1=c\leqq c'\leqq K'$ *and* $K/c'>5\cdot 10^{-8}$, *then* $K=1$, $K'=c'$ *and* $M(K)(M'(K')$, *resp.*) *is isometric to* $S^3(1)$ ($S^3(c')$, *resp.*).

Proof. (i) $\mathrm{Spec}\,S^3(c)$ is given by

$$\text{Spec}\, S^3(c)=\{0,\ 3c,\ 8c,\ \cdots,\ k(k+2)c,\ \cdots\},$$

and multiplicities are $1, 4, 9, \cdots, {}_{k+3}C_k-{}_{k+1}C_{k-2}, \cdots$.

(ii) Spec $M(K)$ is a subset of $\{k(k+2)K;\ k=0, 1, 2, \cdots\}$.

(iii) If $3K\in \text{Spec}\, M(K)$, then $M(K)$ is isometric to $S^3(K)$. In fact, this follows from the property of eigenfunctions corresponding to the first eigenvalue $3K$ of $S^3(K)$.

(iv) If $K=1$, then $K'=c'$ by $K+K'=1+c'$. In this case, since Spec $M(1)$ contains 3, $M(1)$ is isometric to $S^3(1)$ by (iii).

(v) From now on in this proof we assume $K<1$. Then $c'<K'$.

(v-1) We show that there exists some integer k such that

$$3=k(k+2)K. \tag{3.3}$$

In fact, since $3K'>3c'\geqq 3$, we see that the first eigenvalue 3 of $S^3(1)$ is contained in Spec $M(K)$. Thus, we get (3.3). This means that $k\geqq 2$, and for any positive integer $t<k$, $t(t+2)K\notin \text{Spec}\, M(K)$.

(v-2) Similarly we get some integer l such that

$$3c'=l(l+2)K, \qquad k\leqq l. \tag{3.4}$$

(v-3) There exists some integer r such that

$$3+3c'=r(r+2)K. \tag{3.5}$$

In fact, for $3+3c'\in \text{Spec}\, [S^3(1)\times S^3(c')]$, there exsist some integers r and s such that

$$3+3c'=r(r+2)K+s(s+2)K'.$$

Since $r\neq 1$ and $K+K'=1+c'$, s must be zero. So, we get (3.5).

(v-4) There exists some integer p such that

$$8=p(p+2)K, \qquad 3\leqq p. \tag{3.6}$$

In fact, for $8\in \text{Spec}\, S^3(1)$, if $8<3K'$, we get (3.6). If $8=3K'$, then noticing that the multiplicities are strictly increasing, we get (3.6). If $8>3K'$ and if $p(p+2)K+3K'=8$, then $3K'\in \text{Spec}\, M'(K')$. In this case, $3K'$ must be of the form $3K'=3+3c'$. However, this contradicts $K+K'=1+c'$. So, in any case we get (3.6).

(v-5) There exists some integer q such that

$$8c'=q(q+2)K. \tag{3.7}$$

In fact, if $c'=1$, (3.7) is clear. If $c'>1$, then $8c'$ is of the form (3.7) or $8c'=q'(q'+2)K+3K'$. We consider the second case.

If $q'\neq 0$, then $8c'>3K'$. Since $3K'-3c'<3$, we get $3K'=a(a+2)\geqq 8$ for some integer a. Furthermore, $3+3K'$ is of the form $b(b+2)+d(d+2)c'$ for some integers b and d. If $d=0$, then $b(b+2)-a(a+2)=3$, which is impossible. If $b=0$, then $3+3K'=8c'$. In this case $K=1+c'-K'=(11-5K')/8$, which contradicts

$3K'\geqq 8$ and $K>0$. Thus, $3+3K'=b(b+2)+3c'$ and $b\geqq 2$. However, $3K'-3c'=b(b+2)-3\geqq 5$, which is a contradiction.

If $q'=0$, considering the multiplicities we get (3.7).

(v-6) By (3.3) and (3.6) we get

$$K=\frac{3}{k(k+2)}=\frac{8}{p(p+2)}. \tag{3.8}$$

By (3.4) and (3.7) we get

$$\frac{K'}{c'}=\frac{3}{l(l+2)}=\frac{8}{q(q+2)}. \tag{3.9}$$

By (3.3), (3.4) and (3.5) we obtain.

$$k(k+2)+l(l+2)=r(r+2), \qquad k\leqq l. \tag{3.10}$$

We show that there are no integers k, l, p, q, and r satisfying (3.8)~(3.10) for $l\leqq 7800$. $l\leqq 7800$ corresponds to $K/c'\geqq 4.92\cdots\cdot 10^{-8}$. Pairs (k, p) ((l, q), resp.) satisfying (3.8) ((3.9), resp.) are as follows (cf. Remarks 1, 2, below):

(7, 12),	(18, 30),	(78, 128),
(187, 306),	(781, 1276),	(1860, 3038),
(7740, 12640).		

It is verified that for any two pairs chosen from the above, there is no integer r satisfying (3.10). This means that $M(K)\times M'(K')$ and $S^3(1)\times S^3(c')$ are not isospectral for $K<1$ and $K/c'>5\cdot 10^{-8}$. q. e. d.

Remark 1. If one wants to use a computer, a simple BASIC-program for (k, p) is as follows:

For $k=3n$ and $p=2u$;

```
10 FOR N=1 TO 2600
20 A=6*N*N+4*N
30 B=SQR(A): U=INT(B)
40 V=(U+1)*U
50 If A=V THEN PRINT 3*N, 2*U
60 NEXT
70 END
```

For $k=3n+1$ and $p=2u$, replace 20 and 50 by

```
20 A=6*N*N+8*N+2
50 IF A=V THEN PRINT 3*N+1, 2*U
```

Remark 2. If one wants to apply a method for indeterminate equation, put $x=k+1$ and $y=p+1$. Then (3.8) is

$$8x^2-3y^2=5.$$

To solve this equation, we consider Pell's equation

$$t^2-Du^2=\pm 4, \qquad D=4\cdot 8\cdot 3=96,$$

The smallest solution (t, u) such that $(t+u\sqrt{D})/2>1$ is $(10, 1)$. Then general solutions (t_n, u_n) are given by $[(10+\sqrt{96})/2]^n=(t_n+u_n\sqrt{96})/2$. Therefore

$$t_n=(5+2\sqrt{6})^n+(5-2\sqrt{6})^n,$$

$$u_n=[(5+2\sqrt{6})^n-(5-2\sqrt{6})^n]/\sqrt{96}.$$

Since

$$\frac{t_{n+1}+u_{n+1}\sqrt{96}}{2}=\frac{t_n+u_n\sqrt{96}}{2}\cdot\frac{10+\sqrt{96}}{2}$$

$$=\frac{1}{2}\left(\frac{10t_n+96u_n}{2}+\frac{t_n+10u_n}{2}\sqrt{96}\right),$$

we get

$$\begin{pmatrix} t_{n+1} \\ u_{n+1} \end{pmatrix}=\begin{pmatrix} 5 & 48 \\ 1/2 & 5 \end{pmatrix}\begin{pmatrix} t_n \\ u_n \end{pmatrix}, \qquad \begin{pmatrix} t_0 \\ u_0 \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

and hence

$$\begin{pmatrix} t \\ u \end{pmatrix}=\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 1 \end{pmatrix}, \begin{pmatrix} 98 \\ 10 \end{pmatrix}, \begin{pmatrix} 970 \\ 99 \end{pmatrix}, \begin{pmatrix} 9602 \\ 980 \end{pmatrix}, \cdots$$

The matrix corresponding to (t, u) is

$$\begin{pmatrix} t/2 & 3u \\ 8u & t/2 \end{pmatrix}$$

and hence

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 8 \\ 13 \end{pmatrix}, \begin{pmatrix} 79 \\ 129 \end{pmatrix}, \begin{pmatrix} 782 \\ 1277 \end{pmatrix}, \begin{pmatrix} 7741 \\ 12641 \end{pmatrix}, \cdots$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 19 \\ 31 \end{pmatrix}, \begin{pmatrix} 188 \\ 307 \end{pmatrix}, \begin{pmatrix} 1861 \\ 3039 \end{pmatrix}, \begin{pmatrix} 18422 \\ 30083 \end{pmatrix}, \cdots$$

LEMMA 3.3. *Let $M(K)$ and $M'(K')$ be 3-dimensional compact Riemannian manifolds of constant curvature K and K'. For $0<K<1=c\leqq c'<K'$ and $K+K'=c+c'$, if $K/c'\leqq 5\cdot 10^{-6}$, then $M(K)\times M'(K')$ and $S^3(1)\times S^3(c')$ are not isospectral.*

Proof. Suppose that Spec $[M(K)\times M'(K')]=$ Spec $[S^3(1)\times S^3(c')]$.

(i) First we show that $48\in$ Spec $S^3(1)$ is expressed as

$$48=x(x+2)K \tag{3.11}$$

for some integer x. In fact, assume that

$$48=x(x+2)K+y(y+2)K'$$

for some integers x and y; $1\leqq y\leqq 5$. In this case, $K'\leqq 16$. Furthermore, we get

$$(3.12)\qquad \begin{aligned} x(x+2)K&=u(u+2)+v(v+2)c',\\ y(y+2)K'&=w(w+2)+z(z+2)c' \end{aligned}$$

for some integers u, v, w, and z. Therefore

$$[v(v+2)+z(z+2)]c'=48-u(u+2)-w(w+2).$$

We put $a=v(v+2)+z(z+2)$.

(i-1) First we assume $a>0$. Then we get $a\leqq 48$ and ac' is an integer. By (3.12), $ay(y+2)K'$ is an integer. On the other hand, we get

$$ay(y+2)K'=ay(y+2)+y(y+2)ac'-y(y+2)aK.$$

Since $y(y+2)a\leqq 1680$, $c'<K'\leqq 16$ and $K/c'\leqq 5\cdot 10^{-6}$, we get $y(y+2)aK<0.1344$ and it is not an integer. This is a contradiction.

(i-2) Next, if $a=0$, then 48 must be a sum of two numbers using $\{0, 3, 8, 15, 24, 35, 48\}$. First we show that $24\notin \operatorname{Spec} M'(K')$. If $24=y(y+2)K'\in \operatorname{Spec} M'(K')$, then $1\leqq y\leqq 3$ and $3+y(y+2)K'=27=b(b+2)+d(d+2)c'$ for some integers b and d. If $d\neq 0$, one gets a contradiction similarly as in (i-1). So, $d=0$, and $b(b+2)\neq 27$ is clear.

Thus, we get $48=x(x+2)K$ or $48=y(y+2)K'$. In the second case, considering the multiplicities we get (3.11).

(ii) As we have seen before, 3 is expressed as $3=k(k+2)K$. By (3.11) we obtain

$$(x+1)^2=(4(k+1))^2-15.$$

This equation has only two solutions $(k, x)=(0, 0)$ and $(1, 6)$. Therefore, $k=1$ and $K=1$. This is a contradiction. q. e. d.

Proof of Theorem A. The first part of Theorem A follows from Proposition 3.1. To prove the second part we can assume that $c=1\leqq c'$ and $K\leqq K'$. If $1<K\leqq K'<c'$, we see that 3 is contained in $\operatorname{Spec} S^3(1)$ but not in $\operatorname{Spec}[(M, g)\times(M', g')]$. This is a contradiction. If $K\leqq 1$, by Lemmas 3.2 and 3.3, proof is completed.

THEOREM A′. *Let (M, g) and (M', g') be 3-dimensional compact and simply connected Riemannian manifolds. Assume that $(M, g)\times(M', g')$ and $S^3(c)\times S^3(c')$ are isospectral, then $(M, g)\times(M', g')$ is isometric to $S^3(c)\times S^3(c')$.*

Proof. By Proposition 3.1 we see that (M, g) and (M', g') are constant curvature K and K' such that $K+K'=c+c'$. Since M and M' are simply connected, (M, g) ((M', g'), resp.) is isometric to $S^3(K)$ ($S^3(K')$, resp.). Comparing the volumes of $S^3(K)\times S^3(K')$ and $S^3(c)\times S^3(c')$, we see that $K=c$ and $K'=c'$ (or $K=c'$ and $K'=c$). q. e. d.

§4. Proof of Theorem B.

We prove Theorem B in a more general setting.

PROPOSITION B. *Let (M, g, J) and (M', g', J') be (complex) 3-dimensional compact Kählerian manifolds. Let $N(H)$ and $N'(H')$ be (complex) 3-dimensional compact Kählerian manifolds of constant holomorphic sectional curvature H and H'. Assume that*

(i) $H+H'\neq 0$,

(ii) *for* $i=0, 1, 2,$ *and* 3

$$a_i[(M, g, J)\times(M', g', J')]=a_i[N(H)\times N'(H')].$$

Then (M, g, J) ((M', g', J'), resp.) is of constant holomorphic sectional curvature H or H'.

Proof. We denote the Riemannian products by

$$M^*=(M, g, J)\times(M', g', J'),\qquad M_o^*=N(H)\times N'(H').$$

By the same argument as in §3 we obtain (3.1). By Lemma 2.2 and $n=n'=3$, $a_2(M^*)=a_2(M_o^*)$ is equivalent to

$$\int_{M^*}[2(|B|^2+|B'|^2)+(6/5)(|G|^2+|G'|^2)+5(S+S')^2]=\int_{M_o^*}5(S_o+S_o')^2.$$

Therefore we obtain

$$B=B'=0,\quad G=G'=0,\quad S+S'=S_o+S_o'.$$

Consequently, (M, g, J) and (M', g', J') are of constant holomorphic sectional curvature L and L'. Here $S=n(n+1)L=12L$.

Since R (R', resp.) is parallel and

$$|R|^2=2n(n+1)L^2=24L^2=(1/2)n(n+1)^2L^2=|\rho|^2,$$

$$(R, R, R)=n(n+1)(n+3)L^3=72L^3,$$

$$(\rho; R, R)=n(n+1)^2L^3=48L^3,$$

$$(\rho;\rho;R)=(1/4)n(n+1)^3L^3=48L^3=(\rho\rho\rho),$$

$a_3(M^*)=a_3(M_o^*)$ implies

$$\int_{M^*}[(5/9)(S+S')^3+(64/7)(L^3+L'^3)]=\int_{M_o^*}[(5/9)(S_o+S_o')^3+(64/7)(H^3+H'^3)].$$

Therefore we get $L^3+L'^3=H^3+H'^3$. Since $H+H'\neq 0$ and $L+L'=H+H'$, we obtain $L=H$ and $L'=H'$ (or $L=H'$ and $L'=H$). q.e.d.

REFERENCES

[1] M. BERGER, Le spectre des variétés riemanniennes, Rev. Roum. Math. Pure Appl., **13** (1969), 915-931.

[2] H.F. MCKEAN AND I.M. SINGER, Curvature and the eigenvalues of the Laplacian, Journ. Diff. Geom., **1** (1967), 43-69.

[3] V.K. PATODI, Curvature and the fundamental solution of the heat operator, Journ. Indian Math. Soc., **34** (1970), 269-285.

[4] T. SAKAI, On eigenvalues of Laplacian and curvature of Riemannian manifold, Tohoku Math. Journ., **23** (1971), 589-603.

[5] S. TANNO, Eigenvalues of the Laplacian of Riemannian manifolds, Tohoku Math. Journ., **25** (1973), 391-403.

[6] S. TANNO, A characterization of the canonical spheres by the spectrum, Math. Zeit., **175** (1980), 267-274.

[7] S. TANNO, A characterization of a complex projective space by the spectrum, Kodai Math. Journ., **5** (1982), 230-237.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY