A CERTAIN PROPERTY OF GEODESICS OF THE
FAMILY OF RIEMANNIAN MANIFOLDS O_n^2 (VII)

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§ 0. Introduction.

This is exactly a continuation of Part (VI) ([16]) with the same title written by the present author which proved the following conjecture is true for $9.7 \leq n \leq 16$. On the methods used in it, the lower bound 9.7 of this effective interval is near the crucial values from the argument in it. We shall show that this conjecture is also true for $5 \leq n \leq 9.7$ in the present paper by improving them and some new ideas. As the previous one we shall use the numerical data obtained by computors in the verification. We shall also use the same notation in the previous ones, Parts (I)∼(VI).

The period T of any non-trivial solution $x(t)$ of the non-linear differential equation of order 2:

$$(E) \qquad n x(1-x^2)\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + (1-x^2)(nx^2-1) = 0$$

with a constant $n>1$ such that $x^2+x'^2<1$ is given by the integral:

$$(0.1) \qquad T = \sqrt{nc}\int_{x_0}^{x_1} \frac{dx}{x\sqrt{(n-x)\{x(n-x)^{n-1}-c\}}},$$

where $x_0 = n\{\min x(t)\}^2$, $x_1 = n\{\max x(t)\}^2$, $0<x_0<1<x_1<n$ and $c = x_0(n-x_0)^{n-1} = x_1(n-x_1)^{n-1}$.

CONJECTURE C. The period T as function of $\tau = (x_1-1)/(n-1)$ and n is monotone decreasing with respect to n (>2) for any fixed τ $(0<\tau<1)$.

Here the author thanks heartily Professor Naoto Abe for his cooperation in the numerical computations by computors.

§ 1. The fundamental principle to attain the purpose.

Setting $T = \Omega(\tau, n)$, we have the formulas

$$(1.1) \qquad \frac{\partial\Omega(\tau, n)}{\partial n} = -\frac{\sqrt{c}}{2b^2n\sqrt{n}}\int_{x_0}^1 \frac{(1-x)\sqrt{x(n-x)^{n-1}-c}\,V(x, x_1)dx}{x^2(n-x)^n}$$

((7.4) and Proposition 3 in (III)), where $b=\sqrt{B-c}$, $B=(n-1)^{n-1}$ and $V(x, x_1)$ are defined as follows:

(1.2) $$V(x, x_1):=\frac{x^2 N(x, x_1)}{(1-x)^5\sqrt{n-x}}+\frac{X^2 N(X, x_1)}{(X-1)^5\sqrt{n-X}},$$

where

(1.3) $$f_0(z):=(2n-1-z)B-(n-z)^{n-1}\{n-z+(n-1)z^2\},$$

(1.4) $$f_1(z):=\{4n-1-(2n+1)z\}B-(n-z)^{n-1}\{n+(2n-1)z-(n+1)z^2\},$$

(1.5) $$F_2(z):=-P_2(z)B+(n-z)^{n-1}P_3(z),$$

(1.6) $$P_2(z):=(2n+1)z^2-2(2n^2+5n-4)z+16n^2-16n+3,$$

(1.7) $$P_3(z):=-(n-1)z^3+(2n^2-7n+8)z^2+(n-3)(4n-1)z+3n(2n-1),$$

(1.8) $$\lambda(z):=\log(n-z)+\frac{n-1}{n-z},\qquad \psi(z):=z(n-z)^{n-1},$$

(1.9) $$\tilde{\lambda}(z):=\lambda(z)-\frac{n}{n-1}\cdot\frac{(z-1)^2}{z(n-z)}=\log(n-z)+\frac{nz-1}{(n-1)z},$$

(1.10) $$N(z, x_1):=(n-z)F_2(z)\{\lambda(z)-\tilde{\lambda}(x_1)\}+3(z-1)^2 f_0(z)-2n(z-1)^3\{B-\psi(z)\},$$

and $X=X_n(x)$, $0<x<1<X<n$, defined by $\psi(x)=\psi(X)$.

$V(x, x_1)$ is increasing with respect to x_1 in $X_n(x)\leqq x_1<n$ for $0<x<1$ by Lemma 8.1 in (III). If we have $V(x, X_n(x))>0$ for $0<x<1$, then we obtain $V(x, x_1)>0$ for $0<x<1$ and $X_n(x)\leqq x_1<n$ and so from (1.1) the inequality

$$\frac{\partial\Omega(\tau, n)}{\partial n}<0.$$

We know that $V(x, X_n(x))<0$ near $x=0$ when $2<n<(5+\sqrt{13})/4=2.15138\cdots$ and near $x=1$ when $2<n<(1+\sqrt{13})/2=2.30277\cdots$ by Lemma 8.1 in (III). But we shall show that the above argument is effective for $5\leqq n\leqq 9.7$.

W have the formula (8.1) in (III):

(1.11) $$V(x, X(x))=\frac{x^2\sqrt{n-x}\,F_2(x)\{\lambda(x)-\tilde{\lambda}(X)\}}{(1-x)^5}-\frac{3x^2 f_0(x)}{(x-1)^3\sqrt{n-x}}$$

$$+\frac{2nx^2\{B-\psi(x)\}}{(1-x)^2\sqrt{n-x}}+\frac{n}{n-1}\cdot\frac{XF_2(X)}{(X-1)^3\sqrt{n-X}}+\frac{3X^2 f_0(X)}{(X-1)^3\sqrt{n-X}}$$

$$-\frac{2nX^2\{B-\psi(X)\}}{(X-1)^2\sqrt{n-X}},$$

where $X=X(x)=X_n(x)$, and by Lemma 2.5 in (IV) and (2.4) and (2.5) in (VI)

$$(1.12) \quad V(x,\ X(x)) > \frac{2n(n-X)^{2(n-1)}\{B-\psi(x)\}\sqrt{B}}{\{B-(X-1)^{n-1}\}^2 \phi \sqrt{nB-(X-1)^{n-1}}} + \frac{2nX\sqrt{n-X}\,B}{(n-1)(X-1)^3}H(X)$$

$$+ \frac{x^2\sqrt{n-x}\,F_2(x)\{\lambda(x)-\tilde{\lambda}(X)\}}{(1-x)^5} \quad \text{for} \quad 0 < x < 1,$$

where

$$(1.13) \quad H(X) := \left(\frac{n-X}{n-1}\right)^{n-1}\{(n-1)X^2+(2n-5)X+3n\} - \{8n-5-(2n+1)X\}.$$

For $n > 5/2$ we defined the constant $b=b_n$ by $H(b)=0$, $1 < b < n$, and proved that

$$(1.14) \quad \frac{2n+10}{2n+1} < b_n < 2.268 \quad \text{for} \quad n \geq 4$$

and b_n is decreasing with respect to n (≥ 4) by Table 1, Lemma 4.1 and Lemma 4.3 in (VI).

Since we have

$$H(X) > 0 \quad \text{for} \quad b_n < X < n,$$

we obtain

$$(1.15) \quad V(x,\ X(x)) > 0 \quad \text{for} \quad 0 < x \leq X_n^{-1}(b_n)$$

from (1.2), when $n > 5/2$. Therefore we shall prove the inequality:

$$V(x,\ X(x)) > 0 \quad \text{for} \quad X_n^{-1}(b_n) < x < 1,$$

when $n \geq 5$.

In the following, we set for simplicity

$$U_0(x) := \frac{x^2\sqrt{n-x}}{(1-x)^5}F_2(x)\{\lambda(x)-\tilde{\lambda}(X(x))\}, \qquad U_1(x) := \frac{3x^2 f_0(x)}{(x-1)^3\sqrt{n-x}},$$

$$(1.16) \quad U_2(x) := \frac{2nx^2\{B-\psi(x)\}}{(1-x)^2\sqrt{n-x}}, \qquad U_3(x) := \frac{nxF_2(x)}{(n-1)(x-1)^3\sqrt{n-x}},$$

$$U_4(x) := U_3(X(x)), \qquad U_5(x) := U_1(X(x)), \qquad U_6(x) := U_2(X(x)),$$

then

$$(1.17) \quad V(x,\ X(x)) = U_0(x) - U_1(x) + U_2(x) + U_4(x) + U_5(x) - U_6(x).$$

By means of Proposition 1, Lemma 2.1 in (II) and Lemma 7.1 in (III), we see that $U_i(x)$, $i=0, 1, 2, 4, 5, 6$, are all positive for $0 < x < 1$.

§2. Certain evaluations of $X_n^{-1}(X)$.

LEMMA 2.1. *When $n > 2$ and $k \geq (n+1)^2/(5n-1)$, we have*

$$x = X_n^{-1}(X) < \left(\frac{n-X}{n-1}\right)^{n-k} \quad \text{for} \quad 1 < X < n.$$

Proof. Let $1 < k < n-1$, then the above ineqnality is equivalent to

$$\left(\frac{n-X}{n-1}\right)^{n-k}\left\{n-\left(\frac{n-X}{n-1}\right)^{n-k}\right\}^{n-1} > X(n-X)^{n-1},$$

which can be written as

(2.1) $\dfrac{1}{(n-1)^{n-k}}\left\{n-\left(\dfrac{n-X}{n-1}\right)^{n-k}\right\}^{n-1} > X(n-X)^{k-1}$ for $1 < X < n$.

Both sides of (2.1) have the same value $(n-1)^{k-1}$ at $X=1$. Since we have

$$\left(\frac{1}{(n-1)^{n-k}}\left\{n-\left(\frac{n-X}{n-1}\right)^{n-k}\right\}^{n-1}\right)' = \frac{n-k}{(n-1)^{n-k}}\left\{n-\left(\frac{n-X}{n-1}\right)^{n-k}\right\}^{n-2}\left(\frac{n-X}{n-1}\right)^{n-k-1},$$

$$\left(\frac{1}{(n-1)^{n-k}}\left\{n-\left(\frac{n-X}{n-1}\right)^{n-k}\right\}^{n-1}\right)'' = \frac{n-k}{(n-1)^{n-k+1}}\left\{n-\left(\frac{n-X}{n-1}\right)^{n-k}\right\}^{n-3}\left(\frac{n-X}{n-1}\right)^{n-k-2}$$

$$\times\left[\{n^2-(k+1)n+k-1\}\left(\frac{n-X}{n-1}\right)^{n-k}-n(n-k-1)\right]$$

and

$$(X(n-X)^{k-1})' = (n-kX)(n-X)^{k-2},$$

$$(X(n-X)^{k-1})'' = -(k-1)(2n-kX)(n-X)^{k-3},$$

we obtain

$$\left[\left(\frac{1}{(n-1)^{n-k}}\left\{n-\left(\frac{n-X}{n-1}\right)^{n-k}\right\}^{n-1}\right)'\right]_{X=1} = [(X(n-X)^{k-1})']_{X=1} = (n-k)(n-1)^{k-2}$$

and

$$\left[\left(\frac{1}{(n-1)^{n-k}}\left\{n-\left(\frac{n-X}{n-1}\right)^{n-k}\right\}^{n-1}\right)''\right]_{X=1} - [(X(n-X)^{k-1})'']_{X=1}$$

$$= n(n-1)^{k-4}(k-1)(2n-k-1) > 0,$$

which implies that (2.1) holds near $X=1$.

Now, we assume that (2.1) does not hold in $1 < X < n$. Then there exists a value ζ $(1 < \zeta < n)$ such that

$$\frac{1}{(n-1)^{n-k}}\left\{n-\left(\frac{n-\zeta}{n-1}\right)^{n-k}\right\}^{n-1} = \zeta(n-\zeta)^{k-1}$$

and

$$\frac{n-k}{(n-1)^{n-k}}\left\{n-\left(\frac{n-\zeta}{n-1}\right)^{n-k}\right\}^{n-2}\left(\frac{n-\zeta}{n-1}\right)^{n-k-1} \leqq (n-k\zeta)(n-\zeta)^{k-2},$$

which imply

$$\{1+(n-k-1)\zeta\}\left(\frac{n-\zeta}{n-1}\right)^{n-k} \leqq n-k\zeta.$$

Here, we introduce an auxiliary function

(2.2)
$$\rho_k(X) := \frac{n-kX}{1+(n-k-1)X} \cdot \left(\frac{n-1}{n-X}\right)^{n-k},$$

then the above inequality becomes

(2.3)
$$\rho_k(\zeta) \geqq 1.$$

We obtain easily

$$\rho_k(1) = 1, \qquad \rho_k\left(\frac{n}{k}\right) = 0, \qquad \lim_{X \to n-0} \rho_k(X) = -\infty$$

and

(2.4)
$$\rho_k'(X)$$
$$= -\frac{(n-k)\{n(n-2)-(n+1)(n-k-1)X+k(n-k-1)X^2\}}{(n-1)\{1+(n-k-1)X\}^2}\left(\frac{n-1}{n-X}\right)^{n-k+1}.$$

From $1 < k < n-1$ we have $1+(n-k-1)X > 0$ for $1 < X < n$. Regarding the quadratic polynomial of X:

$$n(n-2)-(n+1)(n-k-1)X+k(n-k-1)X^2,$$

its discriminant is

$$D = (n+1)^2(n-k-1)^2 - 4n(n-2)k(n-k-1)$$
$$= (n-1)(n-k-1)\{(n+1)^2-(5n-1)k\}.$$

Hence, if $k \geqq (n+1)^2/(5n-1)$, then it must be $D \leqq 0$, therefore

$$\rho_k'(X) \leqq 0 \qquad \text{for} \quad 1 \leqq X < n,$$

which implies

$$\rho_k(X) < 1 \qquad \text{for} \quad 1 < X < n.$$

This inequality contradicts to (2.3). On the other hand we see that $(n+1)^2/(5n-1) < n-1$ is equivalent to $n(n-2) > 0$, which holds since $n > 2$. Thus, we have proved the claim of this lemma. Q.E.D.

LEMMA 2.2. *When $n > 2$, let C and k be*

(2.5)
$$1 < C \leqq \frac{5n-1}{2(n+1)}$$

and

(2.6)
$$\frac{n+1+(n-1)C-\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}}{2C} \leqq k < n-1,$$

then we have

$$x = X_n^{-1}(X) < \left(\frac{n-X}{n-1}\right)^{n-k} \qquad \text{for} \quad 1 < X \leqq C.$$

Proof. First we notice that $(5n-1)/2(n+1) < \min(5/2, n)$ and the left hand side of (2.6) is the small root of the quadratic equation of t:

$$C^2t^2 - C\{n+1+(n-1)C\}t + (n^2-1)C - n(n-2) = 0.$$

Since we have

$$[C^2t^2 - C\{n+1+(n-1)C\}t + (n^2-1)C - n(n-2)]_{t=1}$$
$$= (n-2)(C-1)(n-C) > 0$$

and

$$[C^2t^2 - C\{n+1+(n-1)C\}t + (n^2-1)C - n(n-2)]_{t=n-1}$$
$$= -n(n-2) < 0,$$

we obtain the inequality

(2.7) $$1 < \frac{n+1+(n-1)C - \sqrt{(n-1)\{(n-1)C^2 - 2(n+1)C + 5n-1\}}}{2C} < n-1.$$

Therefore we can use the first half of the proof of Lemma 2.1. Since $1 < k < n-1$, the inequality (2.1) holds near $X=1$. Assume that (2.1) does not hold in the interval $1 < X \leq C$. Then, there exists a value ζ $(1 < \zeta \leq C)$ such that

$$\rho_k(\zeta) \geq 1.$$

By means of Lemma 2.1, we may suppose that $k < (n+1)^2/(5n-1)$. As for the quadratic polynomial of X of the right hand side of (2.4):

$$k(n-k-1)X^2 - (n+1)(n-k-1)X + n(n-2),$$

we have

$$\frac{n+1}{2k} > \frac{5n-1}{2(n+1)} \geq C$$

and

$$[k(n-k-1)X^2 - (n+1)(n-k-1)X + n(n-2)]_{X=1} = (n-1)^2 - (n-k)^2 > 0.$$

If we have

$$k(n-k-1)C^2 - (n+1)(n-k-1)C + n(n-2)$$
$$= -C^2k^2 + C\{n+1+(n-1)C\}k - (n^2-1)C + n(n-2) > 0$$

then

$$k(n-k-1)X^2 - (n+1)(n-k-1)X + n(n-2) > 0 \quad \text{for} \quad 1 \leq X \leq C.$$

From the property of the quadratic polynomial of t:

$$C^2t^2 - C\{n+1+(n-1)C\}t + (n^2-1)C - n(n-2)$$

stated in the begining, we see that the above fact holds from (2.6). Thus, we see that $\rho_k(X)$ is decreasing in $1 < X \leq C$, and hence

$$\rho_k(X) < 1 \quad \text{for} \quad 1 < X \leq C,$$

which contradicts to $\rho_k(\zeta) \geqq 1$. Q. E. D.

Here, we set

(2.8) $\qquad k_n(C) := \dfrac{n+1+(n-1)C-\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}}{2C}$,

then

(2.9) $\qquad n-k_n(C) = \dfrac{(n+1)(C-1)+\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}}{2C}$.

LEMMA 2.3. (i) $k_n(C)$ *is increasing with respect to C in* $1 < C \leqq (5n-1)/(2(n+1))$, *with $n > 2$, and $k_n(1)=1$ and $k_n((5n-1)/(2(n+1)))=(n+1)^2/(5n-1)$.*

(ii) $k_n(C)$ *and $n-k_n(C)$ are increasing with respect to n ($\geqq 2.5$), with $1 < C \leqq (5n-1)/(2(n+1))$.*

Proof. (i) From (2.8) we obtain

$$\frac{\partial k_n(C)}{\partial C} = \frac{1}{2C^2\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}}$$
$$\times [(n-1)\{5n-1-(n+1)C\}-(n+1)\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}].$$

Since we have

$$5n-1-(n+1)C \geqq 5n-1-\frac{5n-1}{2}=\frac{5n-1}{2}>0$$

and

$$(n-1)^2\{5n-1-(n+1)C\}^2-(n+1)^2(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}$$
$$=4n(n-1)(n-2)\{5n-1-2(n+1)C\}>0,$$

we obtain

$$\frac{\partial k_n(C)}{\partial C}>0 \quad \text{for} \quad 1<C<\frac{5n-1}{2(n+1)}, \quad \text{with} \quad n>2.$$

(ii) Differentiating the numerator of (2.8) with respect to n, we obtain

$$\frac{(C+1)\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}-\{(n-1)C^2-2nC+5n-3\}}{\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}}$$

and

$$(C+1)^2(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}-\{(n-1)C^2-2nC+5n-3\}^2$$
$$=4(C-1)\{(n-1)^2C^2-(2n^2-1)C+5n^2-6n+2\}>0 \quad \text{for} \quad n\geqq 2.5,$$

because $C>1$ and

$$(2n^2-1)^2-4(n-1)^2(5n^2-6n+2)=-(16n^4-64n^3+80n^2-40n+7)<0$$

for $n\geqq 2.5$.

The claim on $n-k_n(C)$ is evident from the above computation. Q. E. D.

LEMMA 2.4. *When $3<n\leqq 10$, we have*

$$x = X_n{}^{-1}(X) < \left(\frac{n-X}{n-1}\right)^{n-2} \quad \text{for} \quad 1 < X \leqq b_n .$$

Proof. By the same argument used in the proof of Lemma 2.1, it is sufficient to prove that

$$n(n-2) - (n+1)(n-3)X + 2(n-3)X^2 \geqq 0 \quad \text{for} \quad 1 < X \leqq b_n .$$

The discriminant of the quadratic polynomial of X is

$$D = (n-1)(n-3)(n^2 - 8n + 3) ,$$

which is non positive for $3 < n \leqq 4 + \sqrt{13} = 7.6055 \cdots$. Therefore the claim is true for $3 < n \leqq 4 + \sqrt{13}$.

Now, let assume $n > 4 + \sqrt{13}$, then the polynomial is positive for

$$1 \leqq X < \frac{1}{4}\left(n + 1 - \sqrt{\frac{(n-1)(n^2 - 8n + 3)}{n-3}}\right),$$

and $(n+1)/4 < (n/3)$ and

$$[n(n-2) - (n+1)(n-3)X + 2(n-3)X^2]_{X=n/3} = -\frac{1}{9}n(n^2 - n + 9) < 0 .$$

Therefore, if

$$\rho_2\left(\frac{n+1}{4}\right) = \frac{2}{n-1}\left(\frac{4(n-1)}{3n-1}\right)^{n-2} < 1$$

and $b_n \leqq (n+1)/4$, then the claim is true for this n.

We can prove easily that $1/(n-1)(4(n-1)/(3n-1))^{n-2}$ is increasing with respect to n for $n > 5.2$. Hence, we have for $4 + \sqrt{13} < n \leqq 8.5$

$$\rho_2\left(\frac{n+1}{4}\right) \leqq \rho_2\left(\frac{9.5}{4}\right) = \frac{2}{7.5}\left(\frac{4 \times 7.5}{24.5}\right)^{6.5} = 0.9466 \cdots < 1$$

and by means of Lemma 4.3 in (VI) we have

$$b_n < b_7 = 1.7949 \cdots < 2 < \frac{n+1}{4} .$$

(Here we notice that $\rho_2((8.6+1)/4) = 1.00897 \cdots > 1$.)

Next, let assume $n > 8.5$. We obtain

$$\rho_2\left(\frac{n}{5}\right) = \frac{3n}{n^2 - 3n + 5}\left(\frac{5(n-1)}{4n}\right)^{n-2}$$

and we can prove that this is increasing with respect to n for $n \geqq 8$. Hence, we have for $8.5 \leqq n \leqq 9.5$

$$\rho_2\left(\frac{n}{5}\right) \leqq \rho_2(1.9) = \frac{3 \times 9.5}{66.75}\left(\frac{5 \times 8.5}{4 \times 9.5}\right)^{7.5} = 0.98840 \cdots < 1$$

and

$$b_n \leq b_{8.5} = 1.6654 \cdots < \frac{8.5}{5} = 1.7 \leq \frac{n}{5} \quad \left(< \frac{n}{3} \right).$$

Therefore, the claim is true for $8.5 < n \leq 9.5$.

Finally, let assume $n > 9.5$, then we obtain

$$\rho_2 \left(\frac{n}{5.5} \right) = \frac{7n}{2n^2 - 6n + 11} \left(\frac{11(n-1)}{9n} \right)^{n-2}$$

and we can prove that this is increasing with respect to n for $n \geq 9$. Hence, we have for $9.5 \leq n \leq 10$

$$\rho_2 \left(\frac{n}{5.5} \right) = \rho_2 \left(\frac{10}{5.5} \right) = \frac{70}{151} \left(\frac{11 \times 9}{9 \times 10} \right)^8 = 0.99371 \cdots < 1$$

and

$$b_n \leq b_{9.5} = 1.5998 \cdots < \frac{9.5}{5.5} = \frac{19}{11} = 1.\dot{7}\dot{2} \leq \frac{n}{5.5}.$$

Therefore, the claim is also true for $9.5 < n \leq 10$. \hfill Q. E. D.

LEMMA 2.5. (i) *When* $3 < n \leq 10$, *we have*

$$\frac{X-1}{1-x} < e_{n-2} = \left(\frac{n-1}{n-2} \right)^{n-2} \qquad for \quad 1 < X \leq \min(2, b_n).$$

(ii) *When* $4 \leq n \leq 4.5$, *we have*

$$\frac{X-1}{1-x} < \left(\frac{n-1}{n-1.225} \right)^{n-1.169} \qquad for \quad 1 < X \leq 1.225.$$

(iii) *When* $4.5 \leq n \leq 5$, *we have*

$$\frac{X-1}{1-x} < \left(\frac{n-1}{n-1.2} \right)^{n-1.165} \qquad for \quad 1 < X \leq 1.163.$$

Proof. (i) By means of Lemma 2.4 and the means value theorem, for $1 < X \leq \min(2, b_n)$ we have

$$\frac{X-1}{1-x} > \frac{X-1}{1 - \left(\frac{n-X}{n-1} \right)^{n-2}} < \frac{n-1}{n-2} \cdot \frac{1}{\left(\frac{n-X}{n-1} \right)^{n-3}} \leq \left(\frac{n-1}{n-2} \right)^{n-2} = e_{n-2}.$$

(ii) and (iii) By means of Lemma 2.2, for $1 < C \leq (5n-1)/(2(n+1))$, $k = k_n(C)$, $1 < X \leq C$, we have

$$\frac{X-1}{1-x} < \frac{X-1}{1 - \left(\frac{n-X}{1-n} \right)^{n-k}} < \frac{n-1}{n-k} \cdot \frac{1}{\left(\frac{n-X}{n-1} \right)^{n-k-1}} \leq \frac{(n-1)^{n-k}}{(n-k)(n-C)^{n-k-1}}.$$

When $4 \leq n \leq 4.5$, we have $(5n-1)/2(n+1) \geq (19/10)$. For $1 < C \leq 1.225$, we obtain by Lemma 2.3

$$k_n(C) \leq k_{4.5}(1.225) = 1.21248 \cdots < 1.213.$$

Hence we obtain

$$\frac{(n-1)^{n-k}}{(n-k)(n-C)^{n-k-1}} < \left(\frac{n-1}{n-1.225}\right)^{n-k_n(C)}$$

and
$$k_n(1.225) \geqq k_4(1.225) = 1.16940 \cdots > 1.169.$$

When $4.5 \leqq n \leqq 5$, we have $(5n-1)/(2(n+1)) \geqq 43/22$. For $1 < C \leqq 1.163$, we have

$$k_n(C) \leqq k_5(1.163) = 1.19931 \cdots < 1.2.$$

Hence we obtain

$$\frac{(n-1)^{n-k}}{(n-k)(n-C)^{n-k-1}} \leqq \left(\frac{n-1}{n-1.2}\right)^{n-k_n(C)}.$$

and
$$k_n(1.163) \geqq k_{4.5}(1.163) = 1.16581 \cdots > 1.165. \qquad \text{Q. E. D.}$$

§ 3. Some evaluations related with $U_i(x)$.

As for $F_2(x)$ defined by (1.5), we have the formulas

$$(3.1) \qquad F_2'(x) = -2B\{(2n+1)x - (2n^2+5n-4)\} - (n-x)^{n-2}Q_3(x),$$

where

$$(3.2) \qquad Q_3(x) := 2n^2(n+2) + n(n-13)x + 2(n^3-n^2-n+4)x^2 - (n-1)(n+2)x^3$$

by (3.2), (3.3) and (3.5) in (II), and

$$(3.3) \qquad F_2''(x) = -2B(2n+1) - (n-x)^{n-3}\{(n-x)Q_3'(x) - (n-2)Q_3(x)\}$$
$$= -2B(2n+1) - (n-x)^{n-3}\{-n^2(2n^2-n+5) + n(4n^3-5n^2+10n+3)x$$
$$-n(n+1)(2n^2-n+2)x^2 + (n-1)(n+1)(n+2)x^3\},$$

and

$$(3.4) \qquad F_2'''(x) = -n(n-1)(n-x)^{n-4}\widetilde{Q}(x),$$

where

$$(3.5) \qquad \widetilde{Q}_3(x) := 6n(n^2-n+2) - (8n^3-3n^2+19n+6)x + 2(n+1)(n^2+n+4)x^2$$
$$- (n+1)(n+2)x^3,$$

and

$$(3.6) \qquad F_2^{(4)}(x) = -n(n-1)(n-x)^{n-5}\widetilde{R}_3(x),$$

where

$$(3.7) \qquad \widetilde{R}_3(x) := -n(14n^3-33n^2+55n-42) + (12n^4-19n^3+48n^2-35n-18)x$$
$$- (2n^4+3n^3+11n^2-6n-16)x^2 + (n-1)(n+1)(n+2)x^3.$$

LEMMA 3.1. *When $n > 2$, $F_2(x)$ is decreasing in $0 < x < 1$.*

Proof. We get from (3.5)

$$\tilde{Q}_3{}'(x) = -(8n^3 - 3n^2 + 19n + 6) + 4(n+1)(n^2 + n + 4)x - 3(n+1)(n+2)x^2$$

and

$$1 < \frac{2(n^2 + n + 4)}{3(n+2)} < n \quad \text{for} \quad n > 2 .$$

Since we have

$$\tilde{Q}_3{}'(0) = -(8n^3 - 3n^2 + 19n + 6) < 0 ,$$

$$\tilde{Q}_3{}'(1) = -4(n-1)(n^2 - n + 1) < 0 ,$$

we obtain

$$\tilde{Q}_3{}'(x) < 0 \quad \text{for} \quad 0 \leqq x \leqq 1 .$$

Since we have

$$\tilde{Q}_3(0) = 6n(n^2 - n + 2) > 0 \quad \text{and} \quad \tilde{Q}_3(1) = 0 ,$$

we obtain

$$\tilde{Q}_3(x) > 0 \quad \text{for} \quad 0 \leqq x < 1$$

and so

$$F_2{}'''(x) < 0 \quad \text{for} \quad 0 \leqq x < 1 .$$

Hence $F_2{}''(x)$ is decreasing in $0 \leqq x < 1$. Now, we have

$$F_2{}''(0) = -2B(2n+1) + n^{n-1}(2n^2 - n + 5)$$
$$= (n-1)^{n-1} \{ e_{n-1}(2n^2 - n + 5) - 2(2n+1) \} ,$$

where $e_{n-1} = (1 + (1/n - 1))^{n-1}$. When $n > 2$, we have

$$e_{n-1}(2n^2 - n + 5) - 2(2n+1) > 2(2n^2 - n + 5) - 2(2n+1) = 2(2n^2 - 3n + 4) > 0 ,$$

and hence

$$F_2{}''(0) > 0 .$$

Since we see easily that $F_2{}''(1) = 0$, it must be

$$F_2{}''(x) > 0 \quad \text{for} \quad 0 \leqq x < 1 .$$

Hence $F_2{}'(x)$ is increasing in $0 \leqq x < 1$. We can prove easily that $F_2{}'(1) = 0$. Hence it must be

$$F_2{}'(x) < 0 \quad \text{for} \quad 0 \leqq x < 1 ,$$

which implies the claim of this lemma.   Q.E.D.

LEMMA 3.2. $x^2\sqrt{n-x}/(1-x)^5$ *is increasing with respect to* x $(0 < x < 1)$, *when* $n > 1$.

Proof. We have easily

$$\frac{d}{dx} \cdot \frac{x^2\sqrt{n-x}}{(1-x)^5} = \frac{x\{4n + (6n-5)x - 5x^2\}}{2(1-x)^6\sqrt{n-x}} > 0 \quad \text{for} \quad 0 < x < 1 ,$$

when $n > 1$.

Regarding Lemma 2.1 we have   Q.E.D.

$$n-\frac{(n+1)^2}{5n-1}=\frac{(n-1)(4n+1)}{5n-1},$$

and so here we introduce two auxiliary functions as

$$(3.8) \qquad y=y(n,\,X):=\left(\frac{n-X}{n-1}\right)^{(n-1)(4n+1)/(5n-1)},$$

$$(3.9) \qquad w=w(n,\,X):=\left(\frac{n-X}{n-1}\right)^{n-1}.$$

When $n>2$, we have by Lemma 2.1 and Lemma 2.1 in (IV), respectively, the inequalities

$$(3.10) \qquad w(n,\,X)<x=X_n^{-1}(X)<y(n,\,X) \qquad \text{for} \quad 1<X<n.$$

LEMMA 3.3. *When $n>2$, we have*

$$(3.11) \qquad \frac{U_0(x)}{B}>\frac{w^2\sqrt{n-w}}{(1-w)^5}\left\{-P_2(y)+\left(\frac{n-y}{n-1}\right)^{n-1}P_3(y)\right\}$$

$$\times\left\{\log\frac{n-y}{n-X}+\frac{nXy-(2n-1)X+n-y}{(n-1)X(n-y)}\right\} \qquad \text{for} \quad 0<x<1.$$

Proof. By means of (1.6), Lemma 3.1, Lemma 3.2, (3.10) and the fact that $\lambda(x)$ is decreasing in $0<x<1$, and $\tilde{\lambda}(X)$ is also decreasing in $1<X<n$, and we have

$$\lambda(x)-\tilde{\lambda}(X)>\lambda(y)-\tilde{\lambda}(X)=\log(n-y)+\frac{n-1}{n-y}-\log(n-X)-\frac{nX-1}{(n-1)X}$$

$$=\log\frac{n-y}{n-X}+\frac{nXy-(2n-1)X+n-y}{(n-1)X(n-y)},$$

we obtain immediately (3.11).   Q. E. D.

LEMMA 3.4. *When $n>1$, $f_0(x)$ is negative in $0<x<1$ and it is increasing there and $f_0(1)=0$.*

Proof. We get from (1.3)

$$f_0'(x)=-B+(n-x)^{n-2}\{(n^2-1)x^2-n(2n-1)x+n^2\},$$

$$f_0''(x)=-n(n-1)(x-1)\{(n+1)x-3n\}(n-x)^{n-3},$$

and hence

$$f_0''(x)<0 \qquad \text{for} \quad 0<x<1,$$

therefore $f_0'(x)$ is decreasing there and $f_0'(1)=0$. Hence, we see that $f_0'(x)>0$ for $0<x<1$ and so $f_0(x)$ is increasing in $0<x<1$. Since $f_0(1)=0$, we obtain that $f_0(x)<0$ for $0<x<1$.   Q. E. D.

LEMMA 3.5. *When $n>2$, we have*

$$(3.12) \qquad -\frac{U_1(x)}{B}>\frac{3y^2}{(1-y)^3\sqrt{n-y}}\left[2n-1-w-\left(\frac{n-w}{n-1}\right)^{n-1}\{n-w+(n-1)w^2\}\right].$$

Proof. We obtain easily this inequality from (1.16), Lemma 3.4 and (3.10).

Q. E. D.

We can easily prove the following

LEMMA 3.6. *When* $n > 2$, *we have*

$$(3.13) \qquad \frac{U_2(x)}{B} > \frac{2nw^2(1-Xw)}{(1-w)^2\sqrt{n-w}} \qquad \text{for} \quad 0 < x < 1.$$

Since we can represent $U_4(x)$, $U_5(x)$ and $U_6(x)$ as

$$(3.14) \qquad \begin{cases} \dfrac{U_4(x)}{B} = \dfrac{nX}{(n-1)(X-1)^3\sqrt{n-X}} \cdot \{-P_2(X) + wP_3(X)\}, \\[2mm] \dfrac{U_5(x)}{B} = \dfrac{3X^2}{(X-1)^3\sqrt{n-X}} \cdot [(2n-1-X) - w\{n-X+(n-1)X^2\}], \\[2mm] \dfrac{U_6(x)}{B} = \dfrac{2nX^2(1-Xw)}{(X-1)^2\sqrt{n-X}} \qquad \text{with} \quad X = X_n(x), \end{cases}$$

when $n > 2$, we obtain from (3.11), (3.12), (3.13) and (3.14) the following inequality

$$\frac{V(x, X(x))}{B} > \frac{w^2\sqrt{n-w}}{(1-w)^5}\left\{-P_2(y) + \left(\frac{n-y}{n-1}\right)^{n-1}P_3(y)\right\}$$

$$\times \left\{\log\frac{n-y}{n-X} + \frac{nXy - (2n-1)X + n - y}{(n-1)X(n-y)}\right\}$$

$$+ \frac{3y^2}{(1-y)^3\sqrt{n-y}}\left[2n-1-w-\left(\frac{n-w}{n-1}\right)^{n-1}\{n-w+(n-1)w^2\}\right]$$

$$+ \frac{2nw^2(1-Xw)}{(1-w)^2\sqrt{n-w}} + \frac{nX}{(n-1)(X-1)^3\sqrt{n-X}}\{-P_2(X) + wP_3(X)\}$$

$$+ \frac{3X^2}{(X-1)^3\sqrt{n-X}}[2n-1-X-w\{n-X+(n-1)X^2\}] - \frac{2nX^2(1-Xw)}{(X-1)^2\sqrt{n-X}}$$

for $0 < x < 1$. Arranging the right hand side of this inequality, we obtain a fundamental formula for our purpose.

PROPOSITION 1. *When* $n > 2$, *we have*

$$(3.15) \qquad V(x, X(x)) > (n-1)^{n-1}K(n, X(x)) \qquad \text{for} \quad 0 < x < 1,$$

where

$$(3.16) \qquad K(n, X) := \frac{w^2\sqrt{n-w}}{(1-w)^5} \cdot \left[\left\{-P_2(y) + \left(\frac{n-y}{n-1}\right)^{n-1}P_3(y)\right\}\right.$$

$$\times \left\{\log\frac{n-y}{n-X} + \frac{nXy - (2n-1)X + n - y}{(n-1)X(n-y)}\right\} + \left.\frac{2n(1-w)^3(1-Xw)}{n-w}\right]$$

$$+ \frac{3y^2}{(1-y)^3\sqrt{n-y}} \cdot \left[2n-1-w-\left(\frac{n-w}{n-1}\right)^{n-1}\{n-w+(n-1)w^2\}\right]$$

$$-\frac{X}{(n-1)(X-1)^3\sqrt{n-X}}\cdot[Q_2(X)+wQ_3(X)]\,,$$

(3.17) $Q_2(z):=(4n^2+2n-3)z^2-(4n^3+18n^2-19n+3)z+n(4n-1)(4n-3)\,,$

(3.18) $Q_3(z):=(n-1)(2n-3)z^3-(2n^3-9n^2+13n-3)z^2-2n(2n^2-8n+3)z$
$$-3n^2(2n-1)\,,$$

and $w=w(n,\,X)$, $y=y(n,\,X)$ by (3.8), (3.9).

Proof. We show only the computation related with the last term in (3.16). In fact, from the last three terms in the inequality described above we have

$$Q_2(X)=nP_2(X)-3(n-1)X(2n-1-X)+2n(n-1)X(X-1)$$
$$=n(2n+1)X^2-2n(2n^2+5n-4)X+n(4n-1)(4n-3)$$
$$+3(n-1)X^2-3(2n^2-3n+1)X+2n(n-1)X^2-2n(n-1)X$$
$$=(4n^2+2n-3)X^2-(4n^3+18n^2-19n+3)X+n(4n-1)(4n-3)\,,$$

$$Q_3(X)=-nP_3(X)+3(n-1)X\{n-X+(n-1)X^2\}-2n(n-1)X^2(X-1)$$
$$=n(n-1)X^3-n(2n^2-7n+8)X^2-n(n-3)(4n-1)X-3n^2(2n-1)$$
$$+3(n-1)^2X^3-3(n-1)X^2+3n(n-1)X-2n(n-1)X^3+2n(n-1)X^2$$
$$=(n-1)(2n-3)X^3-(2n^3-9n^2+13n-3)X^2-2n(2n^2-8n+3)X-3n^2(2n-1)\,.$$

<div align="right">Q. E. D.</div>

In the following, we want to show that

$$V(x,\,X(x))>0\qquad\text{for}\quad X_n^{-1}(b_n)<x<1$$

by a finite number of evaluations of $K(n,\,X)$ by computors, that we have used this kind of argument in (VI).

§4. Certain properties of $K(n,\,X)$.

On $H(X)$ we have by (3.1) and (3.2) in (VI)

(4.1) $H'(X)=-\dfrac{1}{n-1}\Big(\dfrac{n-X}{n-1}\Big)^{n-2}\{(n^2-1)X^2-3nX+n(n+2)\}+2n+1\,,$

(4.2) $H''(X)=\dfrac{n}{n-1}\Big(\dfrac{n-X}{n-1}\Big)^{n-3}(X-1)\{(n+1)X-(n+4)\}$

and

$$H(1)=H'(1)=H''(1)=0\,.$$

Hence $H'(X)$ is negative and decreasing in $1<X<(n+4)/(n+1)$ and increasing in $(n+4)/(n+1)<X<n$ and $H'(n)=2n+1>0$. Therefore, we can define a constant a_n by

(4.3) $$H'(a_n)=0, \qquad 1<a_n<n$$

for each $n>2$. On a_n we know the following fact ([18]).

PROPOSITION 2. $H(X)$ is decreasing in $1<X<a_n$ and increasing in $a_n<X<n$ and a_n has the properties:

(4.4) $$\frac{n+4}{n+1}<a_n<n \qquad for \quad n>2,$$

(4.5) $$a_n<\frac{2n+10}{2n+1} \qquad for \quad n>2.5,$$

(4.6) $$\frac{2n+9}{2n+1}<a_n \qquad for \quad n\geqq\frac{1+\sqrt{137}}{4}=3.17617\cdots$$

and a_n is decreasing as function of n in $3.5\leqq n<\infty$.

LEMMA 4.1. $w(n, X)$ is increasing with respect to n for $1<n$ $(1<X<n)$ and decreasing with respect to X for $1<X<n$.

Proof. Since we have

$$\frac{\partial}{\partial n}\log\left(\frac{n-X}{n-1}\right)^{n-1}=\log\frac{n-X}{n-1}+\frac{X-1}{n-X}\to 0 \qquad (\text{as } n\to\infty)$$

and

$$\frac{\partial^2}{\partial n^2}\log\left(\frac{n-X}{n-1}\right)^{n-1}=-\frac{(X-1)^2}{(n-1)(n-X)^2}<0 \qquad for \quad 1<X<n,$$

we obtain immediately this lemma. Q. E. D.

LEMMA 4.2. $P_2(x)$ and $P_3(x)$ are increasing with respect to n for $1<n$ as functions of n for each fixed x $(0\leqq x\leqq 1)$.

Proof. From (1.6) we obtain

$$\frac{\partial P_2(x)}{\partial n}=2x^2-2(4n+5)x+16(2n-1),$$

which becomes $24(n-1)$ (>0) at $x=1$ and $(4n+5)/2>1$. Hence we have $\partial P_2(x)/\partial n>0$ for $n>1$ and $0\leqq x\leqq 1$.

Next, from (1.7) we obtain

$$\frac{\partial P_3(x)}{\partial n}=-x^3+(4n-7)x^2+(8n-13)x+3(4n-1)$$

and

$$\frac{\partial^2 P_3(x)}{\partial x\partial n}=-3x^2+2(4n-7)x+8n-13,$$

hence we have

$$\left[\frac{\partial^2 P_3(x)}{\partial x\partial n}\right]_{x=0}=8n-13, \qquad \left[\frac{\partial^2 P_3(x)}{\partial x\partial n}\right]_{x=1}=16n-30$$

and

$$\left[\frac{\partial P_3(x)}{\partial n}\right]_{x=0}=3(4n-1)>0\,, \qquad \left[\frac{\partial P_3(x)}{\partial n}\right]_{x=1}=24(n-1)>0\,.$$

From these we see that if $n\geqq 13/8$, then

$$\frac{\partial P_3(x)}{\partial n}>0 \qquad \text{for}\quad 0\leqq x\leqq 1\,.$$

If $1<n<13/8$, then $(4n-7)/3<-(1/6)$ and so $\partial^2 P_3(x)/\partial x\partial n<0$ for $0\leqq x\leqq 1$. There-fore, it holds the same inequality above.   Q.E.D.

LEMMA 4.3. *$y(n, X)$ is increasing with respect to n for $1<n$ $(1<X<n)$ and decreasing with respect to X for $1<X<n$.*

Proof. We have

$$\frac{\partial}{\partial n}\log\Big(\frac{n-X}{n-1}\Big)^{(n-1)(4n+1)/(5n-1)}=\frac{4(5n^2-2n+2)}{(5n-1)^2}\log\frac{n-X}{n-1}+\frac{(4n+1)(X-1)}{(5n-1)(n-X)}$$

$$=\frac{4(5n^2-2n+2)}{(5n-1)^2}\Big[\log\frac{n-X}{n-1}+\frac{(X-1)(4n+1)(5n-1)}{4(5n^2-2n+2)(n-X)}\Big]\,.$$

We have

$$\log\frac{n-X}{n-1}+\frac{(X-1)(4n+1)(5n-1)}{4(5n^2-2n+2)(n-X)}\to 0 \qquad (\text{as } n\to\infty)$$

and

$$\frac{\partial}{\partial n}\Big[\log\frac{n-X}{n-1}+\frac{(X-1)(4n+1)(5n-1)}{4(5n^2-2n+2)(n-X)}\Big]$$

$$=\frac{X-1}{(n-1)(n-X)}-\frac{(X-1)\{100n^4+10n^3-57n^2+4n-2-45n(n-2)X\}}{4(5n^2-2n+2)^2(n-X)^2}$$

and

$$4(5n^2-2n+2)^2(n-X)-(n-1)(100n^4+10n^3-57n^2+4n-2)+45n(n-1)(n-2)X$$

$$=10n^4+163n^3-93n^2+22n-2-X(100n^4-125n^3+231n^2-122n+16)\,.$$

Now, regarding the last expression we see that

$$100n^4-125n^3+231n^2-122n+16>206n^2-122n+16>0 \qquad \text{for}\quad n>1$$

and so it is decreasing with respect to X when $n>1$ and it becomes at $X=1$

$$-90n^4+288n^3-324n^2+144n-18=-18(n-1)^2(5n-1)<0\,.$$

Therefore it is negative for $1<X<n$, when $n>1$, from which we obtain

$$\frac{\partial}{\partial n}\Big[\log\frac{n-X}{n-1}+\frac{(X-1)(4n+1)(5n-1)}{4(5n^2-2n+2)(n-X)}\Big]<0 \qquad \text{for}\quad 1<X<n$$

and so

$$\log\frac{n-X}{n-1}+\frac{(X-1)(4n+1)(5n-1)}{4(5n^2-2n+2)(n-X)}>0$$

and hence

$$\frac{\partial}{\partial n} y(n, X) > 0 \quad \text{for} \quad 1 < X < n,$$

when $n > 1$.

The second claim is evident. Q. E. D.

LEMMA 4.4. $\lambda(y(n, X)) - \tilde{\lambda}(X)$ *is increasing with respect to* X $(1 < X < n)$, *when* $n > 1$, *and* $\lambda(x) - \tilde{\lambda}(Y)$ *is decreasing with respect to* n *for* $n > 1$ *with* $0 < x < 1$ *and* $1 < Y < n$.

Proof. The first part holds from Lemma 4.3 and the facts that $\lambda(x)$ is decreasing with respect to x in $0 < x < 1$ and $\tilde{\lambda}(X)$ is decreasing with respect to X in $1 < X < n$.

Next, for fixed x $(0 < x < 1)$ and Y $(1 < Y < n)$ we obtain from (1.8) and (1.9)

$$\frac{\partial}{\partial n} \lambda(x) - \frac{\partial}{\partial n} \tilde{\lambda}(Y) = \frac{n+1-2x}{(n-x)^2} - \frac{Y^2 + n(n-3)Y + n}{(n-1)^2 Y(n-Y)}$$

and

$$\frac{\partial}{\partial x} \frac{n+1-2x}{(n-x)^2} = \frac{2(1-x)}{(n-x)^3} > 0,$$

$$\frac{\partial}{\partial Y} \frac{Y^2 + n(n-3)Y + n}{Y(n-Y)} = \frac{n(Y-1)\{(n-2)Y+n\}}{Y^2(n-Y)^2} > 0.$$

Hence

$$\frac{n+1-2x}{(n-x)^2} - \frac{Y^2 + n(n-3)Y + n}{(n-1)^2 Y(n-Y)} < \frac{n-1}{(n-1)^2} - \frac{1 + n(n-3) + n}{(n-1)^3} = 0,$$

which implies the second claim. Q. E. D.

LEMMA 4.5. $n(1-w)^3(1-Xw)/(n-w)$ *is increasing with respect to* X *in* $1 < X < n$ *with* $n > 1 + 1/\sqrt{2}$ *and decreasing with respect to* n *in* $1 < n$ *with* $1 < X < n$.

Proof. First we show $1 - Xw > 0$. In fact from (3.9) we get

$$\frac{\partial}{\partial X} Xw = \left(\frac{n-X}{n-1}\right)^{n-1} - X\left(\frac{n-X}{n-1}\right)^{n-2} = -\frac{n(X-1)}{n-1}\left(\frac{n-X}{n-1}\right)^{n-2} < 0$$

and so

$$1 - Xw > 1 - \lim_{X \to 1} Xw = 1 - 1 = 0.$$

Now, we have

$$\frac{\partial}{\partial X} \frac{(1-w)^3(1-Xw)}{n-w} = -\frac{(1-w)^2\{3n-1-2w+X(n-4nw+3w^2)\}}{(n-w)^2} \frac{\partial w}{\partial X} - \frac{w(1-w)^3}{n-w}$$

$$= \frac{(n-1)w(1-w)^2\{3n-1-2w+X(n-4nw+3w^2)\}}{(n-X)(n-w)^2} - \frac{w(1-w)^3}{n-w}$$

$$= \frac{w(1-w)^2}{(n-X)(n-w)^2} \cdot \tilde{\rho}(w),$$

where

(4.7) $\tilde{\rho}(w) := 2n^2 - 4n + 1 + (n^2 - n + 2)w - nw^2$

$$+ X\{n^2 - (4n^2 - 3n + 1)w + (3n - 2)w^2\}.$$

We shall show $\tilde{\rho}(w) \geqq 0$ for $0 < w < 1$, considering X as function of w. Since we have

$$\lim_{w \to 0} X(w) = n \quad \text{and} \quad \lim_{w \to 1} X(w) = 1$$

we obtain

$$\tilde{\rho}(0) = 2n^2 - 4n + 1 + n^3 = (n-1)(n^2 + 3n - 1) > 0, \qquad \tilde{\rho}(1) = 0.$$

Since we have

$$\frac{\partial X}{\partial w} = -\frac{n - X}{(n-1)w}, \qquad \frac{\partial^2 X}{\partial w^2} = \frac{n - X}{(n-1)w^2} + \frac{1}{(n-1)w}\frac{\partial X}{\partial w},$$

we obtain

$$\frac{\partial X}{\partial w}\Big|_{w=1} = -1 \quad \text{and} \quad \frac{\partial^2 X}{\partial w^2}\Big|_{w=1} = \frac{n-2}{n-1}.$$

We obtain from (4.7)

(4.8) $\dfrac{\partial \tilde{\rho}}{\partial w} = n^2 - n + 2 - 2nw + X\{-(4n^2 - 3n + 1) + 2(3n - 2)w\}$

$$- \frac{n - X}{(n-1)w}\{n^2 - (4n^2 - 3n + 1)w + (3n - 2)w^2\},$$

from which we get

$$\frac{\partial \tilde{\rho}}{\partial w}\Big|_{w=1} = 0, \qquad \frac{\partial^2 \tilde{\rho}}{\partial w^2}\Big|_{w=1} = 5n(n-1) > 0.$$

Hence, $\tilde{\rho}(w)$ is positive near $w = 1$ in $0 < w < 1$.

Here, we suppose that $\tilde{\rho}(w) \geqq 0$ does not hold in $0 < w < 1$. Then there exist ξ, η $(0 < \xi < \eta < 1)$ such that

$$\tilde{\rho}(\xi) = \tilde{\rho}(\eta) = 0, \qquad \tilde{\rho}'(\xi) \leqq 0, \qquad \tilde{\rho}'(\eta) \geqq 0.$$

Since we have from (4.8)

(4.8)' $(n-1)w\dfrac{\partial \tilde{\rho}}{\partial w} = -n^3 + (5n^3 - 5n^2 + 4n - 2)w - n(5n - 4)w^2$

$$+ X\{n^2 - (4n^3 - 3n^2 + n)w + (6n^2 - 7n + 2)w^2\}.$$

On the other hand, regarding (4.7) we see that for $0 < w < 1$

$$2n^2 - 4n + 1 + (n^2 - n + 2)w - nw^2 > \min\{2n^2 - 4n + 1, 3(n-1)^2\},$$

and so if $n \geqq 1 + 1/\sqrt{2} = 1.7071 \cdots$, then

$$2n^2 - 4n + 1 + (n^2 - n + 2)w - nw^2 > 0 \quad \text{for} \quad 0 < w < 1,$$

and

$$n^2 - (4n^2 - 3n + 1)w + (3n - 2)w^2 < 0 \quad \text{at} \quad w = \xi \text{ and } \eta.$$

Eliminating X from (4.7) and (4.8′) at $w = \xi, \eta$, we obtain respectively

$$\{n^2 - (4n^2 - 3n + 1)w + (3n - 2)w^2\} \{-n^3 + (5n^3 - 5n^2 + 4n - 2)w - n(5n - 4)w^2\}$$

$$- \{2n^2 - 4n + 1 + (n^2 - n + 2)w - nw^2\} \{n^2 - (4n^3 - 3n^2 + n)w + (6n^2 - 7n + 2)w^2\}$$

$$\geqq 0 \quad \text{at} \quad w = \xi \quad \text{and} \quad \leqq 0 \quad \text{at} \quad w = \eta,$$

when $n \geqq 1 + 1/\sqrt{2}$. The left hand side of the above inequalities can be expressed as

$$n(n-1)[-n(n^2 + 3n - 1) + (17n^3 - 14n^2 + 10n - 1)w - (16n^3 + 8n^2 - 13n + 7)w^2$$

$$+ (25n^2 - 15n + 2)w^3 - (9n - 6)w^4].$$

Here we set for simplicity

(4.9) $$R_4(x) := 3(3n - 2)x^4 - (25n^2 - 15n + 2)x^3 + (16n^3 + 8n^2 - 13n + 7)x^2$$

$$- (17n^3 - 14n^2 + 10n - 1)x + n(n^2 + 3n - 1),$$

then it must hold

(4.10) $$R_4(\xi) \leqq 0 \quad \text{and} \quad R_4(\eta) \geqq 0.$$

We have

$$R_4'(x) = 12(3n - 2)x^3 - 3(25n^2 - 15n + 2)x^2 + 2(16n^3 + 8n^2 - 13n + 7)x$$

$$- 17n^3 + 14n^2 - 10n + 1,$$

$$\frac{1}{2}R_4''(x) = 18(3n - 2)x^2 - 3(25n^2 - 15n + 2)x + 16n^3 + 8n^2 - 13n + 7.$$

Since we have

$$R_4''(0) = 16n^3 + 8n^2 - 13n + 7 > 0, \qquad R_4''(1) = 2(n-1)^2(16n - 35)$$

and

$$R_4'(0) = -(17n^3 - 14n^2 + 10n - 1) < 0, \qquad R_4'(1) = 15(n-1)^3 > 0,$$

therefore $R_4(x)$ must be decreasing first and then turns into increasing in $0 < x < 1$. We have

$$R_4(0) = n(n^2 + 3n - 1) > 0 \quad \text{and} \quad R_4(1) = 0.$$

From these facts it is impossible to have (4.10), when $n \geqq 1 + 1/\sqrt{2}$. Thus we obtain

$$\tilde{\rho}(w) \geqq 0 \quad \text{for} \quad 0 < w < 1, \quad \text{when} \quad n \geqq 1 + 1/\sqrt{2}.$$

Finally, regarding the second part we have

$$\frac{\partial}{\partial n}\log\frac{n(1-w)^3(1-Xw)}{n-w} = \frac{1}{n} - \frac{1}{n-w} + \left(-\frac{3}{1-w} - \frac{X}{1-Xw} + \frac{1}{n-w}\right)\frac{\partial w}{\partial n}$$

$$= -\frac{w}{n(n-w)} - \left\{\frac{3n-1-2w}{(1-w)(n-w)} + \frac{1}{1-Xw}\right\}\frac{\partial w}{\partial n} < 0$$

by Lemma 4.1, when $n>1$.   Q. E. D.

LEMMA 4.6. $f_0(x)/B = 2n-1-x-(n-x/(n-1))^{n-1}\{n-x+(n-1)x^2\}$ *is increasing with respect to x in $0<x<1$ with $n>1$ and decreasing with respect to n in $2<n$ with $0<x<1$.*

Proof. The first part is evident from Lemma 3.4. We have

$$\frac{\partial}{\partial n}\frac{f_0(x)}{B} = 2 - \left(\frac{n-x}{n-1}\right)^{n-1}\left[(1+x^2) + \left(\log\frac{n-x}{n-1} - \frac{1-x}{n-x}\right)\right.$$

$$\left.\times\{n-x+(n-1)x^2\}\right]$$

and

$$\frac{(n-1)^{n-1}}{(n-x)^{n-3}}\frac{\partial^2}{\partial x\partial n}\frac{f_0(x)}{B} = n(1-x)^2(n-2x)$$

$$+(n-x)\{n^2-n(2n-1)x+(n^2-1)x^2\}\left(\log\frac{n-x}{n-1} - \frac{1-x}{n-x}\right).$$

We see easily the following inequalities

$$n(1-x)^2(n-2x)>0 \quad\text{and}\quad n^2-n(2n-1)x+(n^2-1)x^2>0$$

$$\text{for}\quad 0<x<1 \quad\text{and}\quad n>2.$$

On the other hand we have

$$\lim_{n\to\infty}\left(\log\frac{n-x}{n-1} - \frac{1-x}{n-x}\right)=0$$

and

$$\frac{\partial}{\partial n}\left(\log\frac{n-x}{n-1} - \frac{1-x}{n-x}\right)= -\frac{(1-x)^2}{(n-1)(n-x)^2}<0,$$

hence

$$\log\frac{n-x}{n-1} - \frac{1-x}{n-x}>0.$$

Thus we see that

$$\frac{\partial^2}{\partial x\partial n}\frac{f_0(x)}{B}>0 \quad\text{for}\quad 0<x<1 \quad\text{and}\quad n>2,$$

and so

$$\frac{\partial}{\partial n}\frac{f_0(x)}{B}<\lim_{x\to1}\frac{\partial}{\partial n}\frac{f_0(x)}{B}=0.$$ Q. E. D.

LEMMA 4.7 $Q_2(X)$ *is decreasing with respect to X in $1\leq X\leq2.5$ with $n\geq3$*

and increasing with respect to n in $3 \leq n$ with $1 \leq X \leq 2.5$, and positive there.

Proof. From (3.17) we get for $1 \leq X \leq 2.5$

$$Q_2'(X) = 2(4n^2 + 2n - 3)X - (4n^3 + 18n^2 - 19n + 3)$$

$$\leq 5(4n^2 + 2n - 3) - (4n^3 + 18n^2 - 19n + 3)$$

$$= -(4n^3 - 2n^2 - 29n + 18) < 0 \quad \text{for} \quad n \geq 3$$

and

$$Q_2(2.5) = 6n^3 - 36n^2 + 63n - 26.25 \geq 9n - 26.25 > 0 \quad \text{for} \quad n \geq 3 \,.$$

From these we obtain the first part and

$$Q_2(X) > 0 \quad \text{for} \quad 1 \leq X \leq 2.5, \quad \text{when} \quad n \geq 3 \,.$$

Next, we have

$$\frac{\partial Q_2(X)}{\partial n} = 2(4n+1)X^2 - (12n^2 + 36n - 19)X + 48n^2 - 32n + 3$$

and

$$\frac{12n^2 + 36n - 19}{4(4n+1)} > 3 \quad \text{for} \quad n \geq 3 \,,$$

and then $\partial Q_2(X)/\partial n$ is decreasing with respect to X in $1 < X < 3$. Since we have

$$\frac{\partial Q_2(X)}{\partial n} \bigg|_{X=2.5} = 18n^2 - 72n + 63 \geq 9 \quad \text{for} \quad n \geq 3 \,,$$

we obtain

$$\frac{\partial Q_2(X)}{\partial n} > 0 \quad \text{for} \quad n \geq 3 \quad \text{with} \quad 1 \leq X \leq 2.5 \,. \qquad \text{Q. E. D.}$$

LEMMA 4.8.   (i) $Q_3(X) < 0$ *for* $1 \leq X \leq 2.5$, *with* $n \geq 3.5$.

(ii) $Q_3(X)$ *is decreasing with respect to X in* $1 < X < 2.2$, *with* $n \geq 4.46$, *and is decreasing in* $1 < X < \gamma_n$ *and increasing in* $\gamma_n < X < 2.5$, *with* $3.5 \leq n < 4.46$, *where*

$$\gamma_n := \frac{2n^3 - 9n^2 + 13n - 3 + \sqrt{4n^6 - 12n^5 - 23n^4 + 66n^3 - 11n^2 - 24n + 9}}{3(n-1)(2n-3)}$$

(iii) $Q_3(X)$ *is decreasing with respect to n in* $3 \leq n$, *with* $1 \leq X \leq 2.5$.

Proof. Proof of (i). We get from (3.18)

$$Q_3(X) = (2n^2 - 5n + 3)X^3 - (2n^3 - 9n^2 + 13n - 3)X^2 - 2n(2n^2 - 8n + 3)X - 3n^2(2n-1)$$

$$= -(2X^2 + 4X + 6)n^3 + (2X^3 + 9X^2 + 16X + 3)n^2 - (5X^3 + 13X^2 + 6X)n$$

$$\quad + 3X^3 + 3X^2,$$

$$Q_3'(X) = 3(2n^2 - 5n + 3)X^2 - 2(2n^3 - 9n^2 + 13n - 3)X - 2n(2n^2 - 8n + 3) \,,$$

$$Q_3(1) = -6(n-1)^2(2n-1) < 0 \,,$$

$$Q_3'(1) = -(8n^3 - 40n^2 + 47n - 15) < 0 \quad \text{for} \quad n \geq 3.5 \,,$$

$$Q_3(2.5) = -28.5n^3 + 130.5n^2 - 174.375n + 65.625 < 0 \quad \text{for} \quad n \geq 3 \,,$$

which imply $Q_3(X) < 0$ for $1 \leq X \leq 2.5$, with $n \geq 3.5$.

Proof of (ii). We have

$$Q_3'(2.2) = -(12.8n^3 - 84.64n^2 + 135.8n - 56.76)$$

$$\leq -12.91808n + 56.76 \leq -0.8546368 \quad \text{for} \quad n \geq 4.46 \,,$$

which implies $Q_3'(X) < 0$ in $1 \leq X \leq 2.2$ with the above evaluation of $Q_3'(1)$. (Here we notice $\gamma_{4.4} = 2.1803 \cdots < 2.2$).

Now, for $n \geq 3.5$, let γ_n be the large root of the quadratic equation $Q_3'(X) = 0$ of X, then we obtain its expression above. The rest part is evident. (Here, we notice

$$\gamma_{3.5} = \frac{18 + \sqrt{219}}{30} = 1.093 \cdots, \qquad \gamma_4 = \frac{33 + \sqrt{2196}}{45} = 1.7682 \cdots) \,.$$

Proof of (iii). We have from (3.18)

$$\frac{\partial Q_3(X)}{\partial n} = -3(2X^2 + 4X + 6)n^2 + 2(2X^3 + 9X^2 + 16X + 3)n - (5X^3 + 13X^2 + 6X) \,,$$

$$\frac{\partial^2 Q_3(X)}{\partial X \partial n} = -12(X+1)n^2 + 2(6X^2 + 18X + 16)n - (15X^2 + 26X + 6) \,,$$

$$\frac{\partial^2 Q_3(X)}{\partial X \partial n}\bigg|_{X=1} = -(24n^2 - 80n + 47) < 0 \,,$$

$$\frac{\partial Q_3(X)}{\partial n}\bigg|_{X=1} = -12(n-1)(3n-2) < 0 \,,$$

$$\frac{\partial Q_3(X)}{\partial n}\bigg|_{X=2.5} = -85.5n^2 + 261n - 174.375 < 0 \quad \text{for} \quad n \geq 3 \,,$$

which imply easily

$$\frac{\partial Q_3(X)}{\partial n} < 0 \quad \text{for} \quad 1 \leq X \leq 2.5 \quad \text{and} \quad n \geq 3. \qquad \text{Q. E. D.}$$

LEMMA 4.9. $X/(X-1)^3\sqrt{n-X}$ *is decreasing with respect to* X *in* $1 < X \leq 2.5$ *with* $n \geq 3.$

Proof. We have easily

$$\frac{\partial}{\partial X} \frac{X}{(X-1)^3\sqrt{n-X}} = -\frac{-5X^2 + (4n-1)X + 2n}{2(X-1)^4(n-X)^{3/2}}$$

and

$$[-5X^2 + (4n-1)X + 2n]_{X=1} = 6(n-1) > 0 \,,$$

$$[-5X^2 + (4n-1)X + 2n]_{X=2.5} = 12n - 33.75 \geq 2.25 \quad \text{for} \quad n \geq 3 \,.$$

Hence we obtain

$$-5X^2+(4n-1)X+2n>0 \quad \text{in} \quad 1\leqq X\leqq 2.5 \quad \text{with} \quad n\geqq 3.$$

<div align="right">Q. E. D.</div>

PROPOSITION 3. $K(n, X)$ defined by (3.16) satisfies the following inequality: Let $n\geqq 3$ and $1<X<b_n$, and n_1, n_2, X_1, X_2 be such that

$$3\leqq n_1\leqq n\leqq n_2, \qquad a_n\leqq X_1\leqq X\leqq X_2<b_n,$$

then

$$K(n, X)\geqq K(n_1, n_2; X_1, X_2),$$

where

$$(4.11) \quad K(n_1, n_2; X_1, X_2)=\frac{(w(n_1, X_2))^2\sqrt{n_1-w(n_1, X_2)}}{(1-w(n_1, X_2))^5}$$

$$\times\left[\left\{-P_2(n_2, y(n_2, X_1))+\left(\frac{n_1-y(n_2, X_1)}{n_1-1}\right)^{n_1-1}P_3(n_1, y(n_2, X_1))\right\}\right.$$

$$\times\left\{\log\frac{n_2-y(n_2, X_1)}{n_2-X_1}+\frac{n_2X_1y(n_2, X_1)-(2n_2-1)X_1+n_2-y(n_2, X_1)}{(n_2-1)X_1(n_2-y(n_2, X_1))}\right.$$

$$\left.+\frac{2n_2\{1-w(n_2, X_1)\}^3\{1-X_1w(n_2, X_1)\}}{n_2-w(n_2, X_1)}\right]$$

$$+\frac{3(y(n_2, X_1))^2}{(1-y(n_2, X_1))^3\sqrt{n_1-y(n_2, X_1)}}\left[2n_2-1-w(n_1, X_2)-\left(\frac{n_2-w(n_1, X_2)}{n_2-1}\right)^{n_2-1}\right.$$

$$\left.\left.\times\{n_2-w(n_1, X_2)+(n_2-1)(w(n_1, X_2))^2\}\right]\right]$$

$$-\frac{X_1Q_2(n_2, X_1)}{(n_1-1)(X_1-1)^3\sqrt{n_1-X_1}}-\frac{X_2w(n_1, X_2)Q_3(n_1, X^*)}{(n_2-1)(X_2-1)^3\sqrt{n_2-X_2}}$$

and $X^*=X_1$ for $n\geqq 4.46$ and $X^*=X_2$ when $X>\gamma_n$ and $X^*=X_1$ when $X\leqq\gamma_n$ for $(8+\sqrt{34})/4\leqq n<4.46$, and $X^*=X_2$ for $3\leqq n<(8+\sqrt{34})/4$ and $P_2(n, x)$, $P_3(n, x)$, $Q_2(n, x)$, $Q_3(n, x)$ stand for $P_2(x)$, $P_3(x)$, $Q_2(x)$, $Q_3(x)$ with n by (1.6), (1.7), (3.17), (3.18) respectively.

Proof. By Lemma 4.1 and Lemma 4.2 we have first

$$(4.12) \qquad w(n_1, X_2)\leqq w(n, X)\leqq w(n_2, X_1),$$

$$(4.13) \qquad y(n_1, X_2)\leqq y(n, X)\leqq y(n_2, X_1).$$

From Lemma 3.2 we obtain

$$(4.14) \qquad \frac{(w(n, X))^2\sqrt{n-w(n, X)}}{(1-w(n, X))^5}\geqq\frac{(w(n_1, X_2))^2\sqrt{n_1-w(n_1, X_2)}}{(1-w(n_1, X_2))^5}>0.$$

From Lemma 3.1 and the above inequalities (4.13) we have

$$-P_2(n, y(n, X))+\left(\frac{n-y(n, X)}{n-1}\right)^{n-1}P_3(n, y(n, X))$$

$$\geqq -P_2(n, \ y(n_2, \ X_1)) + \Big(\frac{n-y(n_2, \ X_1)}{n-1}\Big)^{n-1} P_3(n, \ y(n_2, \ X_1))$$

(by Lemma 4.2)

$$\geqq -P_2(n_2, \ y(n_2, \ X_1)) + \Big(\frac{n-y(n_2, \ X_1)}{n-1}\Big)^{n-1} P_3(n_1, \ y(n_2, \ X_1)).$$

On the other hand we see easily that $w(n, \ x) = (n-x/(n-1))^{n-1}$ with $0 < x < 1$ is increasing with respect to n (>1) by the same argument in Lemma 4.1. Hence we obtain

(4.15) $\qquad -P_2(n, \ y(n, \ X)) + \Big(\frac{n-y(n, \ X)}{n-1}\Big)^{n-1} P_3(n, \ y(n, \ X))$

$$\geqq -P_2(n_2, \ y(n_2, \ X_1)) + \Big(\frac{n_1-y(n_2, \ X_1)}{n_1-1}\Big)^{n_1-1} P_3(n_1, \ y(n_2, \ X_1)).$$

Next, denoting $\lambda(x)$ and $\tilde{\lambda}(x)$ by (1.8) and (1.9) by $\lambda_n(x)$ and $\tilde{\lambda}_n(x)$ respectively, by Lemma 4.4 we obtain

(4.16) $\qquad \lambda_n(y(n, \ X)) - \tilde{\lambda}_n(X) \geqq \lambda_n(y(n_2, \ X_1)) - \tilde{\lambda}_n(X_1)$

$$\geqq \lambda_{n_2}(y(n_2, \ X_1)) - \tilde{\lambda}_{n_2}(X_1)$$

$$= \log \frac{n_2-y(n_2, \ X_1)}{n_2-X_1} + \frac{n_2 X_1 y(n_2, \ X_1) - (2n_2-1)X_1 + n_2 - y(n_2, \ X_1)}{(n_2-1)X_1(n_2-y(n_2, \ X_1))} > 0.$$

From Lemma 4.5 we obtain

(4.17) $\quad \dfrac{n(1-w(n, \ X))^3(1-Xw(n, \ X))}{n-w(n, \ X)} \geqq \dfrac{n_2(1-w(n_2, \ X_1))^3(1-X_1w(n_2, \ X_1))}{n_2-w(n_2, \ X_1)} > 0.$

Then, we proceed to the second part of (3.16). Noticing $f_0(x) < 0$ for $0 < x < 1$, by (4.12), (4.13) and Lemma 4.6 we obtain

(4.18) $\quad \dfrac{(y(n, \ X))^2}{(1-y(n, \ X))^3 \sqrt{n-y(n, \ X)}} \cdot \Big[2n-1-w(n, \ X) - \Big(\dfrac{n-w(n, \ X)}{n-1}\Big)^{n-1}$

$$\times \{n-w(n, \ X) + (n-1)(w(n, \ X))^2\}\Big]$$

$$\geqq \dfrac{(y(n_2, \ X_1)^2}{(1-y(n_2, \ X_1))^3 \sqrt{n_1-y(n_2, \ X_1)}} \cdot \Big[2n_2-w(n_1, \ X_2) - \Big(\dfrac{n_2-w(n_1, \ X_2)}{n_2-1}\Big)^{n_2-1}$$

$$\times \{n_2-w(n_1, \ X_2) + (n_2-1)(w(n_1, \ X_2))^2\}\Big].$$

Lastly we deal with the third part of (3.16). By Lemma 4.7, Lemma 4.8 and Lemma 4.9 we obtain

(4.19) $\qquad -\dfrac{X}{(n-1)(X-1)^3 \sqrt{n-X}} \cdot [Q_2(n, \ X) + w(n, \ X)Q_3(n, \ X)]$

$$\geq -\frac{X_1}{(n_1-1)(X_1-1)^3\sqrt{n_1-X_1}}\cdot Q_2(n_2,\,X_1)$$

$$-\frac{X_2}{(n_2-1)(X_2-1)^3\sqrt{n_2-X_2}}\cdot w(n_1,\,X_2)Q_3(n_1,\,X^*)\,,$$

where $X^*=X_1$ for $n\geq 4.46$, and $X^*=X_1$ when $X\leq \gamma_n$ and $X^*=X_2$ when $X>\gamma_n$ for $(8+\sqrt{34})/4\leq n<4.46$, and $X^*=X_2$ for $3\leq n<(8+\sqrt{34})/4$.

Using these inequalities $(4.14)\sim(4.19)$, we obtain

$$K(n,\,X)\geq K(n_1,\,n_2;\,X_1,\,X_2)\,.\qquad\qquad \text{Q. E. D.}$$

§5. Evaluation of $V(x,\,X_n(x))$ near $x=1$.

Refering to the argument used in §8 of (III), we shall try more fine evaluation of $V(x,\,X_n(x))$ near $x=1$ than it.

LEMMA 5.1. *When $4\leq n\leq 10$, we have*

$$U_0(x)>\frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}\cdot(X-1)\left\{1-\frac{(12n-13)p_n+6n-10}{6(n-1)}(X-1)\right.$$

$$\left.+\frac{(4n-5)p_n(p_n+3n-5)}{6(n-1)^2}(X-1)^2\right\}\qquad \text{for }1<X\leq C\leq \min(2,\,b_n)\,,$$

where if $C=\min(2,\,b_n)$, then $p_n=e_{n-2}=(1+1/(n-2))^{n-2}$; if $4\leq n\leq 4.5$ and $C=1.225$, then $p_n=((n-1)/(n-1.225))^{n-1.169}$; and if $4.5\leq n\leq 5$ and $C=1.163$, then $p_n=((n-1)/(n-1.2))^{n-1.165}$, $X=X_n(x)$.

Proof. We obtain $F_2(1)=F_2'(1)=F_2''(1)=F_2'''(1)=0$ from $(3.1)\sim(3.5)$ and so for x $(0<x<1)$ there exists x_1 $(x<x_1<1)$ such that

$$F_2(x)=-\frac{n(n-1)}{24}(x-1)^4\tilde{R}_3(x_1)$$

by (3.6). We have from (3.7)

$$\tilde{R}_3'(x)=12n^4-19n^3+48n^2-35n-18-2(2n^4+3n^3+11n^2-6n-16)x$$

$$+3(n-1)(n+1)(n+2)x^2>0\,,$$

because

$$\tilde{R}_3'(0)>0\,,\qquad \tilde{R}_3'(1)>0\,,\qquad \frac{2n^4+3n^3+11n^2-6n-16}{3(n-1)(n+1)(n+2)}>1\,,$$

when $n\geq 2$. Therefore we have

$$\tilde{R}_3(x)<\tilde{R}_3(1)=-4(n-1)^2(n^2-n+1)<0\,,$$

and hence we obtain the inequality:

$$(5.1)\qquad F_2(x)>\frac{1}{6}n(n-1)^{n-2}(n^2-n+1)(1-x)^4\qquad \text{for }0<x<1\,.$$

Now, since we have from (1.8) and (1.9)

$$\lambda'(x) = -\frac{1-x}{(n-x)^2}, \qquad \tilde{\lambda}'(x) = -\frac{(x-1)\{n+(n-1)x\}}{(n-1)x^2(n-x)},$$

$$\tilde{\lambda}''(x) = -\frac{2}{(n-1)x^3} - \frac{1}{(n-x)^2}, \qquad \frac{dX}{dx} = -\frac{(1-x)(n-x)^{n-2}}{(X-1)(n-X)^{n-2}},$$

we can obtain

$$\frac{d}{dX}\{\lambda(x) - \tilde{\lambda}(X)\} = \frac{(X-1)(n-X)^{n-2}}{(n-x)^n} + \frac{(X-1)\{n+(n-1)X\}}{(n-1)X^2(n-X)},$$

which becomes 0 at $X=1$, and

$$\frac{d^2}{dX^2}\{\lambda(x) - \tilde{\lambda}(X)\} = \frac{(n-1)(2-X)(n-X)^{n-3}}{(n-x)^n} - \frac{n(X-1)^2(n-X)^{2n-4}}{(1-x)(n-x)^{2n-1}}$$
$$+ \frac{2}{(n-1)X^3} + \frac{1}{(n-X)^2},$$

and

$$\frac{d}{dX}\frac{(2-X)(n-X)^{n-3}}{(n-x)^n} = -\frac{(n-2)(3-X)(n-X)^{n-4}}{(n-x)^n} - \frac{n(X-1)(2-X)(n-X)^{2n-5}}{(1-x)(n-x)^{2n-1}}$$
$$-\frac{(n-2)2(n-1)^{n-4}}{(n-1)^n} - \frac{n(n-1)^{2n-5}}{(n-1)^{2n-1}}p_n = -\frac{2(n-2)+np_n}{(n-1)^4},$$

where p_n denotes the right hand side of (i), (ii) and (iii) in Lemma 2.5 according to these cases.   Hence we obtain

$$\frac{(n-1)(2-X)(n-X)^{n-3}}{(n-x)^n} > \frac{1}{(n-1)^2} - \frac{2(n-2)+np_n}{(n-1)^3}(X-1).$$

We obtain also

$$\frac{n(X-1)^2(n-X)^{2n-4}}{(1-x)(n-x)^{2n-1}} < \frac{n(X-1)(n-1)^{2n-4}}{(n-1)^{2n-1}}p_n = \frac{np_n}{(n-1)^3}(X-1),$$

and

$$\frac{2}{(n-1)X^3} > \frac{2}{n-1} - \frac{6}{n-1}(X-1), \qquad \frac{1}{(n-X)^2} > \frac{1}{(n-1)^2} + \frac{2}{(n-1)^3}(X-1).$$

Using these inequalities in the right hand side of the above equality, we obtain for $n \geq 4$

$$\frac{d^2}{dX^2}\{\lambda(x) - \tilde{\lambda}(X)\} > \frac{2n}{(n-1)^2} - \frac{2n(p_n+3n-5)}{(n-1)^3}(X-1).$$

which implies

(5.2) $$\lambda(x)-\tilde{\lambda}(X)>\frac{n}{(n-1)^2}(X-1)^2-\frac{n(p_n+3n-5)}{3(n-1)^3}(X-1)^3$$

$$\text{for } 1<X\leqq C.$$

Last, since we have

$$\frac{d}{dX}x^2\sqrt{n-x}=-\frac{x(4n-5x)(n-X)^{n-2}}{2(n-x)^{n-3/2}}\cdot\frac{X-1}{1-x},$$

we obtain the following inequality:

(5.3) $$x^2\sqrt{n-x}>\sqrt{n-1}-\frac{(4n-5)p_n}{2\sqrt{n-1}}(X-1) \quad \text{for } 1<X\leqq C.$$

Using these inequalities (5.1)$\sim$(5.3) for the expression of $U_0(x)$ in (1.16) and $X-1>1-x$ by Proposition 4 in (IV), we obtain

$$U_0(x)=\frac{x^2\sqrt{n-x}}{(1-x)^5}F_2(x)\{\lambda(x)-\tilde{\lambda}(X(x))\}$$

$$>\frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}\left\{1-\frac{(4n-5)p_n}{2(n-1)}(X-1)\right\}\left\{1-\frac{p_n+3n-5}{3(n-1)}(X-1)\right\}(X-1),$$

which becomes the required one in the statement of the lemma.　　Q. E. D.

　　We prepare some formulas on $f_0(x)$.　We obtain from (1.3)

(5.4) $$\begin{cases} f_0(x)=(2n-1-x)B-(n-x)^{n-1}\{(n-1)x^2-x+n\}, \\ f_0'(x)=-B+(n-x)^{n-2}\{(n^2-1)x^2-n(2n-1)x+n^2\}, \\ f_0''(x)=-n(n-1)(x-1)\{(n+1)x-3n\}(n-x)^{n-3}, \\ f_0'''(x)=n(n-1)\{(n^2-1)x^2-(6n^2-5n-2)x+n(7n-8)\}(n-x)^{n-4}, \\ f_0^{(4)}(x)=-n(n-1)(n-2)\{(n^2-1)x^2-(8n^2-7n-3)x+n(13n-15)\}(n-x)^{n-5}, \\ f_0^{(5)}(x)=n(n-1)(n-2)(n-3)\{(n^2-1)x^2-(10n^2-9n-4)x+3n(7n-8)\}(n-x)^{n-6}, \\ f_0^{(6)}(x)=-n(n-1)(n-2)(n-3)(n-4)\{(n^2-1)x^2-(12n^2-11n-5)x+n(31n-35)\} \\ \qquad\qquad \times(n-x)^{n-7}. \end{cases}$$

LEMMA 5.2.　*When $3\leqq n$, we have*

$$U_6(x)>\frac{n(2n-1)(n-1)^{n-5/2}}{2}+\frac{n(5n^2-3n+1)(n-1)^{n-7/2}}{4}(X-1)$$

$$+\left\{\frac{n(n-2)(n-3)q_n(n-1)^{1/2}(n-\varepsilon_0)^{n-6}}{40}-\frac{n(n-2)(3n-1)(4n-3)(n-1)^{n-9/2}}{8}\right\}$$

$$\times(X-1)^2+\frac{n(n-2)(n-3)(4n-3)q_n(n-\varepsilon_0)^{n-6}}{80\sqrt{n-1}}(X-1)^3$$

$$\text{for } 1<X\leqq C \ (\leqq2),$$

where we set

(5.5) $q_n = q_n(C) = (n^2-1)C^2 - (10n^2-9n-4)C + 3n(7n-8)$

and $\varepsilon_0 = 1$ for $3 \leq n < 6$, $\varepsilon_0 = 2$ for $n \geq 6$.

   Proof. We have $f_0(1) = f_0'(1) = f_0''(1) = 0$ and

$$f_0'''(1) = n(2n-1)(n-1)^{n-2}, \qquad f_0^{(4)}(1) = -2n(n-2)(3n-1)(n-1)^{n-3},$$

hence for X $(1 < x \leq C)$ there exists X_1 $(1 < X_1 < X)$ such that

$$f_0(X) = \frac{n(2n-1)B}{6(n-1)}(X-1)^3 - \frac{n(n-2)(3n-1)B}{12(n-1)^2}(X-1)^4$$

$$+ \frac{(X-1)^5}{120} n(n-1)(n-2)(n-3)\{(n^2-1)X_1^2 - (10n^2-9n-4)X_1 + 3n(7n-8)\}$$

$$\times (n-X_1)^{n-6}.$$

We have

$$[(n^2-1)x^2 - (10n^2-9n-4)x + 3n(7n-8)]_{x=1} = 3(n-1)(4n-1) > 0$$

and

$$(10n^2-9n-4)/2(n^2-1) > 2 \qquad \text{for} \quad n > 3/2,$$

hence, assuming $C \leq 2$, we obtain

$$(n^2-1)X_1^2 - (10n^2-9n-4)X_1 + 3n(7n-8)$$

$$\geq (n^2-1)C^2 - (10n^2-9n-4)C + 3n(7n-8) := q_n.$$

Thus we obtain

(5.6) $$f_0(X) > \frac{n(2n-1)B}{6(n-1)}(X-1)^3 - \frac{n(n-2)(3n-1)B}{12(n-1)^2}(X-1)^4$$

$$+ \frac{n(n-1)(n-2)(n-3)q_n(n-\varepsilon_0)^{n-6}}{120}(X-1)^5 \qquad \text{for} \quad 1 < x \leq C \ (\leq 2),$$

where $\varepsilon_0 = 1$ for $3 \leq n < 6$ and $\varepsilon_0 = 2$ for $n \geq 6$.
   Next, since we have

(5.7) $$\frac{d}{dX}\frac{X^2}{\sqrt{n-X}} = \frac{X(4n-3X)}{2(n-X)^{3/2}} \quad \text{and} \quad \frac{d^2}{dX^2}\frac{X^2}{\sqrt{n-X}} = \frac{3X^2-8nX+8n^2}{4(n-X)^{5/2}} > 0,$$

we obtain

(5.8) $$\frac{X^2}{\sqrt{n-X}} > \frac{1}{n-1}\left\{1 + \frac{4n-3}{2(n-1)}(X-1)\right\}$$

Using (5.7) and (5.8), we obtain from (1.16)

$$U_5(x) = \frac{3X^2 f_0(X)}{(X-1)^3 \sqrt{n-X}} > \frac{3}{\sqrt{n-1}} \left\{ 1 + \frac{4n-3}{2(n-1)}(X-1) \right\} \left\{ \frac{n(2n-1)B}{6(n-1)} \right.$$

$$\left. - \frac{n(n-2)(3n-1)B}{12(n-1)^2}(X-1) + \frac{n(n-1)(n-2)(n-3)q_n(n-\varepsilon_0)^{n-6}}{120}(X-1)^2 \right\},$$

which becomes the required one in the statement of the lemma.   Q. E. D.

LEMMA 5.3.   *When $n > 4$, we have*

$$-U_1(x) > -\frac{n(2n-1)(n-1)^{n-5/2}}{2} + \frac{n(5n^2-3n+1)(n-1)^{n-7/2}}{4}(1-x)$$

$$+ \left\{ \frac{n(8n^3-50n^2+44n-9)(n-1)^{n-9/2}}{16} - \frac{3(n-2)(n-3)(7n-8)(n-1)^{n-1/2}e_{n-1}}{40n^3} \right\}$$

$$\times (1-x)^2 - \frac{n(n-2)(3n-1)(8n^2-8n+3)(n-1)^{n-11/2}}{32}(1-x)^3$$

$$\text{for } 0 < x < 1.$$

Proof.   For x $(0 < x < 1)$ there exists x_1 $(x < x_1 < 1)$ such that

$$f_0(x) = -\frac{n(2n-1)B}{6(n-1)}(1-x)^3 - \frac{n(n-2)(3n-1)B}{12(n-1)^2}(1-x)^4$$

$$- \frac{(1-x)^5}{120} n(n-1)(n-2)(n-3) \{(n^2-1)x_1^2 - (10n^2-9n-4)x_1 + 3n(7n-8)\}$$

$$\times (n-x_1)^{n-6}.$$

We see easily that $f_0^{(6)}(x) < 0$ for $0 < x < 1$, when $n > 4$, and $f_0^{(5)}(x_1) < f_0^{(5)}(0)$ $= 3n^{n-4}(n-1)(n-2)(n-3)(7n-8)$, which implies

(5.8) $$f_0(x) > -\frac{n(2n-1)B}{6(n-1)}(1-x)^3 - \frac{n(n-2)(3n-1)B}{12(n-1)^2}(1-x)^4$$

$$- \frac{(n-1)(n-2)(n-3)(7n-8)n^{n-4}}{40}(1-x)^5.$$

From (5.7) we obtain

$$\frac{d^3}{dx^3}\frac{x^2}{\sqrt{n-x}} = \frac{3(x-2n)(x-4n)}{8(n-x)^{7/2}} > 0 \quad \text{for } 0 < x < n.$$

Using this and (5.7) with x in place of X, we obtain

(5.9) $$\frac{x^2}{\sqrt{n-x}} < \frac{1}{\sqrt{n-1}} - \frac{4n-3}{2(n-1)^{3/2}}(1-x) + \frac{8n^2-8n+3}{8(n-1)^{5/2}}(1-x)^2.$$

Using (5.8) and (5.9), we obtain from (1.16)

$$-U_1(x) = \frac{3x^2 f_0(x)}{(1-x)^3 \sqrt{n-x}} > -\frac{1}{\sqrt{n-1}} \left\{ 1 - \frac{4n-3}{2(n-1)}(1-x) + \frac{8n^2-8n+3}{8(n-1)^2}(1-x)^2 \right\}$$

$$\times \left\{ \frac{n(2n-1)B}{2(n-1)} + \frac{n(n-2)(3n-1)B}{4(n-1)^2}(1-x) + \frac{3(n-1)(n-2)(n-3)(7n-8)e_{n-1}B}{40n^3} \right\}(1-x)^2$$

$$= -\frac{n(2n-1)(n-1)^{n-5/2}}{2} + \frac{n(5n^2-3n+1)(n-1)^{n-7/2}}{4}(1-x)$$

$$+ \left\{ \frac{n(8n^3-50n^2+44n-9)(n-1)^{n-9/2}}{16} - \frac{3(n-2)(n-3)(7n-8)(n-1)^{n-1/2}e_{n-1}}{40n^3} \right\}(1-x)^2$$

$$- \frac{n(n-2)(3n-1)(8n^2-8n+3)(n-1)^{n-11/2}}{32}(1-x)^3$$

$$+ \frac{3(n-2)(n-3)(7n-8)e_{n-1}(n-1)^{n-5/2}}{320n^3}(1-x)^3$$

$$\times [4(n-1)(4n-3) - (8n^2-8n+3)(1-x)],$$

in which we see about the last part in the blackets that

$$4(n-1)(4n-3) - (8n^2-8n+3)(1-x) > 4(n-1)(4n-3) - (8n^2-8n+3)$$

$$= 8n^2-20+9 > 0.$$

Therefore, if we omit this part in the above expression, then we can obtain the required inequality.   Q.E.D.

We prepare here some formulas on $\phi(x)$. We have

(5.10)
$$\begin{cases} \phi(x) = x(n-x)^{n-1}, \quad \phi'(x) = n(1-x)(n-x)^{n-2}, \\ \phi''(x) = -n(n-1)(2-x)(n-x)^{n-3}, \\ \phi'''(x) = n(n-1)(n-2)(3-x)(n-x)^{n-4}, \\ \phi^{(4)}(x) = -n(n-1)(n-2)(n-3)(4-x)(n-x)^{n-5}. \end{cases}$$

LEMMA 5.4. *When $n > 3$, we have*

$$U_2(x) > n^2(n-1)^{n-5/2} - \frac{n^2(8n-1)(n-1)^{n-7/2}}{6}(1-x)$$

$$- \frac{n^2(n-2)(4n-3)(n-1)^{n-9/2}}{3}(1-x)^2 \quad for \quad 0 < x < 1.$$

Proof. For x $(0 < x < 1)$ there exists x_1 $(x < x_1 < 1)$ such that

$$B - x(n-x)^{n-1} = \frac{n(n-1)^{n-2}}{2}(1-x)^2 + \frac{n(n-1)(n-2)(3-x_1)(n-x_1)^{n-4}}{6}(1-x)^3,$$

from which, assuming $n > 3$, we obtain

(5.11) $$\frac{B-x(n-x)^{n-1}}{(1-x)^2} > \frac{n(n-1)^{n-2}}{2} + \frac{n(n-2)(n-1)^{n-3}}{3}(1-x).$$

As in the proof of Lemma 5.2, we obtain

(5.12) $$\frac{x^2}{\sqrt{n-x}} > \frac{1}{\sqrt{n-1}}\left\{1 - \frac{4n-3}{2(n-1)}(1-x)\right\}.$$

Using (5.11) and (5.12), we obtain from (1.16)

$$U_2(x) = \frac{2nx^2\{B-\phi(x)\}}{(1-x)^2\sqrt{n-x}}$$

$$> n^2(n-1)^{n-5/2}\left\{1 - \frac{4n-3}{2(n-1)}(1-x)\right\}\left\{1 + \frac{2(n-2)}{3(n-1)}(1-x)\right\},$$

which becomes the required one in the statement. Q. E. D.

LEMMA 5.5. *When $n>4$, we have*

$$-U_6(x) > -n^2(n-1)^{n-5/2} - \frac{n^2(8n-1)(n-1)^{n-7/2}}{6}(X-1)$$

$$+ \left\{\frac{n^2(n-2)(13n-3)(n-1)^{n-9/2}}{12} - \frac{n^2(8n^2-8nC+3C^2)(n-1)^{n-2}}{8(n-C)^{5/2}}\right\}(X-1)^2$$

$$- \frac{n^2(n-2)(n-3)(4n-3)(n-1)^{n-11/2}}{8}(X-1)^3 \quad \text{for } 1<X\leqq C\leqq\min(2, b_n).$$

Proof. For X $(1<X\leqq C)$ there exists X_1 $(1<X_1<X)$ such that

$$B-X(n-X)^{n-1} = \frac{n(n-1)^{n-2}}{2}(X-1)^2 - \frac{n(n-2)(n-1)^{n-3}}{3}(X-1)^3$$

$$+ \frac{n(n-1)(n-2)(n-3)(4-X_1)(n-X_1)^{n-5}}{24}(X-1)^4.$$

Since we have $((4-x)(n-x)^{n-5})' = -(n-4)(5-x)(n-x)^{n-6} < 0$ for $n>4$ and $x<4$, we obtain easily

(5.13) $$\frac{B-X(n-X)^{n-1}}{(X-1)^2} < \frac{n(n-1)^{n-2}}{2}\left\{1 - \frac{2(n-2)}{3(n-1)}(X-1) + \frac{(n-2)(n-3)}{4(n-1)^2}(X-1)^2\right\}.$$

Then there exists X_2 $(1<X_2<X)$ such that

$$\frac{X^2}{\sqrt{n-X}} = \frac{1}{\sqrt{n-1}} + \frac{4n-3}{2(n-1)^{3/2}}(X-1) + \frac{3X_2^2-8nX_2+8n^2}{8(n-X_2)^{5/2}}(X-1)^2$$

by means of (5.7). Since we have

$$\frac{d}{dX}\frac{3X^2-8nX+8n^2}{(n-X)^{5/2}} = \frac{3(X^2-4nX+8n^2)}{2(n-X)^{7/2}} > 0,$$

we obtain

(5.14) $\dfrac{X^2}{\sqrt{n-X}} < \dfrac{1}{\sqrt{n-1}}\Big\{1 + \dfrac{4n-3}{2(n-1)}(X-1) + \dfrac{8n^2-8nC+3C^2}{8(n-C)^2}\sqrt{\dfrac{n-1}{n-C}}(X-1)^2\Big\}$.

Using (5.13) and (5.14), we obtain from (1.16)

$$-U_6(x) = -\frac{2nX^2\{B-\phi(X)\}}{(X-1)^2\sqrt{n-X}}$$

$$> -n^2(n-1)^{n-5/2}\Big\{1 + \frac{4n-3}{2(n-1)}(X-1) + \frac{8n^2-8nC+3C^2}{8(n-C)^2}\sqrt{\frac{n-1}{n-C}}(X-1)^2\Big\}$$

$$\times\Big\{1 - \frac{2(n-2)}{3(n-1)}(X-1) + \frac{(n-2)(n-3)}{4(n-1)^2}(X-1)^2\Big\}$$

$$= -n^2(n-1)^{n-5/2} - \frac{n^2(8n-1)(n-1)^{n-7/2}}{6}(X-1)$$

$$+\Big\{\frac{n^2(n-2)(13n-3)(n-1)^{n-9/2}}{12} - \frac{n^2(8n^2-8nC+3C^2)(n-1)^{n-2}}{8(n-C)^{5/2}}\Big\}(X-1)^2$$

$$-\frac{n^2(n-2)(n-3)(4n-3)(n-1)^{n-11/2}}{8}(X-1)^3$$

$$+n^2(n-1)^{n-5/2}\cdot\frac{8n^2-8nC+3C^2}{8(n-C)^2}\sqrt{\frac{n-1}{n-C}}\cdot\frac{n-2}{n-1}\Big[\frac{2}{3} - \frac{n-3}{4(n-1)}(X-1)\Big](X-1)^3,$$

in which we see about the last part in the blackets that

$$\frac{2}{3} - \frac{n-3}{4(n-1)}(X-1) > \frac{2}{3} - \frac{n-3}{4(n-1)} = \frac{5n+1}{12(n-1)} > 0 .$$

Therefore, if we omit this part in the above expression, then we can obtain the required inequality.   Q. E. D.

LEMMA 5.6. *When* $n\geq 4$, *we have*

$$U_4(x) > \frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}(X-1) - \Big\{\frac{n^2(2n^3-4n^2+6n-3)(n-1)^{n-9/2}}{12}$$

$$+ \frac{n^2(n-5)(n^2-n+1)(n-1)^{n-9/2}}{6}\Big(\frac{n-\varepsilon_4}{n-1}\Big)^{n-6}\Big\}(X-1)^2 + \frac{n^2(n-1)^{n-11/2}}{12}$$

$$\times\Big\{(n-5)(2n^3-4n^2+6n-3)\Big(\frac{n-\varepsilon_4}{n-1}\Big)^{n-6} - \frac{(2n-1)(4n^3-7n^2+9n-4)}{2}\Big\}(X-1)^3$$

$$+ \frac{1}{24}n^2(2n-1)(n-5)(4n^3-7n^2+9n-4)(n-1)^{n-13/2}\Big(\frac{n-\varepsilon_4}{n-1}\Big)^{n-6}(X-1)^4$$

$$\textit{for}\ \ 1 < X \leq C \leq 2 ,$$

where we put ε_4 *as follows: if* $n\geq 6$, $\varepsilon_4 = 1$; *if* $5 < n < 6$, $\varepsilon_4 = C$; *if* $4\leq n\leq 5$, $\varepsilon_4 = -\infty$ *and* $C = 1.225$ *for* $4\leq n < 4.5$ *and* $C = 1.163$ *for* $4.5\leq n\leq 5$ *as special cases.*

Proof. By means of (3.1)~(3.7), for $1 < X \leq C (\leq 2)$ there exists X_1 $(1 < X_1 < X)$

such that

$$F_2(X) = \frac{(X-1)^4}{24} F_2^{(4)}(X_1) = -\frac{n(n-1)}{24}(X-1)^4(n-X_1)^{n-5}\tilde{R}_3(X_1).$$

On the other hand, from (3.7) we obtain

$$\tilde{R}_3''(x) = 6(n-1)(n+1)(n+2)x - 2(n+1)(2n^3+n^2+10n-16)$$

$$= 2(n+1)\{3(n-1)(n+2)x - 2n^3-n^2-10n+16\}$$

and

$$3(n-1)(n+2)x - 2n^3-n^2-10n+16 \leqq -(2n^3-5n^2+4n-4)$$

$$= -(n-2)(2n^2-n+2) < 0 \qquad \text{for} \quad 1 \leqq x \leqq 2, \quad \text{when} \quad n > 2.$$

Hence, $\tilde{R}_3(x)$ is convex upward in $1 \leqq x \leqq 2$. Since we have $\tilde{R}'_3(1) > 0$, we obtain

$$\tilde{R}_3(X_1) < \tilde{R}_3(1) + (X_1-1)\tilde{R}_3'(1) < \tilde{R}_3(1) + (X-1)\tilde{R}_3'(1)$$

$$= -4(n-1)^2(n^2-n+1) + 2(n-1)(4n^3-7n^2+9n-4)(X-1),$$

which becomes $2(n-1)(2n^3-3n^2+5n-2) > 0$ at $X = 2$ and $-(4/5)(n-1)(n^3-3n+n-1) < 0$ at $X = 1.4$, respectively. We obtain

(5.15) $\quad F_2(X) > \dfrac{n(n-1)^3(n^2-n+1)}{6}(X-1)^4\left\{1 - \dfrac{4n^3-7n^2+9n-4}{2(n-1)(n^2-n+1)}(X-1)\right\}(n-X_1)^{n-5}.$

There exists X_2 $(1 < X_2 < X_1)$ such that

$$(n-X_1)^{n-5} = (n-1)^{n-5} - (n-5)(n-X_2)^{n-6}(X_1-1),$$

from which we obtain

(5.16) $\quad (n-X_1)^{n-5} > (n-1)^{n-5}\left\{1 - \dfrac{n-5}{n-1}\left(\dfrac{n-\varepsilon_4}{n-1}\right)^{n-6}(X-1)\right\} \qquad \text{for} \quad 1 < X \leqq C,$

where if $n \geqq 6$, $\varepsilon_4 = 1$; if $5 < n < 6$, $\varepsilon_4 = C$; if $4 \leqq n \leqq 5$, $\varepsilon_4 = -\infty$.
 Since we have

$$\frac{d}{dx}\frac{x}{\sqrt{n-x}} = \frac{2n-x}{2(n-x)^{3/2}}, \qquad \frac{d^2}{dx^2}\frac{x}{\sqrt{n-x}} = \frac{4n-x}{4(n-x)^{5/2}},$$

we obtain easily

(5.17) $\qquad \dfrac{X}{\sqrt{n-X}} > \dfrac{1}{\sqrt{n-1}}\left\{1 + \dfrac{2n-1}{2(n-1)}(X-1)\right\} \qquad \text{for} \quad 1 < X < n.$

Now using (5.15)~(5.17) we obtain from (1.16)

$$U_4(x) = \frac{nXF_2(X)}{(n-1)(X-1)^3\sqrt{n-X}}$$

$$> \frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}(X-1)\left\{1-\frac{4n^3-7n^2+9n-4}{2(n-1)(n^2-n+1)}(X-1)\right\}$$

$$\times\left\{1+\frac{2n-1}{2(n-1)}(X-1)\right\}\left\{1-\frac{n-5}{n-1}\left(\frac{n-\varepsilon_4}{n-1}\right)^{n-6}(X-1)\right\}$$

$$= \frac{n^2(n^2-n+1)(n-1)^{n-7/2}}{6}(X-1)-\left\{\frac{n^2(2n^3-4n^2+6n-3)(n-1)^{n-9/2}}{12}\right.$$

$$\left. +\frac{n^2(n-5)(n^2-n+1)(n-1)^{n-9/2}}{6}\left(\frac{n-\varepsilon_4}{n-1}\right)^{n-6}\right\}(X-1)^2$$

$$+\frac{n^2(n-1)^{n-11/2}}{12}\left\{(n-5)(2n^3-4n^2+6n-3)\left(\frac{n-\varepsilon_4}{n-1}\right)^{n-6}\right.$$

$$\left. -\frac{(2n-1)(4n^3-7n^2+9n-4)}{2}\right\}(X-1)^3$$

$$+\frac{1}{24}n^2(2n-1)(n-5)(4n^3-7n^2+9n-4)(n-1)^{n-13/2}\left(\frac{n-\varepsilon_4}{n-1}\right)^{n-6}(X-1)^4.$$

<div align="right">Q. E. D.</div>

Finally, we shall give a main evaluation formula on $V(x, X_n(x))$ near $x=1$. Using the inequalities in Lemmas 5.1~5.6 in $V(x, X_n(x))=-U_1(x)+U_2(x)+U_5(x)-U_6(x)+U_0(x)+U_4(x)$, and noticing the part with $-U_1(x)+U_2(x)$ expressed by a polynomial of $(1-x)$:

$$-\frac{n(n^2+7n-3)(n-1)^{n-7/2}}{12}(1-x)$$

$$-\left\{\frac{n(40n^3-26n^2-36n+27)(n-1)^{n-9/2}}{48}\right.$$

$$\left. +\frac{3(n-2)(n-3)(7n-8)(n-1)^{n-1/2}e_{n-1}}{40n^3}\right\}(1-x)^2$$

$$-\frac{n(n-2)(3n-1)(8n^2-8n+3)(n-1)^{n-11/2}}{32}(1-x)^3,$$

whose coefficients of $(1-x)$, $(1-x)^2$ and $(1-x)^3$ are always negative when $n\geqq3$ and so becomes more smaller than it if we replace $1-x$ with $X-1$ since $X-1>1-x$ by Proposition 4 in (IV), and hence doing these replacements, we obtain the following

PROPOSITION 4. *When $n>4$, we have*

$$(5.18) \quad V(x, X_n(x)) > \frac{n(2n-1)(n^2-n-3)(n-1)^{n-7/2}}{6}(X-1)$$

$$-\left[\frac{n(8n^4+44n^3-108n^2+102n-9)}{48}+\frac{3e_{n-1}(n-2)(n-3)(7n-8)(n-1)^4}{40n^3}\right.$$

$$+\frac{n^2(8n^2-8nC+3C^2)}{8}\left(\frac{n-1}{n-C}\right)^{5/2}+\frac{n^2(n^2-n+1)\{(12n-13)p_n+6n-10\}}{36}$$

$$+\frac{n^2(n-5)(n^2-n+1)}{6}\left(\frac{n-\varepsilon_4}{n-1}\right)^{n-6}\right](n-1)^{n-9/2}(X-1)^2$$

$$+\left[-\frac{n(32n^5+48n^4-416n^3+643n^2-311n+18)}{96}+\frac{n^2(4n-5)(n^2-n+1)p_n(p_n+3n-5)}{12}\right.$$

$$+\frac{n^2(n-5)(2n^3-4n^2+6n-3)}{12}\left(\frac{n-\varepsilon_4}{n-1}\right)^{n-6}\right](n-1)^{n-11/2}(X-1)^3$$

$$+\left[\left[\left\{1+\frac{4n-3}{2(n-1)}(X-1)\right\}\frac{n(n-2)(n-3)}{40}q_n\sqrt{n-1}\,(n-\varepsilon_0)^{n-6}(X-1)^2\right.\right.$$

$$+\frac{1}{24}n^2(2n-1)(n-5)(4n^3-7n^2+9n-4)(n-1)^{n-13/2}\left(\frac{n-\varepsilon_4}{n-1}\right)^{n-6}(X-1)^4\right]\right]$$

$$for \quad 1<X\leqq C,$$

where C, p_n, q_n, ε_0 and ε_4 are constants as in Lemmas 5.1~5.5.

Remark. In the above expression, the quantity in the blackets 〚 〛 is positive.

§ 6. Proof of Theorem C for $6\leqq n\leqq 9.7$.

LEMMA 6.1. *When* $6\leqq n\leqq 9.7$, *we have*

$$V(x, X_n(x))>\frac{n(n-1)^{n-11/2}}{6}(X-1)\Big[(n-1)^2(2n-1)(n^2-n-3)$$

$$-\frac{1}{120}(n-1)(980n^4+411n^3-6052n^2+9615n-4457)(X-1)$$

$$+\frac{1}{32}(124n^5-807n^4+1905n^3-2315n^2+1032n-36)(X-1)^2\Big]$$

$$for \quad 1<X\leqq b_n.$$

Proof. Omitting the last part in the blackets 〚 〛 of (5.18), we shall evaluate the coefficients of $(X-1)^2$ and $(X-1)^3$ of the right hand side. First, for $6\leqq n\leqq 9.7$ we put $\varepsilon_4=1$ and $C=b_n$ ($\leqq b_6=1.9117\cdots$) by Lemma 5.6 and have

$$\frac{3e_{n-1}(n-1)^4}{n^3}=3ne_{n-1}\left(\frac{n-1}{n}\right)^4\leqq 3ne_{8.7}\left(\frac{8.7}{9.7}\right)^4=3n\left(\frac{9.7}{8.7}\right)^{4.7}$$

$$=n\times 5.002739\cdots,$$

$$\frac{8n^2-8nC+3C^2}{8}\left(\frac{n-1}{n-C}\right)^{5/2}<\frac{8n^2-16n+12}{8}\left(\frac{n-1}{n-2}\right)^{5/2}$$

$$\leqq\frac{2n^2-4n+3}{2}\left(\frac{5}{4}\right)^{5/2}=(2n^2-4n+3)\times0.873464\cdots<2n^2-4n+3\,,$$

since $(8n^2-8nC+3C^2)/(n-C)^{5/2}$ is increasing with respect to C, and

$$(12n-13)e_{n-2}+6n-10\leqq(12n-13)e_{7.7}+6n-10$$

$$\leqq(12n-13)\times2.5604698\cdots+6n-10$$

$$=36.725637\cdots\times n-43.286107\cdots<37n-43\,.$$

Using these evaluations for the expressions, we obtain

(6.1) $\dfrac{1}{48}(8n^4+44n^3-108n^2+102n-9)+\dfrac{1}{40}(n-2)(n-3)(7n-8)\times5.00274$

$\qquad+n(2n^2-4n+3)+\dfrac{1}{36}n(n^2-n+1)(37n-43)+\dfrac{n(n-5)(n^2-n+1)}{6}$

$\qquad<\dfrac{1}{720}(980n^4+411n^3-6052n^2+9615n-4457)\,,$

which is available for $6\leqq n\leqq9.7$.

Now, we proceed to the treatment of the terms regarding $(X-1)^3$ of (5.18). We have for $6\leqq n\leqq9.7$

$$e_{n-2}(e_{n-2}+3n-5)\geqq e_4(e_4+3n-5)=\left(\frac{5}{4}\right)^8+\left(\frac{5}{4}\right)^4(3n-5)$$

$$=5.96046\cdots+(3n-5)\times2.44140\cdots=7.32481\cdots\times n-6.24657\cdots$$

$$>\frac{3(39n-34)}{16}\,,$$

hence using this evaluation, we obtain

(6.2) $-\dfrac{1}{96}(32n^5+48n^4-416n^3+643n^2-311n+18)$

$\qquad+\dfrac{1}{192}n(4n-5)(n^2-n+1)(39n-34)+\dfrac{1}{12}n(n-5)(2n^3-4n^2+6n-3)$

$\qquad=\dfrac{1}{192}(124n^5-807n^4+1905n^3-2315n^2+1032n-36)\,,$

which is available for $6\leqq n\leqq9.7$.

Combining (6.1) and (6.2) with (5.18), we can obtain the inequality of this lemma.   Q. E. D.

LEMMA 6.2. *When* $6 \leq n \leq 9.7$, *we have*

$$V(x, X_n(x)) > 0 \quad for \quad X_n^{-1}\left(\frac{982-37n}{882-37n}\right) \leq x < 1.$$

This lemma will be supported by Lemma 6.1 and the following proposition which will be proved in another coming paper [18], because

$$\frac{982-37n}{882-37n} < \frac{612}{512} = 1.1953 \cdots < b_{9.7} = 1.5882 \cdots .$$

PROPOSITION 5. *When* $6 \leq n \leq 9.7$, *the quadratic polynomial of t*

$$(n-1)^2(2n-1)(n^2-n-3) - \frac{t}{120}(n-1)(980n^4+411n^3-6052n^2+9615n-4457)$$

$$+ \frac{t^2}{32}(124n^5-807n^4+1905n^3-2315n^2+1032n-36) > 0$$

$$for \quad 0 \leq t \leq \frac{100}{882-37n}.$$

On the other hand, according to Proposition 1 we cite here the approximately calculated values of Y such that $K(n, Y)=0$ with $1<Y<n$ for each n of Table 1.

Table 1.

n	Y	Z	n	Y	Z	n	Y	Z	n	Y	Z
6.0	1.3250	1.1515	7.0	1.3037	1.1605	8.0	1.2846	1.1706	9.0	1.2675	1.1820
.1	.3228	.1523	.1	.3017	.1614	.1	.2828	.1717	.1	.2659	.1833
.2	.3206	.1532	.2	.2997	.1624	.2	.2810	.1728	.2	.2643	.1846
.3	.3184	.1541	.3	.2978	.1634	.3	.2793	.1739	.3	.2627	.1859
.4	.3162	.1549	.4	.2958	.1644	.4	.2775	.1750	.4	.2611	.1871
.5	.3141	.1558	.5	.2939	.1654	.5	.2758	.1762	.5	.2596	.1885
.6	.3119	.1567	.6	.2920	.1664	.6	.2741	.1773	.6	.2581	.1898
.7	.3099	.1577	.7	.2901	.1674	.7	.2724	.1785	.7	.2566	.1911
.8	.3078	.1586	.8	.2883	.1685	.8	.2708	.1797	Y	$K(n, Y)=0$	
.9	.3057	.1595	.9	.2864	.1695	.9	.2691	.1809			

$$Z=(982-37n)/(882-37n)$$

Now, noticing the values of Y in Table 1, we can prove that $K(n, (73-n)/50) > 0$ for $6 \leq n \leq 9.7$ by Proposition 3 and an analogous argument used in the proof of Proposition 2 of (VI), where $(73-n)/50$ is greater than Y for the values of n in the table.

Next, we put $C=(73-n)/50$ in (2.9) and obtain

$$n-k_n\left(\frac{73-n}{50}\right)=\frac{(n+1)(23-n)+\sqrt{(n-1)(n^3-47n^2+10775n-15129)}}{2(73-n)}$$

and we replace $y(n, X)$ defined by (3.8) in Proposition 1 with

$$y=\left(\frac{n-X}{n-1}\right)^{n-k_n(C)}, \qquad C=\frac{73-n}{50},$$

and denote these functions by $K_1(n, X)$ and $K_1(n_1, n_2; X_1, X_2)$. We cite here the approximately calculated values of Y_1 such that $K_1(n, Y_1)=0$ with $1<Y_1<n$ for the n's in Table 1.   Table 2 tells us that

$$Y_1<\frac{982-37n}{882-37n} \qquad \text{for} \quad 7.6\leqq n\leqq 9.7$$

in the tables.   We can also prove this inequality holds for any n in this interval by the modified Proposition 3.

<div align="center">Table 2.</div>

n	Y_1	n	Y_1	n	Y_1	n	Y_1
6.0	1.2125	7.0	1.1814	8.0	1.1554	9.0	1.1334
.1	.2091	.1	.1786	.1	.1531	.1	.1314
.2	.2058	.2	.1759	.2	.1508	.2	.1294
.3	.2025	.3	.1731	.3	.1485	.3	.1274
.4	.1993	.4	.1705	.4	.1462	.4	.1255
.5	.1962	.5	.1679	.5	.1440	.5	.1236
.6	.1931	.6	.1653	.6	.1418	.6	.1217
.7	.1901	.7	.1628	.7	.1397	.7	.1199
.8	.1872	.8	.1603	.8	.1375	$K_1(n, Y_1)=0$	
.9	.1843	.9	.1578	.9	.1355		

Then, noticing the values Y_1 (for $6\leqq n\leqq 7.6$), we can prove that

$$K_1\left(n, \frac{37-n}{25}\right)>0 \qquad \text{for} \quad 6\leqq n\leqq 7.6$$

by the method described above.   We put $C=(37-n)/25$ in (2.9) and obtain

$$n-k_n\left(\frac{37-n}{25}\right)=\frac{(n+1)(12-n)+\sqrt{(n-1)(n^3-25n^2+2768n-3844)}}{2(37-n)}.$$

Next, we replace $y(n, X)$ in Proposition 1 with

$$y=\left(\frac{n-X}{n-1}\right)^{n-k_n(C)}, \qquad C=\frac{37-n}{25}$$

and denote these functions by $K_2(n, X)$ and $K_2(n_1, n_2; X_1, X_2)$. We cite here the approximately calculated values of Y_2 such that $K_2(n, Y_2)=0$ with $1<Y_2<n$ for the n's $(6\leq n\leq 7.6)$ in Table 1. Table 3 tells us that

$$Y_2 < \frac{982-37n}{882-37n} \quad \text{for} \quad 6.3\leq n\leq 7.6$$

in the tables. We can also prove this inequality for any n in this interval by the modified Proposition 3.

Finally, we use C in $k_n(C)$ as

$$C = \frac{141-4n}{100}$$

and can prove

$$K_2\left(n, \frac{141-4n}{100}\right)>0 \quad \text{for} \quad 6\leq n\leq 6.3 \,.$$

Then we obtain

$$n-k_n\left(\frac{141-4n}{100}\right) = \frac{(n+1)(41-4n)+\sqrt{(n-1)(16n^3-344n^2+43609n-58081)}}{2(141-4n)}$$

and, using this function, compose $K_3(n, X)$ and $K_3(n_1, n_2; X_1, X_2)$ by the same method above. And computing approximately the values Y_3 by $K_3(n, Y_3)=0$ for $6\leq n\leq 6.3$, we obtain Table 4, which shows

$$Y_3 < \frac{982-37n}{882-37n} \quad \text{for} \quad 6\leq n\leq 6.3.$$

Thus, we can conclude that

$$V(x, X_n(x))>0 \quad \text{for} \quad 0<x<1 \,,$$

when $6\leq n\leq 9.7$, which assures Theorem C.

	Table 3.		
n	Y_2	n	Y_2
6.0	1.1656	7.0	1.1250
.1	.1612	.1	.1213
.2	.1569	.2	.1177
.3	.1526	.3	.1141
.4	.1485	.4	.1106
.5	.1444	.5	.1071
.6	.1404	.6	.1037
.7	.1364	$K_2(n, Y_2)=0$	
.8	.1326		
.9	.1287		

	Table 4.
n	Y_3
6.0	1.1249
.1	.1205
.2	.1162
.3	.1120

§7. Proof of Theorem C for $5 \leqq n \leqq 6$.

LEMMA 7.1.   *When* $5 \leqq n \leqq 6$, *for* $1 < X \leqq \min(2, b_n)$ *we have*

$$V(x, X_n(x)) > \frac{n(n-1)^{n-11/2}}{6}(X-1)\Big[(n-1)^2(2n-1)(n^2-n-3)$$

$$-\frac{1}{120}(n-1)(1000n^4+92n^3-4864n^2+7441n-3245)(X-1)$$

$$+\frac{1}{32}(130n^5-869n^4+2027n^3-2479n^2+1108n-36)(X-1)^2\Big]$$

$$\text{for} \quad 1 < X \leqq \min(2, b_n).$$

Proof.   We shall evaluate the coefficients of $(X-1)^2$ and $(X-1)^3$ of the right hand side of (5.18).   First, for $5 \leqq n \leqq 6$ we put $\varepsilon_4 = C = \min(2, b_n)$ by Lemma 5.1 and have

$$\frac{3e_{n-1}(n-1)^4}{n^3} = 3n\, e_{n-1}\Big(\frac{n-1}{n}\Big)^4 \leqq 3n\, e_5\Big(\frac{5}{6}\Big)^4 = \frac{18}{5}\, n = 3.6n\,,$$

$$\frac{8n^2-8nC+3C^2}{8}\Big(\frac{n-1}{n-C}\Big)^{5/2} \leqq \frac{8n^2-16n+12}{8}\Big(\frac{n-1}{n-2}\Big)^{5/2} \leqq \frac{2n^2-4n+3}{2}\Big(\frac{4}{3}\Big)^{5/2}$$

$$=(2n^2-4n+3)\times 1.0264004\cdots < (2n^2-4n+3)\times 1.0265\,,$$

$$(12n-13)e_{n-2}+6n-10 \leqq (12n-13)e_4+6n-10$$

$$=(12n-13)\times\frac{625}{256}+6n-10=\frac{9036n-10685}{256}<36n-41$$

and

$$\Big(\frac{n-\varepsilon_4}{n-1}\Big)^{n-6}=\Big(\frac{n-1}{n-C}\Big)^{6-n}\leqq\Big(\frac{n-1}{n-2}\Big)^{6-n}\leqq\Big(\frac{4}{3}\Big)^{6-n}\leqq\frac{4}{3}.$$

Using these evaluations for the expressions in the blackets corresponding to $(X-1)^2$ of (5.18), we obtain

$$(7.1) \quad \frac{1}{48}(8n^4+44n^3-108n^2+102n-9)+\frac{1}{40}(n-2)(n-3)(7n-8)\times 3.6$$

$$+n(2n^2-4n+3)\times 1.0265+\frac{1}{36}n(n^2-n+1)(36n-41)+\frac{2}{9}n(n-5)(n^2-n+1)$$

$$<\frac{1}{720}(1000n^4+92n^3-4864n^2+7441n-3245)\,.$$

Next, we proceed to the treatment of the terms regarding $(X-1)^3$.   We have for $5 \leqq n \leqq 6$

$$e_{n-2}(e_{n-2}+3n-5)\geqq e_3(e_3+3n-5)=\frac{4^3}{3^6}(81n-71)\,.$$

Using this evaluation, we obtain

(7.2) $-\dfrac{1}{96}(32n^5+48n^4-416n^3+643n^2-311n+18)$

$\qquad +\dfrac{1}{36}\cdot\dfrac{4^3}{3^6}\,n(4n-5)(n^2-n+1)(81n-71)+\dfrac{1}{12}\cdot\dfrac{4}{3}\,n(n-5)(2n^3-4n^2+6n-3)$

$\qquad =\dfrac{1}{2^5\cdot3^8}(142560n^5-950244n^4+2216736n^3-2710593n^2+1211837n-39366)$

$\qquad >\dfrac{1}{192}(130n^5-869n^4+2027n^3-2479n^2+1108n-36)\,.$

Combining (7.1) and (7.2) with (5.18), we can obtain the inequality of this lemma.   Q. E. D.

LEMMA 7.2. *When* $5\leqq n\leqq6$, *we have*

$$V(x,\,X_n(x))>0\qquad for\quad X_n^{-1}\!\Big(\dfrac{57-3n}{52-3n}\Big)\leqq x<1\,.$$

This lemma will be supported by Lemma 7.1 and the following proposition which will be proved also in [18], because

$$\dfrac{57-3n}{52-3n}\leqq\dfrac{39}{34}=1.14770\cdots<b_6=1.9117\cdots\leqq b_n\qquad for\quad 5\leqq n\leqq6\,.$$

PROPOSITION 6. *When* $5\leqq n\leqq6$, *the quadratic polynomial of* t.

$$(n-1)^2(2n-1)(n^2-n-3)-\dfrac{t}{120}\,(n-1)(1000n^4+92n^3-4864n^2+7441n-3245)$$

$$+\dfrac{t^2}{32}(130n^5-869n^4+2027n^3-2479n^2+1108n-36)>0$$

$$for\quad 0\leqq t\leqq\dfrac{5}{52-3n}\,.$$

Now, we cite here the approximately calculated values of Y analogous to Table 1 for $5\leqq n\leqq6$, in Table 1', in which $Z_1=(57-3n)/(52-3n)$. Noticing the values of Y and we can also prove

$$K\!\Big(n,\,\dfrac{73-n}{50}\Big)>0\qquad for\quad 5\leqq n\leqq6\,.$$

Using the revised $K_1(n,\,X)$ as in §6, we cite the approximately calculated values of Y_1 analogous to Table 2 in Table 2'.

Noticing the values of Y_1 in Table 2', we can also prove that

$$K_1\!\Big(n,\,\dfrac{37-n}{25}\Big)>0\qquad for\quad 5\leqq n\leqq6\,.$$

Then, using $K_2(n,\,X)$ as in §6, we cite the approximately calculated values of Y_2 analogous to Table 3.

<center>Table 1'.</center>

n	Y	Z_1
5.0	1.3492	1.1351
.1	.3466	.1362
.2	.3441	.1373
.3	.3416	.1385
.4	.3391	.1396
.5	.3367	.1408
.6	.3343	.1420
.7	.3319	.1432
.8	.3296	.1445
.9	.3273	.1457
6.0	1.3250	1.1470

<center>Table 2'.</center>

n	Y_1
5.0	1.2507
.1	.2406
.2	.2423
.3	.2383
.4	.2344
.5	.2305
.6	.2268
.7	.2231
.8	.2195
.9	.2159
6.0	1.2125

Noticing Table 3', we can prove

$$K_2\left(n, \frac{76-3n}{50}\right)>0 \quad \text{for } 5\leqq n\leqq 5.5$$

and

$$K_2\left(n, \frac{141-4n}{100}\right)>0 \quad \text{for } 5.5\leqq n\leqq 6.$$

Then, using $K_3(n, X)$ for $5.5\leqq n\leqq 6$, we obtain the approximately calculated values Y_3 as in §6. Here we cite them in Table 4', which shows

$$Y_3<\frac{57-3n}{52-3n}=Z_1 \quad \text{for } 5.7\leqq n\leqq 6.$$

For $5\leqq n\leqq 5.5$, we use

$$k_n(C)=k_n\left(\frac{76-3n}{50}\right).$$

Then, we have

$$n-k_n(C)=\frac{(n+1)(26-3n)+\sqrt{(n-1)(9n^3-165n^2+11432n-15876)}}{2(76-3n)},$$

and, using this function, compose $K_4(n, X)$ and $K_4(n_1, n_2; X_1, X_2)$ by the same method as in §6. And computing approximately the values of Y_4 by $K_4(n, Y_4)=0$ for $5\leqq n\leqq 5.5$ as in Table 4'.

Noticing Table 4', we can prove

$$K_3\left(n, \frac{189-8n}{125}\right)>0 \quad \text{for } 5.5\leqq n\leqq 6,$$

Table 3'.

n	Y_2	n	Y_2
5.0	1.2143	5.6	1.1839
.1	.2090	.7	.1792
.2	.2038	.8	.1746
.3	.1987	.9	.1700
.4	.1937	6.0	1.1656
.5	.1888		

Table 4'.

n	Y_4	n	Y_3
5.0	1.1806	5.5	1.1482
.1	.1739	.6	.1434
.2	.1673	.7	.1386
.3	.1609	.8	.1339
.4	.1545	.9	.1293
.5	.1482	6.0	.1249

$$K_4\left(n, \frac{189-8n}{125}\right)>0 \quad \text{for} \quad 5\leq n\leq 5.5.$$

Next, for $5\leq n\leq 5.7$, we use

$$k_n(C)=k_n\left(\frac{189-8n}{125}\right).$$

Then we obtain

$$n-k_n(C)=\frac{4(n+1)(8-n)+\sqrt{(n-1)(16n^3-272n^2+17905n-24649)}}{189-8n}$$

and, using this function, compose $K_5(n, X)$ and $K_5(n_1, n_2; X_1, X_2)$. Then computing approximately the values of Y_5 by $K_5(n, Y_5)=0$ for $5\leq n\leq 5.7$. We cite here the values Y_5 in Table 5, which shows

$$Y_5<\frac{57-3n}{52-3n}=Z_1 \quad \text{for} \quad 5.4\leq n\leq 5.7.$$

For $5\leq n\leq 5.4$, we can prove

$$K_5\left(n, \frac{1503-68n}{1000}\right)>0$$

by means of $K_5(n_1, n_2; X_1, X_2)$ as the previous case.

Then, taking $C=(1503-68n)/1000$, we obtain

$$n-k_n(C)=\frac{(n+1)(503-68n)+\sqrt{(n-1)(4624n^3-73032n^2+4593417n-6265009)}}{2(1503-68n)}$$

and, using this as the exponent of $y(n, X)$, compose $K_6(n, X)$ and $K_6(n_1, n_2; X_1, X_2)$ as the previous cases. Computing approximately the values of Y_6 by $K_6(n, Y_6)=0$ for $5\leq n\leq 5.4$, we obtain Table 6, which shows

$$Y_6<\frac{57-3n}{52-3n}=Z_1 \quad \text{for} \quad 5.1\leq n\leq 5.4.$$

For $5\leq n\leq 5.1$, we can prove

Table 5.

n	Y_5	n	Y_5
5.0	1.1628	5.4	1.1351
.1	.1556	.5	.1285
.2	.1487	.6	.1220
.3	.1418	.7	.1156

Table 6.

n	Y_6
5.0	1.1427
.1	.1351
.2	.1278
.3	.1205
.4	.1134

Table 7.

n	Y_7
5.0	1.1286
.1	.1201

$$K_6\left(n, \frac{1544-80n}{1000}\right)>0$$

by means of $K_6(n_1, n_2; X_1, X_2)$ as the previous cases.

Finally, for $5\leqq n\leqq 5.1$, taking

$$C=\frac{1544-80n}{1000}=\frac{193-10n}{125},$$

we obtain

$$n-k_n(C)=\frac{(n+1)(68-10n)+\sqrt{(n-1)(100n^3-1460n^2+73484n-101124)}}{2(193-10n)}$$

and, using this as the exponent of $y(n, X)$, compose $K_7(n, X)$ and $K_7(n_1, n_2; X_1, X_2)$ as the previous cases. Computing approximately the values of Y_7 by $K_7(n, Y_7)=0$ for $5\leqq n\leqq 5.1$, we obtain Table 7, which shows

$$Y_7<\frac{57-3n}{52-3n}=Z_1 \quad \text{for} \quad 5\leqq n\leqq 5.1.$$

Thus we can conclude that

$$V(x, X_n(x))>0 \quad \text{for} \quad 0<x<1,$$

when $5\leqq n\leqq 6$, which implies Theorem C for $5\leqq n\leqq 6$.

We have obtained the main theorem of this paper as follows.

THEOREM C. *The period function T as function of τ and n is monotone decreasing with respect to $n\geqq 5$ for any fixed τ $(0<\tau<1)$.*

Remark. When we composed K_i $(n_1, n_2; X_1, X_2)$, we used the following inequalities in place of (4.13) which can be proved by Lemma 2.3, supposing $C=C(n)$ is a decreasing function of n:

$$\left(\frac{n_1-X_2}{n_1-1}\right)^{n_2-k_{n_2}(C(n_2))}\leqq\left(\frac{n-X}{n-1}\right)^{n-k_n(C(n))}\leqq\left(\frac{n_2-X_1}{n_2-1}\right)^{n_1-k_{n_1}(C(n_1))}.$$

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