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KODAI MATH. J.
8 (1985), 330—337

# HOMOTOPY TYPES OF CONNECTED SUMS OF SPHERICAL FIBRE SPACES OVER SPHERES

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## § 1. Introduction.

In the homotopy classification problems of highly-connected Poincaré complexes, the connected sums of spherical fibre spaces over spheres appear frequently. Especially, the manifolds with certain tangential and homotopy properties come to connected sums of sphere bundles over spheres (for example, [12]).

On the other hand, I.M. James and J.H.C. Whitehead classified homotopy types of the total space of sphere bundles over spheres in [3] and [4], and their results were extended to the case of spherical fibre spaces over spheres by S. Sasao in [7].

Motivated by those, H. Ishimoto classified connected sums of sphere bundles over spheres up to homotopy types in [2]. Then the purpose of this paper is to extend Ishimoto's results in [2] to the case of connected sums of spherical fibre spaces over spheres with cross-sections, which is also a generalization of Sasao's Theorem given in [7].

Let $G_{n+1}$ be the space of maps of a $n$-sphere $S^n$ to itself with degree 1 and $F_n$ be the subspace of $G_{n+1}$ consisting of maps preserving the base point $s_0={}^t(1, 0, 0, \cdots, 0)\in S^n$. Let $X$ be a total space of an orientable $n$-spherical fibre space over a $(n+k+1)$-sphere which admits a cross-section, and we denote its characteristic element by $\chi(X)\in\pi_{n+k}(G_{n+1})$. Since $X$ has a cross-section, we may suppose

$$\chi(X)=j_{n*}(\gamma) \quad \text{for some element } \gamma\in\pi_{n+k}(F_n), \tag{1.1}$$

where

$$j_n : F_n \longrightarrow G_{n+1} \quad \text{denotes the inclusion map.}$$

Let

$$\lambda : \pi_{n+k}(F_n) \xrightarrow{\cong} \pi_{n+k}(G_{n+1})$$

be the isomorphism defined by B. Steer in [10], and two maps

$$i' : SO_{n+1} \longrightarrow G_{n+1}$$

Received October 15, 1984

and

$$i'': SO_n \longrightarrow F_n$$

be the inclusion maps, where $SO_m$ denotes the $m$-th rotation group. Clearly, $i'$ induces a fibre map of the fibration $SO_n \to SO_{n+1} \to S^n$ into the fibration $F_n \to G_{n+1} \to S^n$. Thus, we have the following diagram, which is commutative up to sign,

$$\begin{tikzcd}
& \pi_{n+k}(SO_n) \arrow[rr] \arrow[dd, "i''_*"] \arrow[dr, "J"] & & \pi_{n+k}(SO_{n+1}) \arrow[rr] \arrow[dd, "i'_*"] \arrow[dr, "J"] & & {} \\
\pi_{n+k+1}(S^n) \arrow[ur, "\partial'"] \arrow[dr, "\partial"'] \arrow[rr, "P"] & & \pi_{2n+k}(S^n) \arrow[rr, "E"] & & \pi_{2n+k+1}(S^{n+1}) & \\
& \pi_{n+k}(F_n) \arrow[rr] \arrow[ur, "\lambda"', "\cong"] & & \pi_{n+k}(G_{n+1}) \arrow[rr] \arrow[ur] & & {}
\end{tikzcd}$$

Fig. (1.2)

where $J$ means the classical $J$-homomorphism and the homomorphism $P$ is defined by the Whitehead product,

(1.3) $$P(\zeta)=[\zeta, \iota_n] \quad \text{for } \zeta \in \pi_{n+k+1}(S^n).$$

Since the horizontal sequences of (1.2) are exact, the element $\lambda(\gamma) \in \pi_{2n+k}(S^n)$ is uniquely determined up to $P\pi_{n+k+1}(S^n)$, and we define the invariant $\lambda(X)$ by

(1.4) $$\lambda(X)=\{\lambda(\gamma)\} \in \pi_{2n+k}(S^n)/P\pi_{n+k+1}(S^n).$$

By using this invariant, S. Sasao classified the homotopy type of spherical fibre spaces over spheres with cross-sections as follows:

THEOREM 1.5. (S. Sasao, [7]) *Let $n \geqq k+3$, and $k \geqq 1$. Let $X_1$ and $X_2$ be the total spaces of orientable n-spherical fibre spaces over $S^{n+k+1}$ which admit cross-sections. Then $X_1$ and $X_2$ are of the same homotopy type if and only if $\lambda(X_1)=\pm\lambda(X_2)$.*

Let $X_h$ be a total space of an orientable $n$-spherical fibre space over $S^{n+k+1}$ with a cross-section, for $1 \leqq h \leqq r$. We denote by $\#_{h=1}^{r} X_h$ the connected sum of the total spaces $X_h$, $h=1, 2, \cdots, r$. (See Wall [13] about the definition of connected sums of Poincaré complexes.) Let $X'_s$, $s=1, 2, \cdots, r'$, be another set of such spherical fibrations. Then it is easy to see that, if $\#_{h=1}^{r} X_h$ and $\#_{s=1}^{r'} X'_s$ are of the same homotopy type, $r$ must be equal to $r'$ by the homological reason.

Then the aim of this note is to extend the above result to the case of connected sums of total spaces of spherical fibre spaces over spheres which admit a cross-section, and our results are stated as follows:

THEOREM A. *Let $n \geqq k+3$ and $k \geqq 1$. Let $X_h$ and $X'_h$ be total spaces of orientable n-spherical fibre spaces over $(n+k+1)$-spheres which admit cross-sections, for $1 \leqq h \leqq r$. Then the connected sums $\#_{h=1}^{r} X_h$ and $\#_{h=1}^{r} X'_h$ are of the same homotopy type if and only if there exists an unimodular matrix $A \in GL_r(Z)$ such that,* (1.6) $$^t(\lambda(X_1'), \lambda(X_2'), \cdots, \lambda(X_r'))=A\cdot{}^t(\lambda(X_1), \lambda(X_2), \cdots, \lambda(X_r))$$

*where the abelian group* $\pi_{2n+k}(S^n)/P\pi_{n+k+1}(S^n)$ *is considered as a left Z-module.*

In particular, if $X$ is a total space of $n$-sphere bundle over a $(n+k+1)$-sphere with a cross-section, then the invariant $\lambda(X)$ is contained in the subgroup $J\pi_{n+k}(SO_n)/P\pi_{n+k+1}(S^n)$, and we also have

COROLLARY B. (H. Ishimoto, [2]; K. Yamaguchi, [15]) *Let* $n\geqq k+3$ *and* $k\geqq 1$. *Let* $X_h$ *and* $X_h'$ *be total spaces of n-sphere bundles over* $(n+k+1)$*-spheres which admit cross-sections, for* $1\leqq h\leqq r$. *Then the connected sums* $\#_{h=1}^r X_h$ *and* $\#_{h=1}^r X_h'$ *are of the same homotopy type if and only if there exist an unimodular matrix* $A\in GL_r(Z)$ *such that,*

(1.7) $$^t(\lambda(X_1'), \lambda(X_2'), \cdots, \lambda(X_r'))=A\cdot{}^t(\lambda(X_1), \lambda(X_2), \cdots, \lambda(X_r))$$

*where the abelian group* $J\pi_{n+k}(SO_n)/P\pi_{n+k+1}(S^n)$ *is considered as a left Z-module.*

This paper is organized as follows: In § 2 we recall the group of self-homotopy equivalences over the wedge of certain complexes. In § 3 we give the proof of Theorem A.

## § 2. The Group of Self-Homotopy Equivalences.

The set of homotopy classes of self-homotopy equivalences of a based space $X$, which is denoted by $Eq(X)$, is a group with the multiplication defined by the composition of maps. For each based space $X$ and $Y$, we denote by $[X, Y]$ the homotopy set of all based maps from $X$ to $Y$. Furthermore, we denote by $Z\{\kappa\}$ the infinite cyclic group generated by $\kappa$. For example, $\pi_m(S^m)=Z\{\iota_m\}$.

For each $1\leqq h\leqq r$, let $K_h$ denote a $CW$-complex

(2.1) $$K_h=S^n\vee S^{n+k+1}, \quad n\geqq k+3, \quad k\geqq 1,$$

and the maps

$$i_h: S^n \longrightarrow S^n\vee S^{n+k+1}=K_h$$

and

$$\xi_h: S^{n+k+1} \longrightarrow S^n\vee S^{n+k+1}=K_h$$

denote inclusion maps to the first and the second factor, respectively. Similarly, let $p_h: K_h=S^n\vee S^{n+k+1}\to S^{n+k+1}$ be a retraction map to the second factor. In particular, we put

(2.2) $$K=\bigvee_{h=1}^{r} K_h=\bigvee^{r}(S^n\vee S^{n+k+1}).$$

Then it is easy to see that the following relations hold:

(2.3) $$p_h\circ i_h=0, \qquad p_h\circ\xi_h=\iota_{n+k+1}.$$

For each $1 \leqq h, s \leqq r$, we define the maps

$$\sigma_{hs} : K_h \longrightarrow K_s$$

and

$$\lambda_{hs} : K_h \longrightarrow K_s$$

by

$$\sigma_{hs} = id_{S^n \vee S^{n+k+1}} \quad \text{and} \quad \lambda_{hs} = \xi_s \circ p_h .$$

Since $K_h$ is a double suspension space, the homotopy sets $[K_h, K_s]$ and $[K, K]$ become abelian groups with track additions. Moreover, if $n \geqq k+3$, $[K, K]$ becomes a (non-commutative) ring whose addition and multiplication are induced from the track addition and the composition of maps, respectively. Now, we assume $n \geqq k+3$ throughout this paper.

Then we recall several results, which has already been obtained in [15], and we omit the proofs.

PROPOSITION 2.4. ([15], Prop. 2.25)

$$[K_h, K_s] = Z\{\sigma_{hs}\} \oplus Z\{\lambda_{hs}\} \oplus G_{hs} ,$$

*where*

$$G_{hs} = p_h^* i_{s*} \pi_{n+k+1}(S^n) \cong \pi_{n+k+1}(S^n) .$$

PROPOSITION 2.5. ([15], Prop. 5.4)

$$[K, K] \cong \mathrm{Mat}(r, [K_h, K_s]) .$$

Hence, for each element $\theta \in [K, K]$, we can represent it as follows:

(2.6) $$\theta = (\theta_{hs}) \qquad \text{for} \quad \theta_{hs} \in [K_h, K_s] ,$$

where

$$\theta_{hs} = a_{hs}\sigma_{hs} + b_{hs}\lambda_{hs} + g_{hs} ,$$

$$a_{hs} \text{ and } b_{hs} \in Z ,$$

$$g_{hs} = i_s \circ g'_{hs} \circ p_h \in G_{hs} , \quad g'_{hs} \in \pi_{n+k+1}(S^n)$$

$$\text{for } 1 \leqq h, s \leqq r .$$

Then for each element $\theta \in [K, K]$ represented by (2.6), we put

(2.7) $$F(\theta) = \begin{bmatrix} A & 0 & 0 \\ 0 & A+B & \Gamma \\ 0 & 0 & A \end{bmatrix} \qquad \begin{array}{l} \text{for } A = (a_{hs}),\ B = (b_{hs}) \in \mathrm{Mat}(r, Z), \\ \text{and } \Gamma = (g'_{hs}) \in \mathrm{Mat}(r, \pi_{n+k+1}(S^n)) . \end{array}$$

Here we note the formula

$$\begin{bmatrix} A' & 0 & 0 \\ 0 & A'+B' & \Gamma' \\ 0 & 0 & A' \end{bmatrix}\begin{bmatrix} A & 0 & 0 \\ 0 & A+B & \Gamma \\ 0 & 0 & A \end{bmatrix}$$

$$=\begin{bmatrix} A'A & 0 & 0 \\ 0 & A'A+(B'A+A'B+B'B) & (A'+B')\Gamma+\Gamma'A \\ 0 & 0 & A'A \end{bmatrix}$$

Then for each abelian group $G$, let $M(r:G)$ denote the ring of matrices of the form

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A+B & \Gamma \\ 0 & 0 & A \end{bmatrix} \quad \text{for} \quad A,\ B\in \mathrm{Mat}(r,\ Z) \quad \text{and} \quad \Gamma\in\mathrm{Mat}(r,\ G). \tag{2.8}$$

In particular, it follows from (2.7) that we have the group homomorphism

$$F: [K,\ K] \longrightarrow M(r: \pi_{n+k+1}(S^n)).$$

THEOREM 2.9. ([15], Theorem E) *The homomorphism*

$$F: [K,\ K] \xrightarrow{\cong} M(r: \pi_{n+k+1}(S^n))$$

*is a ring isomorphism.*

In particular, we also have

COROLLARY 2.10. ([15], Corollary F)

$$Eq(K)\cong \mathrm{Inv}\,(M(r: \pi_{n+k+1}(S^n)))$$

*where we denote by* $\mathrm{Inv}\,(R)$ *the group of all multiplicative invertible elements of a ring* $R$.

## § 3. The Proof of Theorem A.

In this section, we will show Theorem A.

Let $X_h$ and $X'_h$ be total spaces of orientable $n$-spherical fibre spaces over $(n+k+1)$-spheres which admit cross-sections, for $1\leqq h\leqq r$.

First, we note the following

LEMMA 3.1. (S. Sasao, [7], Lemma 2.3) *The spaces* $X_h$ *and* $X'_h$ *have the cell-decompositions of the forms*

$$X_h=K_h\cup_{\rho_h}e^{2n+k+1}$$

*and*

$$K'_h = K_h \cup_{\rho'_h} e^{2n+k+1},$$

*where*

$$\rho_h = i_{h*}\lambda(\gamma_h) + [\xi_h, \iota_h] \in \pi_{2n+k}(K_h),$$

$$\rho'_h = i_{h*}\lambda(\gamma'_h) + [\xi_h, \iota_h] \in \pi_{2n+k}(K_h),$$

$$\lambda(X_h) = \{\lambda(\gamma_h)\} \quad \text{and} \quad \lambda(X'_h) = \{\lambda(\gamma'_h)\} \in \pi_{2n+k}(S^n)/P\pi_{n+k+1}(S^n).$$

Thus, the connected sums $\#_{h=1}^r X_h$ and $\#_{h=1}^r X'_h$ have the cell-decompositions

$$X = \#_{h=1}^r X_h = K \cup_\rho e^{2n+k+1} \tag{3.2}$$

and

$$X' = \#_{h=1}^r X'_h = K \cup_{\rho'} e^{2n+k+1},$$

where

$$K = \bigvee_{h=1}^r K_h = \bigvee^r (S^n \vee S^{n+k+1}),$$

$$\rho = \sum_{h=1}^r j_{h*}\rho_h \quad \text{and} \quad \rho' = \sum_{h=1}^r j_{h*}\rho'_h \in \pi_{2n+k}(K).$$

and $j_h : K_h \to K$ denotes the inclusion map to the $h$-th factor.

In general, $X$ and $X'$ are of the same homotopy type if and only if there exists a homotopy equivalence $\theta \in Eq(K)$ satisfying $\theta \circ \rho = \pm \rho'$, and for our purpose, it is important to investigate the action

$$Eq(K) \times \pi_{2n+k}(K) \longrightarrow \pi_{2n+k}(K) \tag{3.3}$$

which is induced from the composition of maps.

Let $\theta \in [K, K]$ denote the element of the form (2.6). Then,

$$\theta \circ \rho = \sum_{s=1}^r j_{s*}\Big(\sum_{h=1}^r \theta_{hs} \circ \rho_h\Big). \tag{3.4}$$

Here

$$\theta_{hs} \circ \rho_h = \theta_{hs} \circ \iota_h \circ \lambda(\gamma_h) + \theta_{hs} \circ [\xi_h, i_h]$$

$$= (\theta_{hs} \circ i_h) \circ \lambda(\gamma_h) + [\theta_{hs} \circ \xi_h, \theta_{hs} \circ \iota_h].$$

Since $i_h$ is a suspension element,

$$\theta_{hs} \circ i_h = a_{hs} i_s + b_{hs} \lambda_{hs} \circ i_h + g_{hs} \circ \iota_h$$

$$= a_{hs} i_s + b_{hs}(\xi_s \circ p_h) \circ i_h + (i_s \circ g'_{hs} \circ p_h) \circ \iota_h$$

$$= a_{hs} i_s. \quad \text{(by (2.3))}$$

Similarly, $\xi_h$ is a suspension element,

$$\theta_{hs} \circ \xi_h = a_{hs} \xi_s + b_{hs} \lambda_{hs} \circ \xi_h + g_{hs} \circ \xi_h$$

$$=a_{hs}\xi_s+b_{hs}(\xi_s\circ p_h)\circ\xi_h+(i_s\circ g'_{hs}\circ p_h)\circ\xi_h$$

$$=a_{hs}\xi_s+b_{hs}\xi_s+i_{s*}(g'_{hs}) \qquad \text{(by (2.3))}$$

$$=(a_{hs}+b_{hs})\xi_s+i_{s*}(g'_{hs}).$$

Hence we have

$$\theta\circ\rho=\sum_{s=1}^{r}j_{s*}\left(\sum_{h=1}^{r}a_{hs}i_{s*}\lambda(\gamma_h)+\left(\sum_{h=1}^{r}a_{hs}(a_{hs}+b_{hs})\right)[\xi_s,\ i_s]+\sum_{h=1}^{r}i_{s*}[g'_{hs},\ \iota_n]\right). \tag{3.5}$$

First, we suppose that $X$ and $X'$ are of the same homotopy type. Then there exists a homotopy equivalence $\theta\in Eq(K)$ satisfying $\theta\circ\rho=\pm\rho'$. Hence it follows from (3.2) and (3.5) that we have

$$\lambda(\gamma'_s)=\pm\sum_{h=1}^{r}a_{hs}\lambda(\gamma_h) \qquad \mathrm{mod}\ P\pi_{n+k+1}(S^n), \tag{3.6}$$

and

$$\sum_{h=1}^{r}a_{hs}(a_{hs}+b_{hs})=\pm 1 \qquad \text{for}\quad 1\leqq s\leqq r.$$

Thus, if we put $A=\pm(a_{hs})\in\mathrm{Mat}(r,\ Z)$, then it follows from (3.6) that we have

$${}^t(\lambda(X'_1),\ \lambda(X'_2),\ \cdots,\ \lambda(X'_r))=A\cdot{}^t(\lambda(X_1),\ \lambda(X_2),\ \cdots,\ \lambda(X_r)).$$

Moreover, since $\theta\in Eq(K)$, from Corollary 2.10 we also have $A\in GL_r(Z)$. Therefore, the condition (1.6) is necessary.

Conversely, we suppose that there exists an unimodular matrix $A=(a_{hs})\in GL_r(Z)$ satisfying the condition (1.6). However, considering Theorem 1.5, without loss of generalities we may assume that

$$\lambda(\gamma'_s)=\sum_{h=1}^{r}a_{hs}\lambda(\gamma_h) \qquad \text{for}\quad 1\leqq s\leqq r. \tag{3.7}$$

For each element $B=(b_{hs})\in\mathrm{Mat}(r,\ Z)$, we put $\theta_B=(a_{hs}\sigma_{hs}+b_{hs}\lambda_{hs})\in[K,\ K]$. Then it follows from Corollary 2.10 that we obtain

$$\theta_B\in Eq(K) \quad \text{if and only if}\quad A+B\in GL_r(Z). \tag{3.8}$$

On the other hand, considering (3.5) and (3.7),

$$\theta_B\circ\rho=\sum_{s=1}^{r}j_{s*}\left(\sum_{h=1}^{r}a_{hs}i_{s*}\lambda(\gamma_h)+\left(\sum_{h=1}^{r}a_{hs}(a_{hs}+b_{hs})\right)[\xi_s,\ i_s]\right)$$

$$=\sum_{s=1}^{r}j_{s*}\left(i_{s*}\lambda(\gamma_s)+\left(\sum_{h=1}^{r}a_{hs}(a_{hs}+b_{hs})\right)[\xi_s,\ i_s]\right).$$

Hence it follows from (3.2) that we also have

$$\theta_B\circ\rho=\rho' \quad \text{if and only if}\quad \sum_{h=1}^{r}a_{hs}(a_{hs}+b_{hs})=1 \quad \text{for}\ 1\leqq s\leqq r. \tag{3.9}$$

Now we put $B=({}^tA)^{-1}-A\in\mathrm{Mat}(r, Z)$. Then it is easy to see that the conditions (3.8) and (3.9) are satisfied. Hence $X$ and $X'$ are of the same homotopy type. This completes the proof of Theorem A.

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