K. NIINO
KODAI MATH. J.
8 (1985), 296—306

ON FINITE MODIFICATIONS OF ALGEBROID SURFACES

Dedicated to Professor Yukio Kusunoki on his 60th birthday

BY KIYOSHI NIINO

§1. Introduction.

Let R be an open Riemann surface, $\mathfrak{M}(R)$ the family of non-constant meromorphic functions on R and $P(f)$ the number of values which are not taken by $f\in\mathfrak{M}(R)$. We denote by $P(R)$ the Picard constant of R defined by

$$P(R)=\sup\{P(f)\,;\, f\in\mathfrak{M}(R)\}.$$

In general we have $P(R)\geqq 2$. The significant meaning of this Picard constant lies in the following fact:

THEOREM A (Ozawa [9]). *If $P(R)<P(S)$ for another Riemann surface S, then there is no non-trivial analytic mapping of R into S.*

From now on we shall confine ourselves to finitely sheeted covering algebroid surfaces defined as proper existence domains of algebroid functions. From the theory of algebroid functions we have $P(R_n)\leqq 2n$ for an n-sheeted algebroid surface R_n. An n-sheeted algebroid surface R_n is called regularly branched when it has no branched point other than those of order $n-1$.

Let $\mathfrak{E}_n$ be the family of entire functions having an infinite number of zeros whose orders are coprime to n and $\mathfrak{E}_n^*$ the subfamily of $\mathfrak{E}_n$ consisting of entire functions orders of all zeros of which are less than n.

We denote by R_n and $\tilde{R}_n$ two algebroid surfaces defined by $y^n=G(z)$ and $y^n=\tilde{G}(z)$, respectively, where $G(z)$ and $\tilde{G}(z)$ belong to $\mathfrak{E}_n^*$. If $G(z)$ has the same zeros with the same multiplicity as $\tilde{G}(z)$ in $|z|\geqq r_0$ for a suitable positive number r_0 and has at least one distinct zero with the multiplicity from $\tilde{G}(z)$ in $|z|<r_0$, then we call $\tilde{R}_n$ a finite modification of R_n (cf. Ozawa [11]).

We now consider two n-sheeted, regularly branched algebroid surfaces R_n and $\tilde{R}_n$ and two m-sheeted, regularly branched algebroid surfaces S_m and $\tilde{S}_m$. Suppose that $P(R_n)=2n$, $P(S_m)=2m$ and $\tilde{R}_n$ and $\tilde{S}_m$ are finite modifications of R_n and S_m, respectively. In our previous paper [8] we had a perfect condition for the existence of analytic mappings of R_n into S_m and investigated the structure of the family $\mathfrak{H}(R_n, S_m)$ of projections of analytic mappings of R_n into S_m.

Received September 11, 1984

In the present paper we shall consider the following two problems:

(A) What is $P(\tilde{R}_n)$?
(B) Are there any analytic mappings among R_n, $\tilde{R}_n$, S_m and $\tilde{S}_m$?

And we shall obtain generalizations of results of the author [6].

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and the usual notations such as $T(r, f)$, $N(r, a, f)$, $\overline{N}(r, a, f)$, $m(r, f)$ etc. (see e.g. [5]).

§ 2. A functional equation.

For our purpose we have to consider a functional equation. We firstly prove

THEOREM 1. *Let m be a positive integer. Suppose that two non-constant entire functions $H(z)$ and $M(z)$ with $H(0)=M(0)=0$ and four integers a, b, c and d with $0<a\leqq b<m$ and $0<c\leqq d<m$ satisfy the following functional equation*

$$(e^{M(z)}-\gamma)^a(e^{M(z)}-\delta)^b=F(z)(e^{H(z)}-\sigma)^c(e^{H(z)}-\tau)^d \tag{2.1}$$

with four constants γ, δ, σ and τ and a meromorphic function $F(z)=f_1(z)^m f_2(z)$, where $f_1(z)$ and $f_2(z)$ are meromorphic in $|z|<+\infty$ and

$$T(r, f_2)=o(T(r, e^M)) \quad or \quad T(r, f_2)=o(T(r, e^H)) \qquad r\to\infty \tag{2.2}$$

outside a set of r of finite measure. Then we have

(I) *$a=c$ and $b=d$*

and

(II) *one of the following four cases:*

$$H(z)=M(z),\quad F(z)=1,\quad \gamma=\sigma,\quad \delta=\tau, \tag{2.3}$$

$$H(z)=M(z),\quad F(z)=1,\quad \gamma=\tau,\quad \delta=\sigma, \tag{2.4}$$

$$H(z)=-M(z),\quad F(z)=(-1)^{a+b}\gamma^a\delta^b e^{(a+b)M(z)},\quad \gamma\sigma=\delta\tau=1, \tag{2.5}$$

$$H(z)=-M(z),\quad F(z)=(-1)^{a+b}\gamma^a\delta^b e^{(a+b)M(z)},\quad \gamma\tau=\delta\sigma=1. \tag{2.6}$$

(2.4) *and* (2.6) *may occur in the case $a=b$ only.*

To prove our Theorem 1 we need

LEMMA A ([10]). *Let $H(z)$ be a non-constant entire function and α a non-zero constant. Then we have*

$$N_2(r, 0, e^H-\alpha)\sim m(r, f) \quad and \quad N_1(r, 0, e^H-\alpha)=o(m(r, e^H)) \qquad r\to\infty$$

outside a set of r of finite measure, where $N_2(r, 0, f)$ is the counting function of *simple zeros of the function f and $N_1(r, 0, f)=N(r, 0, f)-\bar{N}(r, 0, f)$.*

Proof of Theorem 1. Firstly we prove that the condition

$$T(r, f_2)=o(T(r, e^M)) \qquad r\to\infty \tag{2.7}$$

outside a set of r of finite measure is equivalent to the condition

$$T(r, f_2)=o(T(r, e^H)) \qquad r\to\infty \tag{2.8}$$

outside a set of r of finite measure. Assume that the condition (2.7) is true. Then we consider simple zeros of $(e^M-\gamma)(e^M-\delta)$. It follows from the functional equation (2.1) that these are zoros of f_2 or $(e^H-\sigma)^c(e^H-\tau)^d$, and consequently

$$N_2(r, 0, (e^M-\gamma)(e^M-\delta))\leqq N(r, 0, f_2)+N(r, 0, (e^H-\sigma)^c(e^H-\tau)^d).$$

Hence Lemma A and (2.7) imply

$$(2+o(1))m(r, e^M)\leqq o(m(r, e^M))+(c+d+o(1))m(r, e^H) \qquad r\to\infty$$

and so

$$m(r, e^M)=O(m(r, e^H)) \qquad r\to\infty$$

outside a set of r of finite measure. Thus we obtain (2.8). Conversely we assume that the condition (2.8) holds. Then from (2.1) we have

$$N_2(r, 0, (e^H-\sigma)(e^H-\tau))\leqq N(r, 0, f_2)+N(r, 0, (e^M-\gamma)^a(e^M-\delta)^b).$$

Hence it follows from Lemma A and (2.8) that

$$(2+o(1))m(r, e^H)\leqq o(m(r, e^H))+(a+b+o(1))m(r, e^M) \qquad r\to\infty$$

and so

$$m(r, e^H)=O(m(r, e^M)) \qquad r\to\infty$$

outside a set of r of finite measure. Hence we have (2.7).

Thus we see that the condition (2.2) can be replace by

$$T(r, f_2)=o(m(r, e^M)) \quad \text{and} \quad T(r, f_2)=o(m(r, e^H)) \qquad r\to\infty \tag{2.9}$$

outside a set of r of finite measure. Further we can deduce from the above discussion that

$$m(r, e^H)=O(m(r, e^M)) \quad \text{and} \quad m(r, e^M)=O(m(r, e^H)) \qquad r\to\infty \tag{2.10}$$

outside a set of r of finite measure.

We now prove (I). Assume that $a<c$. The functional equation (2.1) implies that a simple zero z_1 of $e^M-\gamma$ is a zero of order a of $G(z)\equiv F(z)(e^{H(z)}-\sigma)^c(e^{H(z)}-\tau)^d$. It follows from our assumption of $F(z)$ and $a<c\leqq b<m$ that z_1 is a zero of f_2 or a pole of f_2 or a multiple zero of $(e^H-\sigma)(e^H-\tau)$. Hence Lemma A and (2.9) yield

$$m(r, e^{M})=o(m(r, e^{H})) \qquad r\rightarrow\infty$$

outside a set of r of finite measure, which contradicts (2.10). We next assume that $a>c$. Then we deduce from (2.1) that a simple zero of $e^{H}-\sigma$ is a zero of $1/f_{2}$ or a pole of $1/f_{2}$ or a multiple zero of $(e^{M}-\gamma)(e^{M}-\delta)$. Hence Lemma A and (2.9) imply

$$m(r, e^{H})=o(m(r, e^{M})) \qquad r\rightarrow\infty$$

outside a set of r of finite measure, which contradicts (2.10). Therefore we obtain $a=c$. Similarly considering simple zeros of $e^{M}-\delta$ and $e^{H}-\tau$ and taking Lemma A, (2.9) and (2.10) into account we have $b=d$. Thus (I) is proved.

From (2.1) and (I) we have

$$(2.11) \qquad (e^{M(z)}-\gamma)^{a}(e^{M(z)}-\delta)^{b}=F(z)(e^{H(z)}-\sigma)^{a}(e^{H(z)}-\tau)^{b},$$

$$H(z)\not\equiv\text{const.},\quad M(z)\not\equiv\text{const.},\quad H(0)=M(0)=0,\quad F(z)=f_{1}(z)^{m}f_{2}(z),$$

$$0<a\leqq b<m,\quad \gamma\delta\sigma\tau(\gamma-\delta)(\sigma-\tau)\neq 0.$$

Considering simple zeros of $e^{M}-\gamma$ and $e^{H}-\sigma$, we can deduce from Lemma A and (2.9) that

$$(2.12) \qquad m(r, e^{M})\sim m(r, e^{H}) \qquad r\rightarrow\infty,$$

that is, $m(r, e^{M})/m(r, e^{H})\rightarrow 1$ as $r\rightarrow\infty$, outside a set of r of finite measure. Further we have

$$T(r, F)=O(T(r, e^{M})+T(r, e^{H}))=O(m(r, e^{H})) \qquad r\rightarrow\infty$$

and

$$\begin{aligned} N(r, \infty, F'/F)&\leqq N(r, 0, F)+N(r, \infty, F)\\ &\leqq N_{1}(r, 0, e^{M}-\gamma)+N_{1}(r, 0, e^{M}-\delta)+N_{1}(r, 0, e^{H}-\sigma)\\ &\quad +N_{1}(r, 0, e^{H}-\tau)+N(r, 0, f_{2})+N(r, \infty, f_{2})+O(\log r)\\ &=o(m(r, e^{M})+m(r, e^{H}))=o(m(r, e^{H})) \end{aligned}$$

outside a set of r of finite measure. Hence we obtain

$$\begin{aligned} T(r, F'/F)&=m(r, F'/F)+N(r, \infty, F'/F)\\ &\leqq O(\log rT(r, F))+N(r, \infty, F'/F)\\ &=o(m(r, e^{H})), \end{aligned}$$

and consequently

$$(2.13) \qquad T(r, F'/F)=o(m(r, e^{H})) \qquad r\rightarrow\infty$$

outside a set of r of finite measure. Since (2.12) and (2.13) valid, the proof of Theorem in [7] can be transferred to our case, even if $a+b\neq m$. Thus the proof of (II) follows the lines of that of Theorem in [7, pp. 298-301].

§ 3. Known results.

Further we need some known results.

THEOREM A ([1]). *Let R_n be an n-sheeted regularly branched algebroid surface with $P(R_n)>(3/2)n$. Then we have $P(R_n)=2n$.*

THEOREM B ([1], [8]). *Let R_n be a regularly branched algebroid surface defined by $y^n=G(z)$ $(G\in\mathfrak{E}_n^*)$. If $P(R_n)=2n$, then $G(z)$ satisfies the following functional equation·*

$$G(z)=f(z)^n(e^{H(z)}-\alpha)^k(e^{H(z)}-\beta)^{n-k},$$

$$H(z)\not\equiv\text{const.},\quad H(0)=0,\quad \alpha\beta(\alpha-\beta)\neq 0,\quad (k,n)=1,\quad 1\leqq k\leqq n/2,$$

where $H(z)$ is entire, $f(z)$ is meromorphic, α and β two complex constants and k is an integer.

THEOREM C ([3], [4], [8]). *Let R_n and S_m be two algebroid surfaces defined by $y^n=G(z)$ and $u^m=g(w)$ $(G, g\in\mathfrak{E}_n)$, respectively and further $G(z)$ satisfies the inequality with a constant η*

$$\frac{N_n^*(r,0,G)}{N(r,0,G)}\geqq\eta>0 \tag{3.1}$$

for a set of r of infinite measure, where $N_n^*(r,0,G)$ is the counting function of zeros whose orders are coprime to n. If there is an analytic mapping ϕ of R_n into S_m, then $n=pm$ with a positive integer p and the projection $h(z)$ of ϕ is a single-valued entire function of z and satisfies

$$g(h(z))=f(z)^mG(z)^k,\qquad p\leqq kp\leqq n-1, \tag{3.2}$$

where $f(z)$ is a suitable meromorphic function and k is a suitable positive integer which is coprime to m.

Conversely, if $n=pm$ with a positive integer p and there is an entire function $h(z)$ satisfies (3.2) *with a suitable meromorphic function $f(z)$ and a suitable positive integer k which is coprime to m, there exists an analytic mapping of R_n into S_m whose projection is $h(z)$.*

§ 4. Picard constants.

With respect to the problem (A) we have the following

THEOREM 2. *Let R_n and $\tilde{R}_n$ be two n-sheeted regularly branched algebroid surfaces defined by $y^n=G(z)$ and $y^n=\tilde{G}(z)$, respectively, where $G(z)$ and $\tilde{G}(z)$ are* *two entire functions belonging to* $\mathfrak{E}_n^*$. *If* $P(R_n)=2n$ *and* $\tilde{R}_n$ *is a finite modification of* R_n, *then we have* $P(\tilde{R}_n)\leqq(3/2)n$.

Proof. By the definition of finite modifications of algebroid surfaces we have

$$\tilde{G}(z)=Q(z)G(z), \tag{4.1}$$

where $Q(z)$ is a rational function satisfying the following conditions (M1)—(M6) (Hereafter in this case we simply say that $Q(z)$ satisfies the condition (M) with respect to $G(z)$ and n):

(M1) $Q(z)$ has a form

$$Q(z)=\prod_{i=1}^{\mu}(z-a_i)^{\mu_i}\prod_{j=1}^{\nu}(z-b_j)^{-\nu_j}.$$

(M2) μ, μ_i, ν, ν_j are non-negative integers and $\mu+\nu\geqq1$.
(M3) a_i and b_j are mutually distinct constants and their moduli are less than r_0.
(M4) If a_i is not a zero of $G(z)$, then $0<\mu_i<n$ and $(\mu_i, n)=1$.
(M5) If a_i is a zero of order k_i of $G(z)$, then $k_i+\mu_i<n$ and $(k_i+\mu_i, n)=1$.
(M6) b_j is a zero of order l_j of $G(z)$ satisfying $l_j-\nu_j=0$ or $0<l_j-\nu_j<n$, $(l_j-\nu_j, n)=1$.

Since $P(R_n)=2n$, Theorem B implies that $G(z)$ satisfies

$$G(z)=f_1(z)^n(e^{H(z)}-\alpha)^l(e^{H(z)}-\beta)^{n-l}, \tag{4.2}$$

$$H(z)\not\equiv\text{const.},\quad H(0)=0,\quad \alpha\beta(\alpha-\beta)\neq0,\quad (l, n)=1,\quad 1\leqq l\leqq n/2,$$

where $H(z)$ is an entire function, $f_1(z)$ is a meromorphic function, α and β are two complex constants and l is an integer.

Now suppose, to the contrary, that $P(\tilde{R}_n)>(3/2)n$. Then it follows from Theorem A and Theorem B that $\tilde{G}(z)$ satisfies

$$\tilde{G}(z)=f_2(z)^n(e^{M(z)}-\gamma)^k(e^{M(z)}-\delta)^{n-k}, \tag{4.3}$$

$$M(z)\not\equiv\text{const.},\quad M(0)=0,\quad \gamma\delta(\gamma-\delta)\neq0,\quad (k, n)=1,\quad 1\leqq k\leqq n/2,$$

where $M(z)$ is entire, $f_2(z)$ is meromorphic and γ and δ are two constants. It follows from (4.1), (4.2) and (4.3) that

$$(e^{M(z)}-\gamma)^k(e^{M(z)}-\delta)^{n-k}=Q(z)\{f_1(z)f_2(z)^{-1}\}^n(e^{H(z)}-\alpha)^l(e^{H(z)}-\beta)^{n-l}.$$

Since $Q(z)$ is a rational function, we have $T(r, Q)=o(T(r, e^M))$ as $r\to\infty$. Hence our Theorem 1 yields that

$$Q(z)\{f_1(z)f_2(z)^{-1}\}^n=1 \quad\text{or}\quad =(-1)^n\gamma^k\delta^k e^{nM(z)},$$

which contradicts the condition (M). Therefore we have $P(\tilde{R}_n)\leqq(3/2)n$.

§ 5. Existence of analytic mappings.

Let R_n and S_m be two regularly branched algebroid surfaces defined by $y^n=G(z)$ and $u^m=g(w)$ with $G(z)\in\mathfrak{E}_n^*$ and $g(w)\in\mathfrak{E}_m^*$, respectively. If $P(R_n)=2n$ and $P(S_m)=2m$, then it follows from Theorem B that $G(z)$ and $g(w)$ satisfy

$$G(z)=F(z)^n(e^{H(z)}-\alpha)^l(e^{H(z)}-\beta)^{n-l}, \tag{5.1}$$

$$H(0)=0,\quad \alpha\beta(\alpha-\beta)\neq 0,\quad (l, n)=1,\quad 1\leqq l\leqq n/2$$

and

$$g(w)=f(w)^m(e^{L(w)}-\gamma)^k(e^{L(w)}-\delta)^{m-k}, \tag{5.2}$$

$$L(0)=0,\quad \gamma\delta(\gamma-\delta)\neq 0,\quad (k, m)=1,\quad 1\leqq k\leqq m/2,$$

where H and L are two non-constant entire functions, F and f are two meromorphic functions, l and k are two integers and α, β, γ and δ are four complex constants. Further let $\tilde{R}_n$ and $\tilde{S}_m$ be finite modifications of R_n and S_m defined by $y^n=\tilde{G}(z)$ and $u^m=\tilde{g}(w)$ with $\tilde{G}(z)=Q(z)G(z)\in\mathfrak{E}_n^*$ and $g(w)=\tilde{q}(w)g(w)\in\mathfrak{E}_m^*$, respectively, where

$$Q(z)=\prod_{i=1}^{\mu}(z-a_i)^{\mu_i}\prod_{j=1}^{\nu}(z-b_j)^{-\nu_j} \tag{5.3}$$

and

$$q(w)=\prod_{i=1}^{\sigma}(w-a_i)^{\sigma_i}\prod_{j=1}^{\tau}(w-d_j)^{-\tau_j} \tag{5.4}$$

satisfy the condition (M) with respect to $G(z)$ and n and with respect to $g(w)$ and m, respectively.

Now in this section we consider the problem (B), that is, whether there exist analytic mappings among R_n, $\tilde{R}_n$, S_m and $\tilde{S}_m$. We have already obtained a perfect condition for the existence of analytic mappings of R_n into S_m in [8]. Using Lemma A we here note that G, $\tilde{G}$, g and $\tilde{g}$ satisfy the condition (3.1) in Theorem C and consequently we can apply Theorem C to analytic mappings in this section.

Firstly we have

THEOREM 3. *There exists an analytic mapping ϕ of R_n into $\tilde{S}_m$ if and only if $n=pm$ with a positive integer p and there exist an entire function $h(z)$ and meromorphic functions $f_1^*(z)$ and $f_2^*(z)$ satisfying one of the following equations:*

(a) $H(z)=L(h(z))-L(h(0)),\quad q(h(z))=f_1^*(z)^m,\quad \gamma/\alpha=\delta/\beta=e^{L(h(0))},$

(a′) $H(z)=L(h(z))-L(h(0)),\quad q(h(z))=f_1^*(z)^m,\quad \gamma/\beta=\delta/\alpha=e^{L(h(0))},$

(b) $H(z)=-L(h(z))+L(h(0)),\quad q(h(z))=f_2^*(z)^m,\quad \gamma\alpha=\delta\beta=e^{L(h(0))},$

(b′) $H(z)=-L(h(z))+L(h(0)),\quad q(h(z))=f_2^*(z)^m,\quad \gamma\beta=\delta\alpha=e^{L(h(0))}.$

Proof. Suppose that there is an analytic mapping ψ of R_n into $\tilde{S}_m$. Then it follows from Theorem C that $n=pm$ with a positive integer p and the projection $h(z)$ of ψ is a single-valued entire function of z and satisfies

$$\tilde{g}(h(z))=f_3(z)^m G(z)^\lambda, \tag{5.5}$$

where $f_3(z)$ is a meromorphic function and an integer λ satisfies $(\lambda, m)=1$ and $p\leqq p\lambda\leqq n-1$. We put $l\lambda=am+c$ and $(n-l)\lambda=bm+d$, where a, b, c and d are four integers satisfying $0\leqq c\leqq m-1$, $0\leqq d\leqq m-1$. Then we have $c>0$ and $d>0$ because of $(\lambda, m)=1$ and $(l, n)=1$. From (5.1), (5.2), (5.4) and (5.5) we have

$$(e^{M(z)}-\gamma e^{-L(h(0))})^k(e^{M(z)}-\delta e^{-L(h(0))})^{m-k}=F_1(z)(e^{H(z)}-\alpha)^c(e^{H(z)}-\beta)^d, \tag{5.6}$$

where $M(z)=L(h(z))-L(h(0))$, $F_1(z)=f_1(z)^m f_2(z)$, $f_2(z)=q(h(z))^{-1}$ and $f_1(z)=e^{-L(h(0))}f(h(z))^{-1}f_3(z)F(z)^{p\lambda}(e^{H(z)}-\alpha)^a(e^{H(z)}-\beta)^b$. Since $q(z)$ is rational, we have $T(r, q(h))=O(T(r, h))$ as $r\to\infty$. Since $e^{L(z)}$ is transcendental, Theorem 2 in Clunie [2] implies $T(r, h)=o(T(r, e^{L(h)}))$ and consequently

$$T(r, f_2)=T(r, q(h))=o(T(r, e^{L(h)}))=o(T(r, e^M)) \qquad r\to\infty.$$

Hence $f_2(z)$ satisfies the condition (2.2) in our Theorem 1. Applying Theorem 1 to functional equation (5.6) we have one of the following four cases:

$$H(z)=M(z), \quad F_1(z)=1, \quad \gamma e^{-L(h(0))}=\alpha, \quad \delta e^{-L(h(0))}=\beta, \tag{5.7}$$

$$H(z)=M(z), \quad F_1(z)=1, \quad \gamma e^{-L(h(0))}=\beta, \quad \gamma e^{-L(h(0))}=\alpha, \tag{5.8}$$

$$H(z)=-M(z), \quad F_1(z)=(-1)^m\gamma^k\delta^{m-k}e^{m(M(z)-L(h(0)))}, \quad \gamma\alpha=\delta\beta=e^{L(h(0))}, \tag{5.9}$$

$$H(z)=-M(z), \quad F_1(z)=(-1)^m\gamma^{m-k}\delta^k e^{m(M(z)-L(h(0)))}, \quad \gamma\beta=\delta\alpha=e^{L(h(0))}, \tag{5.10}$$

which correspond, respectively, to (a), (a′), (b) and (b′) in our Theorem 3 with $f_1^*(z)=f_1(z)$ in (a) and (a′) and $f_2^*(z)=-\gamma^{k/m}\delta^{(m-k)/m}f_1(z)e^{L(h(z))-2L(h(0))}$ in (b) and (b′).

Conversely, suppose that $n=pm$ with a positive integer p and there is an entire function $h(z)$ satisfying one of the four cases (a), (a′), (b) and (b′). Firstly we note that for positive integers n, m, p, l and k satisfying $n=pm$, $(l, n)=(k, m)=1$ there are integers λ, a, b, ρ, c and d satisfying

$$l\lambda+am=k, \quad (n-l)\lambda+bm=m-k, \quad (\lambda, m)=1, \quad 1\leqq\lambda\leqq m-1$$

and

$$l\rho+cm=m-k, \quad (n-l)\rho+dm=k, \quad (\rho, m)=1, \quad 1\leqq\rho\leqq m-1.$$

If $h(z)$ satisfies (a), then we have

$$\begin{aligned}\tilde{g}(h(z))&=q(h(z))f(h(z))^m(e^{L(h(z))}-\gamma)^k(e^{L(h(z))}-\delta)^{m-k}\\&=f_1^*(z)^m f(h(z))^m e^{mL(h(0))}(e^{H(z)}-\alpha)^{l\lambda+am}(e^{H(z)}-\beta)^{(n-l)\lambda+bm}\\&=f_1(z)^m\{F(z)^n(e^{H(z)}-\alpha)^l(e^{H(z)}-\beta)^{n-l}\}^\lambda\end{aligned}$$

$$=f_1(z)^m G(z)^\lambda ,$$

where $f_1(z)=e^{L(h(0))}f_1^*(z)f(h(z))F(z)^{-p\lambda}(e^{H(z)}-\alpha)^a(e^{H(z)}-\beta)^b$. Similarly we have

$$\tilde{g}(h(z))=f_2(z)^m G(z)^\rho ,$$

$$f_2(z)=e^{L(h(0))}f_1^*(z)f(h(z))F(z)^{-p\rho}(e^{H(z)}-\alpha)^c(e^{H(z)}-\beta)^d$$

if (a′) is the case, or

$$\tilde{g}(h(z))=f_3(z)^m G(z)^\lambda ,$$

$$f_3(z)=-\gamma^{k/m}\delta^{(m-k)/m}f_2^*(z)f(h(z))e^{-H(z)}F(z)^{-p\lambda}(e^{H(z)}-\alpha)^a(e^{H(z)}-\beta)^b$$

if (b) is the case, or

$$\tilde{g}(h(z))=f_4(z)^m G(z)^\rho ,$$

$$f_4(z)=-\gamma^{k/m}\delta^{(m-k)/m}f_2^*(z)f(h(z))e^{-H(z)}F(z)^{-p\rho}(e^{H(z)}-\alpha)^c(e^{H(z)}-\beta)^d$$

if (b′) is the case. Hence from Theorem C there is an analytic mapping ϕ of R_n into $\tilde{S}_m$ whose projection is $h(z)$.

Thus the proof of our Theorem 3 is complete.

As an immediate consequence of our Theorem 3 and Theorem 2 in [8] we have

COROLLARY 1. *If there is an analytic mapping ϕ of R_n into $\tilde{S}_m$, then there exists an analytic mapping of R_n into S_m whose projection is the same $h(z)$ as that of ϕ.*

If we take $\tilde{R}_n$ as $\tilde{S}_m$ in Theorem 3, then we have $H(z)=\pm H(h(z))\mp H(h(0))$ and consequently $h(z)$ is a linear function $Az+B$ ($A\neq 0$). Hence there is no meromorphic function $f^*(z)$ satisfying $f^*(z)^n=Q(Az+B)$ because of (5.3) and the condition (M). Therefore from Theorem 3 we obtain

COROLLARY 2. *There is no non-trivial analytic mapping of R_n into $\tilde{R}_n$.*

From the arguments in the proof of Theorem 3 we can deduce

THEOREM 4. *There exists an analytic mapping of $\tilde{R}_n$ into $\tilde{S}_m$ if and only if there exist an entire function $h(z)$, two meromorphic functions $f_1^*(z)$ and $f_2^*(z)$ and two positive integers p and λ such that $n=pm$, $(\lambda, m)=1$, $p\leqq p\lambda\leqq n-1$ and one of the following equations holds:*

(a) $H(z)=L(h(z))-L(h(0))$, $q(h(z))=f_1^*(z)^m Q(z)^\lambda$,

$$\frac{\gamma}{\alpha}=\frac{\delta}{\beta}=e^{L(h(0))} \quad or \quad \frac{\gamma}{\beta}=\frac{\delta}{\alpha}=e^{L(h(0))},$$

(b) $H(z)=-L(h(z))+L(h(0))$, $q(h(z))=f_2^*(z)^m Q(z)^\lambda$,

$$\gamma\alpha=\delta\beta=e^{L(h(0))} \quad or \quad \gamma\beta=\delta\alpha=e^{L(h(0))}.$$

It follows from our Theorem 4 and Theorem 2 in [8] that

COROLLARY 3. *If there is an analytic mapping ϕ of $\tilde{R}_n$ into $\tilde{S}_m$, then there exists an analytic mapping of R_n into S_m whose projection is the same $h(z)$ as that of ϕ.*

We can also deduce

THEOREM 5. *There exists an analytic mapping of $\tilde{R}_n$ into S_m if and only if there exist an entire function $h(z)$, two meromorphic functions $f_1^*(z)$ and $f_2^*(z)$ and two positive integers p and λ such that $n=pm$, $(\lambda, m)=1$, $p\leqq p\lambda\leqq n-1$ and one of the following equations holds*

(a) $$H(z)=L(h(z))-L(h(0)), \quad Q(z)^\lambda=f_1^*(z)^m,$$

$$\frac{\gamma}{\alpha}=\frac{\delta}{\beta}e^{L(h(0))} \quad or \quad \frac{\gamma}{\beta}=\frac{\delta}{\alpha}e^{L(h(0))},$$

(b) $$H(z)=-L(h(z))+L(h(0)), \quad Q(z)^\lambda=f_2^*(z)^m,$$

$$\gamma\alpha=\delta\beta=e^{L(h(0))} \quad or \quad \gamma\beta=\delta\alpha=e^{L(h(0))}.$$

COROLLARY 4. *If there is an analytic mapping ϕ of $\tilde{R}_n$ into S_m, then there exists an analytic mapping of R_n into S_m whose projection is the same $h(z)$ as that of ϕ.*

Now we shall give an example which shows existence of an analytic mapping of $\tilde{R}_n$ into S_m.

EXAMPLE. $n=8$, $m=4$. Put $G(z)=(e^{2z}-1)(e^{2z}+1)^7$, $Q(z)=z^4/(z-\pi i/2)^4$, $\tilde{G}(z)=Q(z)G(z)$ and $g(w)=(e^w-1)(e^w+1)^3$. Let R_8, $\tilde{R}_8$ and S_4 be algebroid surfaces defined by $y^8=G(z)$, $y^8=\tilde{G}(z)$ and $u^4=g(w)$, respectively. Then since $z=0$ is a zero of order 5 of $\tilde{G}(z)$ and $z=\pi i/2$ is a zero of order 3 of $\tilde{G}(z)$, it is clear that these surfaces are regularly branched with $P(R_8)=16$ and $P(S_4)=8$ (cf. Theorem B), $\tilde{R}_8$ is a finite modification of R_8 and satisfy (a) of Theorem 5 with

$$H(z)=2z, \quad L(w)=w, \quad h(z)=2z, \quad f_1^*(z)=z/(z-\pi i/2),$$

$$\lambda=1, \quad \gamma=\alpha=1, \quad \delta=\beta=-1.$$

Thus we see that there exists an analytic mapping of $\tilde{R}_8$ into S_4. However we suppose that $(\mu_i, n)=(\nu_j, n)=1$ in (5.3). Then since $(\lambda, m)=1$ and $n=pm$, we have $(\lambda\mu_i, m)=(\lambda\nu_j, m)=1$ and so there is no meromorphic function $f^*(z)$ satisfying $Q(z)^\lambda=f^*(z)^m$. Hence we finally deduce from Theorem 5 that

COROLLARY 5. *Suppose that $(\mu_i, n)=(\nu_j, n)=1$ in (5.3). Then there is no* *analytic mapping of $\tilde{R}_n$ into S_m.*

REFERENCES

[1] AOGAI, H., Picard constant of a finitely sheeted covering surfaces. Kōdai Math. Sem. Rep. **25** (1973), 219-224.

[2] CLUNIE, J., The composition of entire and meromorphic functions. Mathematical Essays dedicated to A. J. Macintyre (Ohio Univ. Press, 1970), 75-92.

[3] HIROMI, G., AND H. MUTŌ, On existence of analytic mappings, II. Kōdai Math. Sem. Rep. **19** (1967), 439-450.

[4] MUTŌ, H., On analytic mappings among algebroid surfaces. Kōdai Math. Sem. Rep. **21** (1969), 191-204.

[5] NEVANLINNA, R., Analytic functions. Springer-Verlag, (1970), pp. 373.

[6] NIINO, K., On finite modifications of two- or three-sheeted covering open Riemann surfaces. Kōdai Math. Sem. Rep. **19** (1967), 415-424.

[7] NIINO, K., On the functional equation $(e^M-\gamma)^k(e^M-\delta)^{m-k}=f^m(e^H-\sigma)(e^H-\tau)^{m-k}$. Aequationes Math. **22** (1981), 293-301.

[8] NIINO, K., On analytic mappings between two algebroid surfaces. Complex Variables Theory Appl. **2** (1983-84), 283-293.

[9] OZAWA, M., On complex analytic mappings. Kōdai Math. Sem. Rep. **17** (1965), 93-102.

[10] OZAWA, M., On ultrahyperelliptic surfaces. Kōdai Math. Sem. Rep. **17** (1965), 103-108.

[11] OZAWA, M., On a finite modification of an ultrahyperelliptic surfaces. Kōdai Math. Sem. Rep. **19** (1967), 312-316.

FACULTY OF TECHNOLOGY
KANAZAWA UNIVERSITY
2-40-20, KODATSUNO
KANAZAWA 920
JAPAN