ON SELF-HOMOTOPY EQUIVALENCES OF
S^3-PRINCIPAL BUNDLES OVER S^n

Dedicated to Professor Nobuo Shimada on his 60-th birthday

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§1. Introduction.

The purpose of this paper is to study the group $\varepsilon(X)$ of homotopy classes of self-homotopy equivalences for the total space of a S^3-principal bundle over S^n ($n \geqq 6$). Since this group acts on the set of all homotopy invariants it must be useful in the homotopy theory to clarify the group structure and it's action. However, as stated in [1], any finite group can be realized as a subgroup of $\varepsilon(X)$ for suitablly chosen space X, so it seems to be difficult that we clarify them in general. Many authors have computed the group $\varepsilon(X)$ for various type of spaces. Especially J. W. Rutter has determined $\varepsilon(X)$ in our case of $n=7$ in [4] and also N. Sawashita and M. Mimura treated our case under some additional conditions in [6] J. W. Rutter's results were complete except one sub-case because he could use the speciality of $n=7$, however, our results are weaker compared with his ones because of generality. In our theorem the group structure of $\varepsilon(X)$ is only clarified up to extension, and to determine extensions is left as problems.

§2. Method and Theorem.

Let $p: X \to S^n$ be a S^3-principal bundle over S^n with the characteristic class $\xi (\in \pi_{n-1}(S^3))$, and let Y^X be the space of continuous maps from X to Y with compact-open topology. Then we have a fibring $p^X: X^X \to S^{nX}$ in the sence that the map p^X satisfies the homotopy lifting condition for CW-complexes. If we take the identity map 1_X and the projection p as the base points of X^X and S^{nX} respectively then the following exact sequence can be obtained as usual

$$\pi_1(X^X, 1_X) \longrightarrow \pi_1(S^{nX}, p) \longrightarrow \pi_0(p^{X-1}(p), 1_X) \longrightarrow \pi_0(X^X, 1_X) \longrightarrow \pi_0(S^{nX}, p).$$

Since, using the free action of S^3 on X, we can easily obtained a homeomorphism

$$\phi: (S^{3X}, *) \longrightarrow (p^{X-1}(p), 1_X) \qquad (*(X)=1)$$

defined by $\phi(f)(x)=xf(x)$ for $f: X \to S^3$, the pair $(p^{X-1}(X), 1_X)$ may be identified

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with the pair $(S^{3X}, *)$. Now it is clear that the space X^X forms a topological semi-group with unit element 1_X under the multiplication defined by composition of maps and also we can give the space S^{3^X} the same structure with unit element $*$ by defining multiplication:

$$f \sharp g(x) = f(x g(x)) g(x).$$

Since these structures are inherited by the set $\pi_0(S^{3^X}, *) = \pi_0(p^{X-1}(P), 1_X)$ and $\pi_0(X^X, 1_X)$, which we may regard as semi-groups, and then the map

$$\pi_0(S^{3^X}, *) \longrightarrow \pi_0(X^X, 1_X)$$

is homomorphic by definitions. For any semi-group G we denote by reg. G the group consisting of regular elements of G. Since it follows from definitions that $\varepsilon(X) = \mathrm{reg.}\, \pi_0(X^X, 1_X)$ and that the boundary

$$\pi_1(S^{nX}, p) \longrightarrow \pi_0(p^{X-1}(P), 1_X) = \pi_0(S^{3^X}, *)$$

is homomorphic, the preceeding sequence can be transformed into the exact sequence

$$\pi_1(X^X, 1_X) \longrightarrow \pi_1(S^{nX}, p) \longrightarrow \mathrm{reg.}\, \pi_0(S^{3^X}, *) \longrightarrow \varepsilon(X) \longrightarrow \pi_0(S^{nX}, p).$$

Thus our purpose is to study

the image of the map $\varepsilon(X) \longrightarrow \pi_0(S^{nX}, p)$ and
the image of the homomorphism $\partial_X : \pi_1(S^{nX}, p) \longrightarrow \mathrm{reg.}\, \pi_0(S^{3^X}, *)$.

Off course these require describing $\pi_0(S^{nX}, p)$ and reg. $\pi_0(S^{3^X}, *)$ by comparablly well-known concepts. These problems shall be treated in §3 and §4. Let $f : K \to L$ be a map and suppose that $H_n(K)$ and $H_n(L)$ are both isomorphic to integers $\mathbf{Z}$. Since the degree of $f_* : H_n(K) \to H_n(L)$ is defined as usual we denote by $d_n(f)$ the degree of f_*. Since our space X has $H_3(X) = H_n(X) = Z$ $d_3(f)$ and $d_n(f)$ are defined. Clearly if f is a homotopoy equivalence we have $d_3(f) = \pm 1$ $d_n(f) = \pm 1$. We denote by $\varepsilon_+(X)$ the kernel of the homomorphism $d = (d_3, d_n)$: $\varepsilon(X) \to Z_2 \times Z_2$, i.e. we have an exact sequence:

$$1 \longrightarrow \varepsilon_+(X) \longrightarrow \varepsilon(X) \overset{d}{\longrightarrow} Z_2 \times Z_2.$$

Remark. d is equivalent to the usual representation $\varepsilon(X) \to \mathrm{Aut}\, H^*(X)$. Let τ be the Blaker-Massey map $S^6 \to S^3$. Then we have

THEOREM A. *Assume that* $\tau \circ E^3 \xi \equiv 0 \mod \xi \circ \pi_{n+2}(S^{n-1})$, *then we have*

$$d : \varepsilon(X) \longrightarrow Z_2 \times Z_2 \quad \text{is onto if } 2\xi = 0 \quad \text{and}$$

$$d : \varepsilon(X) \longrightarrow \{(-1, -1)\} \quad \text{is onto if } 2\xi \neq 0.$$

Remark. If the order of ξ is odd it can be shown that d is trivial in the case of $\tau \circ E^3 \xi \not\equiv 0 \mod \xi \circ \pi_{n+2}(S^{n-1})$.

Let η be the essential map of $\pi_{n+1}(S^n)$ $(n \geq 3)$,

THEOREM B. *If $\xi \circ \eta = 0$ there exists an exact sequence*

$$0 \longrightarrow \pi_n(S^3) \times \pi_{n+3}(S^3)/H_\xi \longrightarrow \varepsilon_+(X) \longrightarrow G_\xi \longrightarrow 0,$$

and if $\xi \circ \eta \neq 0$ we have an exact sequence

$$0 \longrightarrow Z_2 \longrightarrow \pi_n(S^3) \times \pi_{n+3}(S^3)/H_\xi \longrightarrow \varepsilon_+(X) \longrightarrow G_\xi \longrightarrow 0$$

except the case $\xi \circ \eta = \eta \circ E\xi$ where G_ξ and H_ξ are defined in §5.

Remark. We could not get similar results in the exceptional case, but I think that the cases must be determined by $\tau \circ E^3 \xi \circ \eta = 0$ or not. Indeed, I can prove that if $\tau \circ E^3 \xi \circ \eta = 0$ $(= \eta \circ \nu \circ E^4 \xi = \tau \circ \eta \circ E^4 \xi)$ we have the first sequence in the exceptional case.

§3. The set $\pi_0(S^{n X}, p)$

X may be regard as a *CW*-complex of a form $S^3 \cup e^n \cup e^{n+3}$, so we denote by A the subcomplex $S^3 \underset{\xi}{\cup} e^n$. Since S^3 can be considered as a fibre we have the fibring which is obtained from the restriction of maps $r : S^{n X} \rightarrow S^{n S^3}$ $(r(p) = *)$. On the other hand, using dim $X = n+3$ and the homotopy cellular approximation, we know that any map $X \rightarrow S^n \vee S^{n+3}$ can be uniquely determined up to homotopy by a pair of maps: $X \rightarrow S^n$ and $X \rightarrow S^{n+3}$. Let $q : X \rightarrow S^n \vee S^{n+3}$ be the map corresponding to the pair, $p : X \rightarrow S^n$ and the collapsing $X \rightarrow S^{n+3}$, and define the map $\Psi : S^{n S^n \vee S^{n+3}} \rightarrow S^{n X}$ by $\Psi(f) = f \circ q$ for a map $f : S^n \vee S^{n+3} \rightarrow S^n$. Then it is easy that Ψ gives rise a homeomorphism from the space $S^{n S^n \vee S^{n+3}}$ onto the fibre $r^{-1}(*)$ of the above fibring. Since it follows from the assumption $n \geq 6$ that $\pi_1(S^{n S^3}, *) = \pi_0(S^{n S^3}, *) = 0$ we have the bijection

$$[S^n, S^n] \times [S^{n+3}, S^n] = \pi_0(S^{n S^n \vee S^{n+3}}, (1, 0)) \underset{q^*}{\longrightarrow} \pi_0(S^{n X}, p) = [X, S^n].$$

Now we define a multiplication in the set $[S^n, S^n] \times [S^{n+3}, S^n]$ by the formula: $(m, \alpha) \triangledown (n, \beta) = (mn, m\beta + n\alpha)$. Clearly this multiplication gives an abelian semigroup structure with unit element $(1, 0)$. Especially we have

LEMMA 1. reg. $\{[S^n, S^n] \times S^{n+3}, S^n]\} \cong Z_2 \oplus \pi_{n+3}(S^n)$

Proof. Since $(m. \alpha) \triangledown (n, \beta) = (1, 0)$ implies $mn = 1$ and $m\beta + n\alpha = 0$ we have $m = \pm 1$, $n = \pm 1$ and $\alpha + \beta = 0$. Conversely $(\pm 1, \alpha)$ clealy regular for any α and the imbedding $\alpha \rightarrow (1, \alpha)$ is an injective homomorphism and then the decomposition is trivial. These show the proof.

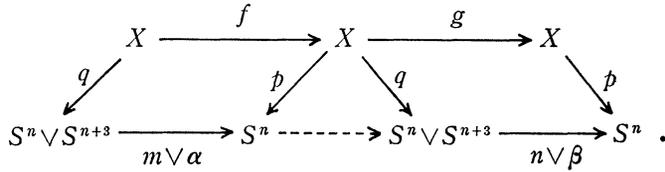Let f, g be maps $X \to X$. Then, by the above bijectivity, we have

$$p \circ f = (m \lor \alpha) \circ q \quad \text{and} \quad p \circ g = (n \lor \beta) \circ q$$

for some integers m, n and $\alpha, \beta (\in \pi_{n+3}(S^n))$.

LEMMA 2. $p(g \circ f) = (mn, (n\alpha + d_3(f)m\beta)) \circ q$

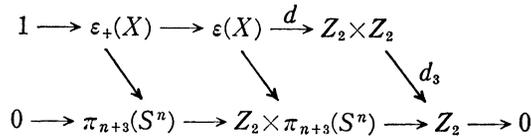*Proof.*   Consider the following diagram

Since we have

$$p(g \circ f) = (n \lor \beta) \circ q \circ f \quad \text{and} \quad q \circ f = \{(m \lor \alpha), (0, d_{n+3}(f))\} \circ q$$

the proof follows from

$$(n \lor \beta)\{(m \lor \alpha), (0, d_{n+3}(f))\} = (mn, (n\alpha + d_{n+3}(f)\beta))$$
$$= (mn, (n\alpha + d_3(f)m\beta)).$$

Now, from lemma 1 and 2, we can obtain the commutative diagram:

Let $\iota_A : A \to X$ and $i_3 : S^3 \to A \ (\subset X)$ be inclusion maps.

LEMMA 3.   *Let λ be the attaching map of the $(n+3)$-cell of X. For a map $h : A \to A$ of type $(-1, -1)$, i. e. $d(h) = (-1, -1)$ we have*

$$i_{A*}(h \circ \lambda) = (s+1)i_{3*}(\tau \circ E^3 \xi)$$

where s is an integer satisfying $2sE^4\xi = 0$.

Proof.   Using the commutative diagram:

$$\{[\bar{\xi},\ \iota_3]_r\} \oplus \bar{\xi}\circ\pi_{n+2}(D^n,\ S^{n-1})$$

$$\|$$

$$\begin{array}{ccccc}
\pi_{n+2}(S^3) & \longrightarrow & \pi_{n+2}(A) & \xrightarrow{\ j_*\ } & \pi_{n+2}(A,\ S^3) \\
\downarrow{-1x}\ {}^{i_{3*}} & & \downarrow{h_*} & & \downarrow{h_*} \\
\pi_{n+2}(S^3) & \longrightarrow & \pi_{n+2}(A) & \longrightarrow & \pi_{n+2}(A,\ S^3) \\
\downarrow{E}\ {}^{\iota_{3*}} & & \downarrow{E}\ {}^{j_*} & & \\
\pi_{n+4}(EA,\ S^4) \xrightarrow{\ \partial\ } \pi_{n+3}(S^4) & \xrightarrow{\ i_*\ } & \pi_{n+3}(EA) & &
\end{array}$$

$$\{[E\bar{\xi},\ \iota_4]_r\} \oplus E\bar{\xi}\circ\pi_{n+4}(D^{n+1},\ S^n)$$

and Hopf map $\nu_4 : S^7 \rightarrow S^4$, we have

$$h_*(\lambda) = \lambda + i_{3*}(\Gamma) \quad \text{for some element } \Gamma \text{ of } \pi_{n+2}(S^3)$$

and

$$E(h\circ(\lambda)) = (Eh)\circ(E\lambda) = (Eh)\circ(i_*(\nu_4\circ E^4\xi)) \quad \text{by (3.1) of [5]}$$

$$= i_*\{(-\iota_4)\circ\nu_4\circ E^4\xi\} = i_*(-\nu_4+[\iota_4,\ \iota_4])\circ E^4\xi$$

$$= i_*(-\nu_4+2\nu_4+E\tau)\circ E^4\xi = i_*(\nu_4\circ E^4\xi) + \iota_*(E\tau\circ E^4\xi) = E\lambda + \iota_* E(\tau\circ E^3\xi).$$

These show that $i_*(E\Gamma) = \iota_* E(\tau\circ E^3\xi)$, i. e. $E(\Gamma - \tau\circ E^3\xi)$ is contained in the ∂-image. Since the

$$\partial\text{-image} = E\xi\circ\pi_{n+3}(S^n) \cup \{[E\xi,\ \iota_4]\}$$

$$= E\xi\pi_{n+3}(S^n) \cup \{(2\nu_4+E\tau)\circ E^4\xi\}$$

we have, for some integer s and $\gamma \in \pi_{n+2}(S^{n-1})$,

$$E(\Gamma-\tau\circ E^3\xi) = E(\xi\circ\gamma) + s(2\nu_4\circ E^4\xi) + sE(\tau\circ E^3\xi).$$

On the other hand, we know the decomposition:

$$\pi_i(S^4) = E\pi_{i-1}(S^3)\oplus\nu_4\circ\pi_i(S^7).$$

Hence we have that $2sE^4\xi = 0$ and $\Gamma = (s+1)\tau\circ E^3\xi + \xi\circ\gamma$, i. e.

$$i_{A_*}(h_*(\lambda)) = (s+1)i_{3*}(\tau\circ E^3\xi).$$

Thus the proof is completed.

Next, let $k : A \rightarrow A$ be another map of type $(-1, -1)$ and let $Q : A \rightarrow A\vee S^n$ be a map collapsing the equator of the n-cell of A to a point. Then it is well-known that k can be represented as a composition:

$$A \xrightarrow{\ Q\ } A\vee S^n \xrightarrow{\ h\vee\sigma\ } X \qquad (\sigma\in\pi_n(X)).$$

By using $Q_*(\lambda) = \lambda + [\sigma,\ \iota_3]$, we can know that

$$i_{A_*}(k\circ\lambda) = (s+1)i_{3*}(\tau\circ E^3\xi) - [\sigma,\ \iota_3].$$

On the other hand, since S^3 acts on X we have $[\sigma, i_3]=0$ for all σ. Hence the proof of Theorem A follows from the above formula and lemma 3 if we note that the existence of a map of type $(1, -1)$ implies $2\xi=0$. Moreover if $2\xi=0$ there exists a bundle map $g: X_\xi \to X_{-\xi}$. Then the composition

$$
\begin{array}{ccccc}
X_\xi & \longrightarrow & X_{-\xi} & \longrightarrow & X_\xi \\
\downarrow p & g & \downarrow p & & \downarrow p \\
S^n & \longrightarrow & S^n & \longrightarrow & S^n \\
& id & & -id &
\end{array}
$$

is clearly of type $(1, -1)$.

Now we must determine the image $\varepsilon_+(X) \to \pi_{n+3}(S^n)$ in the preceeding diagram.

LEMMA 4. *In the following diagram*

$$
\begin{array}{ccc}
X & \cdots\cdots\cdots\cdots\cdots\cdots\to & X \\
\downarrow q & f & \downarrow p \\
S^n \vee S^{n+3} & \xrightarrow{\ id \vee \alpha\ } & S^n
\end{array}
$$

$(1 \vee \alpha) \circ q$ *has a lifting* f *if and only if there exists a map* $\psi: A \to S^3$ *satisfying* $\psi_*(\lambda) = \partial\alpha$ *where* A *is the complex* $S^3 \underset{\xi}{\bigcup} e^n$ *and* $X = A \underset{\lambda}{\bigcup} e^{n+3}$.

Proof. Consider the diagram

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
$$

$(i_A \cdot f = p \mid A)$

Let g be a lifting of $i_A \circ f$. Then there exists a map $\psi: A \to S^3$ satisfing

$$
g(x) = \mu(i_A(x), \psi(x)) \qquad (x \in A)
$$

where μ denote the action of S^3. Clearly the converse is also true, i. e. we have a one to one correspondence

$$
\{\text{liftings of } i_A \circ f\} \longleftrightarrow \{\psi: A \longrightarrow S^3\}.
$$

For a map $\psi: A \to S^3$ we take an extension $\bar{g}: X \to X$ of the map g defined as above. Off course $\bar{g}$ exists if and only if $g_*(\lambda)=0$. Since

$$g_*(\lambda)=\mu_*(\phi_*(\lambda)+\imath_{A*}(\lambda))=\mu_*(\phi_*(\lambda))+\mu_*\imath_{A*}(\lambda)$$

$$=i_*(\phi_*(\lambda))+i_{A*}(\lambda)=\imath_*(\phi_*(\lambda))$$

$g_*(\lambda)=0$ is equivalent to $\phi_*(\lambda)=\partial\sigma$ for some $\sigma\in\pi_{n+3}(S^n)$. Let $\bar\sigma$ be the element of $\pi_{n+3}(X,S^3)$ satisfying $p_*(\bar\sigma)=\sigma$, and define a map g by

$$g\cup\bar\sigma:\ X=A\cup D^{n+3}\longrightarrow X.$$

This is well defined. Then the above diagram shows $\sigma=\alpha$. Thus the proof is completed.

LEMMA 5. *The map* $f=(1\vee\alpha)\circ q:\ X\to S^n$ *has a lifting* $f:X\to X$ *with* $d_3(f)=1$ *and* $d_n(f)=1$ *if and only if* $\partial\alpha=0$, *i. e.* $\xi\circ E^{-1}\alpha=0$.

Proof. The former condition is equivalent by lemma 4 to the condition: there exists a map $\phi:A\to S^3$ with $d_3(\phi)=0$, and $\phi_*(\lambda)=\partial\alpha$ because we have

$$d_3(f)=d_3(\phi\circ\imath_A)=d_3(\phi)+d_3(i_A)=d_3(\phi)+1.$$

This implies that ϕ is represented as a composition $w\circ(p\,|\,A)$ for some $\omega:S^n\to S^3$. Then we have

$$\phi_*(\lambda)=\omega_*(p\,|\,A)_*(\lambda)=\omega_*(0)=0.$$

Thus the proof is completed.

From lemma 4 and lemma 5 we have

PROPOSITION 6. $\alpha\in\pi_{n+3}(S^n)$ *is contained in the image*

$$\varepsilon_+(X)\longrightarrow\pi_{n+3}(S^n)$$

if and only if $\xi\circ E^{-1}\alpha=0$.

§4. reg. $[X,S^3]$

LEMMA 7. *For* $f\in\mathrm{reg.}\,[X,S^3]$, *it holds that* $d_3(f)=0$ *or* -2 *if* $2\xi=0$ *and* $\tau E^3\xi=0$, $d_3(f)=0$ *otherwise.*

Proof. By definition there exists a map $g:X\to S^3$ such that

$$f(xg(x))g(x)=1=g(xf(x))f(x).$$

Restricting each map we have

$$p(q+1)+q=0=q(p+1)+q\qquad(p=d_3(f),\ q=d_3(g))$$

i. e. $(p+1)(q+1)=1$. This shows that $p=q=0$ or $p=q=-2$. If $p=-2$ the extendability of $f\,|\,S^3$ over A gives $2\xi=0$. And moreover the extendability over X gives $(-2\iota_4)J(\xi)=0$, i. e. $\tau\circ E^3\xi=0$. These complete the proof.

Let $f:X\to S^3$ be a map with $d_3(f)=0$. Then it is clear that there exist two

maps $f_1: S^n \to S^3$ and $f_2: S^{n+3} \to S^3$ such that

$$f=(f_1 \vee f_2) \circ q: X \xrightarrow{\quad q \quad} S^n \vee S^{n+3} \xrightarrow{\quad f_1 \vee f_2 \quad} S^3.$$

LEMMA 8. *Let* $g: X \to S^3$ *be another map with* $d_3(g)=0$. *Then we have* $f \sharp g=(f_1 \vee f_2) \circ q \sharp (g_1 \vee g_2) \circ q = \{(f_1+g_1) \vee (f_2+g_2)\} \circ q$, *i. e. this means that if* $d_3(f)=0$ *then* f *is belonging to* reg.$[X, S^3]$ *and*

$$q_*: \pi_n(S^3) \times \pi_{n+3}(S^3)=[S^n \vee S^{n+3}, S^3] \longrightarrow [X, S^3]$$

is an epimorphism onto $d_3^{-1}(0) \subset$ reg.$[X, S^3]$.

Proof. From the diagram:

we can obtain that $f(x g(x))=(f_1 \vee f_2)q(x)=f(x)$. Hence we have $f \sharp g=f(g \vee 1_x) \circ g =f \circ g=\{(f_1 \vee f_2) \circ q\} \{(g_1 \vee g_2) \circ q\}=\{(f_1 \vee f_2) \circ (g_1 \vee g_2)\} \circ q=\{(f_1+g_1) \vee (f_2+g_2)\} q$. Thus the proof is completed.

Now applying lemma 7 and 8 to Puppe sequence for the cofibring $S^3 \to X \to S^n \vee S^{n+3}$ we have an exact sequence, i.e.

PROPOSITION 9. $Z_2 \cong \pi_4(S^3) \xrightarrow[\partial]{} \pi_n(S^3) \times \pi_{n+3}(S^3) \longrightarrow$ reg.$[X, S^3] \xrightarrow[d_3]{} Z_2$

is exact and the following holds

(1) $\partial \eta=(\eta \circ E\xi, \eta \circ J(\xi))=(\eta \circ E\xi, \eta \circ \nu \circ E^4\xi)=(\eta \circ E\xi, \tau \circ \eta \vee E^4\xi)=(\eta \circ E\xi, \tau \circ E^3\xi \circ \eta)$

and

(2) d_3 *is onto if* $2\xi=0=\tau E^3\xi$ *and* $d_3=0$ *otherwise.*

Remark. Lemma 7 shows that $d_3(f)=0$ or -2 for $f \in$ reg.$[X, S^3]$. Let f be a map: $X \to S^3$ with $d_3(f)=-2$. Then $d_3(f \sharp f)=0$ is clear, so $f \sharp f$ is an element of reg.$[X, S^3]$ by lemma 7. This shows $f \in$ reg.$[X, S^3]$ i.e. $f \in$ reg.$[X, S^3]$ is equivalent to $d_3(f)=0$ or -2.
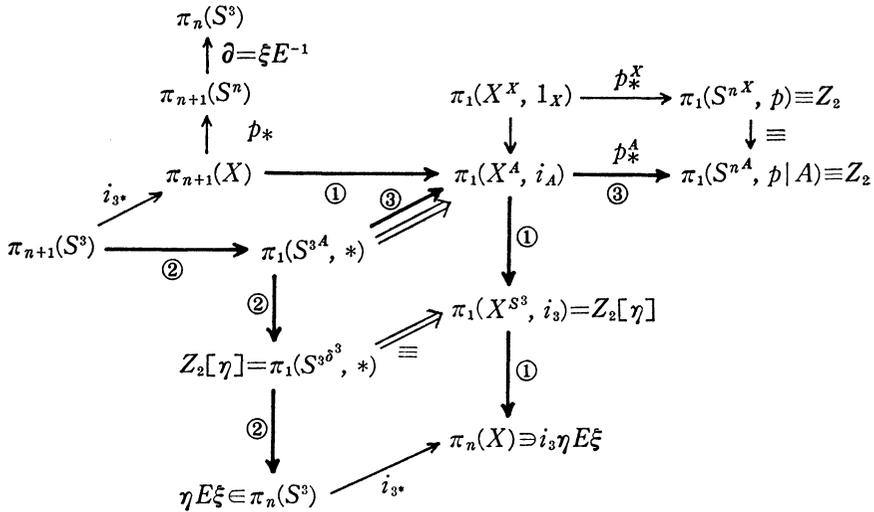
§5. The homomorphism P_*^X

LEMMA 10. $\pi_1(S^{nX}, p) \cong Z_2$.

Proof. Consider a part of the homotopy exact sequence associated with the fibring; $S^{nS^n \vee S^{n+3}} \to S^{nX} \to S^{nS^3}$,

$$\pi_2(S^{nS^3}, *) \longrightarrow \pi_1(S^{nS^n}, 1) \times \pi_1(S^{nS^{n+3}}, *) \longrightarrow \pi_1(S^{nX}, p) \longrightarrow \pi_1(S^{nS^3}, *)$$

$$\| \equiv \qquad\qquad \| \equiv \qquad\qquad \| \equiv$$

$$\pi_5(S^n)=0 \qquad\qquad \pi_{n+4}(S^n)=0 \qquad\qquad \pi_4(S^n)=0$$

Then the proof follows from $\pi_1(S^{nX}, p) \equiv \pi_1(S^{nS^n}, 1) \equiv Z_2$. For studying the homomorphism p_*^X we use the following diagram

where $\Rightarrow$ denotes the homomorphism induced by the map $S^{3Y} \to X^Y$ defined by $\hat{f}(y)=yf(y)$ for $Y=A$ or S^3, and three sequences with gothic arrows are exact. First we consider the case of $\xi \circ \eta \neq 0$. This means that $i_{3*}: \pi_{n+1}(S^3) \to \pi_{n+1}(X)$ is onto. Thus we have that

(1) if $\eta \circ E\xi \notin \{\xi\eta\}$ then p_*^A is zero because $\pi_{n+1}(X) \to \pi_1(X^A, i_A)$ is onto.

(2) if $\eta E\xi = 0$ then p_*^A is zero because $\pi_1(S^{3A}, *) \to \pi_1(X^A, i_A)$ is onto.

(3) if $\eta E\xi = \xi\eta$ then p_*^A is onto because there exist an element of $\pi_1(X^A, i_A)$ which is not contained in the image $\pi_1(S^{3A}, *) \to \pi_1(X^A, i_A)$.

Secondly we suppose $\xi\eta = 0$. This means that there exists an element $\eta_X \in \pi_{n+1}(X)$ such that $p_*(\eta_X) = \eta$. Then from the commutativity of the diagram

$$Z_2 \equiv \pi_{n+1}(S^n) \longrightarrow \pi_1(S^{nA}, p|A) \equiv Z_2$$

$$\uparrow p_* \qquad \cong \qquad \uparrow p_*^A$$

$$\eta_X \in \pi_{n+1}(X) \longrightarrow \pi_1(X^A, i_A) \ni \hat{\eta}_X$$

it follows that p_*^A is onto. Thus p_*^X is onto if $\hat{\eta}_X$ is contained in the image $\pi_1(X^X, 1_X) \to \pi_1(X^A, i_A)$.

LEMMA 11. *$\hat{\eta}_X$ is contained in the image $\pi_1(X^X, 1_X) \to \pi_1(X^A, i_A)$.*

Proof. Let Φ be a map: $S^1 \times A \to S^{n+1} \vee (S^1 \times A)$ which is obtained from pinching the equator of the $(n+1)$-cell of $S^1 \times A$, and let L be the subcomplex $S^1 \times A \cup * \times X$ of $S^1 \times X = L \cup_\omega e^{n+4}$. Our problem is to determine the extendability over $S^1 \times X$ of the map: $L \to X$ defined as follows:

$$L=(S^1 \times A) \cup * \times X \xrightarrow{\Phi \cup 1} (S^{n+1} \vee (S^1 \times A)) \cup * \times X = S^{n+1} \vee L \xrightarrow{\eta_X \vee \text{proj.}} X$$

with $\hat{\Phi}$ to $S^{n+1} \vee L'$

where L' denotes the complex $L/S^1 \times *$. Now we use the decomposition

$$\pi_{n+3}(S^{n+1} \vee L') \cong \pi_{n+3}(S^{n+1}) \oplus \pi_{n+3}(L') \oplus Z[\iota_{n+1}, i_3].$$

The first and second terms of $\Phi_*(\omega)$ are both zero because there exist a map: $EX \to EX/S^4 \cong S^{n+1} \vee S^{n+4} \xrightarrow{\eta_X \vee 0} X$ and the projection: $S^1 \times X \to X$ and, from the cohomology structure of $S^1 \times X$ it follows that the third term is Whitehead product $[\eta_X, i_3]$. Since $[\pi_i(X), i_3] = 0$ for all i. The proof is completed.

From (1), (2), (3) and lemma 11 we have

PROPOSITION 12. *If $\xi \circ \eta = 0$ then p_*^X is onto and if $\xi \circ \eta \neq 0$ p_*^X is zero except the case $\eta \circ E\xi = \xi \circ \eta$.*

Remark. In the exceotional case we know that p_*^A is onto. However we could not determine whether p_*^X is onto or not. Thus, for theorems, it suffices to define subgroups G_ξ and H_ξ. We define $\pi_{n+2}(S^{n-1}) \supset G_\xi = \xi_*^{-1}(0)$ if $2\xi \neq 0$ and $\xi_*^{-1}(\tau \circ E^3\xi)$ if $2\xi = 0$, and

$$\pi_n(S^3) \times \pi_{n+3}(S^3) \supset H_\xi = \{(xE\xi, xJ(\xi)), \ x \in \pi_4(S^3)\}$$

where $\xi_*: \pi_{n+2}(S^{n-1}) \to \pi_{n+2}(S^3)$ is induced by ξ.

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