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ON HOMOTOPY INVARIANCE OF THE SOLVABILITY OF NONLINEAR VARIATIONAL INEQUALITIES

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§1. Introduction

Let E be a Banach space and 2^{E^*} the space of subsets of the dual space E^* of E. Let A be an operator from E into 2^{E^*}. A is said to be monotone if

$$\langle y_1-y_2,\ x_1-x_2\rangle \geqq 0$$

for $y_i \in Ax_i (i=1, 2)$. Let H be a nonempty closed convex subset of E and p be an element of E^*. An element x_0 in E is said to be a solution of the variational inequality with respect to p if there exists $y_0 \in Ax_0$ such that

$$\langle y_0-p,\ x-x_0\rangle \geqq 0 \quad \text{for all} \quad x \in H. \tag{1.1}$$

The variational inequalities of the form (1.1) has been studied by many authors with applications to convex programming and a large class of free-boundary problems. The existence of solutions for the variational inequality (1.1) was investigated by Browder [2], Rockaffeler [7], Stampacchia [8], Takahashi [9] and others.

Our purpose in this paper is to consider invariance of the solvability of the variational inequality (1.1) under a homotopy of monotone operators. Recently Browder established a degree theory for a class of monotone type operators. In [4], he defined a homotopy of maximal monotone operators and proved homotopy invariance of the degree. In this paper we concern a homotopy of monotone operators in the sense of Browder. Our method is based on the method employed in [4] and [6].

§2. Perliminaries and statement of the main result.

Let E be a reflexive Banach space and C, K be nonempty closed convex subsets of E. Then we denote by $\partial_C K$ the set of $z \in K$ such that $U(z) \cap (C-K) \neq \phi$ for every nighborhood of $U(z)$ of z and by $i_C K$ the set of $z \in K$ such that $U(z) \cap (C-K) = \phi$ for some neighborhood $U(z)$ of z. We also denote by $cl(C)$ the closure of C. Let T be a mapping from E into 2^{E^*}. Then we denote by $G(T)$ the graph $G(T)=\{(y, x) \in E^* \times E : y \in Tx\}$ of T and by $R(T)$ the range of T, i.e.,

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$R(T)=\{y\in E^*: y\in Tx$ for some $x\in E\}$. Let A be a monotone operator from E into 2^{E^*}. A is said to be maximal monotone if its graph $G(A)$ is not properly contained in any other graph of monotone operator from E into 2^{E^*}. Let $f: E\to R\cup\{+\infty\}$ be a proper lower semicontinuous convex function. The subdifferential ∂f of f is the mapping defined by

$$\partial f(x)=\{x^*\in E^*: f(x)\leqq f(u)+\langle x^*, x-u\rangle, \text{ for all } u\in E\}.$$

It is well known that ∂f is maximal monotone. Let K be a nonempty closed convex subset of E. Then the indicator function $I_K: E\to R\cup\{+\infty\}$ is defined by

$$I_K(x)=\begin{cases}0 & \text{if } x\in K\\ +\infty & \text{if } x\notin K.\end{cases}$$

The indicator function I_K is a proper lower semicontinuous convex function and $x^*\in\partial I_K x$ if and only if

$$x\in K \text{ and } \langle x^*, x-u\rangle\geqq 0 \quad \text{for all } u\in K. \tag{2.1}$$

By using the subdifferential of the indicator function I_H, the variational inequality (1.1) can be rewritten as

$$p\in Ax_0+\partial I_H x_0. \tag{2.2}$$

Let A be a mapping from H into 2^{E^*}, where E^* is endowed with its weak topology. A is said to be upper semicontinuous from H into 2^{E^*}, if for each $x\in H$ and each neighborhood V of Ax, there exists a nighborbood U of x such that $Au\subset V$ for all $u\in U\cap H$. Suppose that A maps bounded sets of H into bounded sets of E^* and Ax is closed convex subset of E^* for each $x\in H$. Then A is upper semicontiuous if and only if the graph $G(A)$ of A is a closed subset of $E^*\times H$ (cf. [3]). Let $\{A(t): t\in[0, 1]\}$ be a family of monotone operators from H into 2^{E^*}. Then $\{A(t): t\in[0, 1]\}$ is said to be a pseudo-monotone homotopy of monotone operators from H into 2^{E^*} if $\{A(t): t\in[0, 1]\}$ satisfies the following condition (cf. Browder [4]):

(*) Let $\{t_i\}\subset[0, 1]$ be a sequence converging to t and $\{(z_i, x_i)\}\subset E^*\times H$ be a sequence such that $z_i\in A(t_i)x_i$ for each $i\geqq 1$, $x_i\to x$ weakly in E, and $z_i\to z$ weakly in E^*. Suppose that

$$\varlimsup_{i\to\infty}\langle z_i, x_i\rangle\leqq\langle z, x\rangle. \tag{2.3}$$

Then $z\in A(t)x$ and $\langle z_i, x_i\rangle\to\langle z, x\rangle$.

Remark. To add to (*), Browder [4] assumed that $0\in A(t)0$ and $A(t)$ is maximal monotone for each $t\in[0, 1]$.

From the definition of the homotopy $\{A(t): t\in[0, 1]\}$ of monotone operators, we see that for each $t\in[0, 1]$, $A(t)$ is upper-semitinuous from H into 2^{E^*}. Let

$\{A(t): t\in[0, 1]\}$ be a pseudo-monotone homotopy of monotone operators from H into E^*. Then the pseudo-monotone homotopy $\{A(t): t\in[0, 1]\}$ is said to be bounded if for each bounded subset G of H, the set $\cup\{A(t)(G): t\in[0, 1]\}$ is bounded.

The duality mapping J of E into 2^{E^*} is given by

$$J(x)=\{x^*\in E^*: \langle x^*, x\rangle=|x^*||x|=|x|^2\}$$

for each $x\in E$. It is well known that J is a maximal monotone operator from E into 2^{E^*}. It is also known that every reflexive Banach space E can be renormed so that E and E^* are both locally uniformly convex (cf. Diestel [5]). Let J be the duality mapping corresponding to the locally uniformly convex norm of E. Then J has the following property (see proposition 8 of [4]);

(2.4) if $x_i\to x$ weakly in E and $\lim\langle J(x_i), x_i-x\rangle\leqq 0$, then $x_i\to x$ strongly in E and $J(x_i)\to J(x)$ weakly in E^*.

We now state our main result.

THEOREM 1. *Let H be a closed convex subset of a reflexive Banach space E and C be a bounded closed convex subset of H with $i_H C\neq\phi$. Let $\{A(t): t\in[0, 1]\}$ be a bounded pseudo-monotone homotopy of monotone operators from H into 2^{E^*} and $p\in E^*$. Suppose that $p\in(A(0)+\partial I_H)(i_H C)$ and $p\notin cl(\cup\{(A(t)+\partial I_H)(\partial_H C): t\in[0, 1]\})$. Then $p\in(A(t)+\partial I_H)(C)$, for all $t\in[0, 1]$.*

As a direct consequence of Theorem 1, we have the following result which is due to Browder [4] in case when $0\in A(t)0$ for all $t\in[0, 1]$.

COROLLARY. *Let G be a bounded convex and open subset of a reflexive Banach space E and $\{A(t): t\in[0, 1]\}$ be a bounded pseudo-monotone homotopy of maximal monotone operators from E into 2^{E^*}. Let $p\in E^*$. Suppose that $p\in A(0)(G)$ and $p\notin cl(\cup\{A(t)(\partial G): t\in[0, 1]\}$. Then $p\in A(t)(G)$ for all $t\in[0, 1]$.*

§3. Proofs.

In this section, we first state a necessary and sufficient condition for the variational inequality (1.1) to have a solution in H.

THEOREM 2(cf. [6]). *Let H be a closed convex subset of a reflexive Banach space E and A be a monotone and upper-semicontinuous mapping from H into 2^{E^*}, where E^* is endowed with its weak topology. Then the following conditions are equivalent:*

(i) *There exist $x_0\in H$ and $y_0\in Ax_0$ such that*

$$\langle y_0, x-x_0\rangle\geqq 0, \qquad \text{for all} \quad x\in H;$$

(ii) *there exists a bounded closed convex subset K of H such that for each $z\in\partial_H K$ and $w\in Az$, there exists $x\in i_H K$ which satisfies that $\langle w, x-z\rangle\leqq 0$.*

Remark. Sufficient conditions for the existence of solutions of (1.1) were studied by several authors (cf. Browder [2], Stampacchia [8]). Theorem 2 is a version of Theorem 1 of [6] for multivalued monotone operators. The proof of Theorem 1 of [6] is still valid for Theorem 2. Then we omit the proof.

Throughout the rest of this section, we suppose that E, H, C, and $\{A(t): t\in[0, 1]\}$ satisfy the assumption in Theorem 1. For each operator $A: E\rightarrow 2^{E^*}$ and each $\lambda>0$, we denote by A_λ the operator from E into 2^{E^*} given by $A_\lambda=A+(1/\lambda)J$, where J is the duality mapping from E into E^* corresponding to a norm on E in which E and E^* are locally uniformly convex. In the followings, we suppose that $A: H\rightarrow 2^{E^*}$ is a monotone operator satisfying the following condition:

(*)′ Let $\{(z_i, x_i)\}\subset G(A)$ be a sequence such that $x_i\rightarrow x$ weakly in E, $z_i\rightarrow z$ weakly in E^* and

$$\overline{\lim_{i\to\infty}}\langle z_i, x_i\rangle\leqq\langle z, x\rangle .$$

Then $z\in Ax$ and $\langle x_i, z_i\rangle\rightarrow\langle x, z\rangle$.

Remark. The condition (*)′ is the case when $A(t)=A$ in (*). It is obvious from the condition (*) that each $A(t)$ satisfies the condition (*)′.

LEMMA 1. *Let $\lambda>0$. Let $\{t_i\}\subset[0, 1]$ be a sequence converging to t_0 and $\{(y_i, x_i)\}\subset E^*\times E$ be a sequence such that $y_i\in A(t_i)_\lambda x_i$ for each $i\geqq 1$, $x_i\rightarrow x$ weakly in E and $y_i\rightarrow y$ weakly in E^*. Suppose further that*

$$\overline{\lim_{i\to\infty}}\langle y_i, x_i\rangle\leqq\langle y, x\rangle .$$

Then $x_i\rightarrow x$ strongly in E, $\langle y_i, x_i\rangle\rightarrow\langle y, x\rangle$ and $y\in A_\lambda x$.

Proof. Let $\{z_i\}\subset E^*$ be a sequence such that $z_i\in A(t_i)x_i$ and $y_i=z_i+(1/\lambda)Jx_i$ for each $i\geqq 1$. Since $\{z_i\}$ is bounded, we may suppose that $z_i\rightarrow z$ weakly in E^*. Then from the assumption, we have that

$$\overline{\lim_{i\to\infty}}\langle z_i+(1/\lambda)Jx_i, x_i\rangle=\overline{\lim}\langle y_i, x_i\rangle\leqq\langle y, x\rangle=\langle z, x\rangle+\overline{\lim_{i\to\infty}}\langle(1/\lambda)Jx_i, x\rangle . \tag{3.1}$$

Then since $\overline{\lim}\langle Jx_i, x_i\rangle=\overline{\lim}|x_i|^2\geqq|x|^2$, the inequality (3.1) implies that $\overline{\lim}\langle z_i, x_i\rangle\leqq\langle z, x\rangle$. Then from the condition (*) we have that $z\in A(t_0)x$ and $\langle z_i, x_i\rangle\rightarrow\langle z, x\rangle$. Then again by (3.1), it follows that $|x_i|^2\rightarrow|x|^2$, or equivalently $\overline{\lim}\langle Jx_i, x_i-x\rangle\leqq 0$. Then by (2.4), we have that $x_i\rightarrow x$ strongly in E. This completes the proof.

LEMMA 2. *Let $\lambda>0$ and $p\in E^*$. Suppose that $p\notin cl((A_\lambda+\partial I_H)(\partial_H C))$. Then the following conditions are equivalent;*

(a) $p\in(A_\lambda+\partial I_H)(i_H C)$;

(b) *there exists* $\delta>0$ *such that*

$$\inf_{y\in C}\langle z-p,\ y-x\rangle<-\delta,\qquad \text{for all}\quad x\in\partial_H C\quad\text{and}\quad z\in A_\lambda x. \tag{3.2}$$

Proof. Suppose that (b) holds. Then it is easy to see that for each $x\in\partial_H C$ and $z\in A_\lambda x$, there exists $y\in i_H C$ such that $\langle z-p,\ y-x\rangle\leqq 0$. Then by Theorem 2, we have that (a) holds. We next suppese that (a) holds. For the sake of simplicity of the proof, we assume that $p=0$, $0\in i_H C$ and $0\in A_\lambda 0+\partial I_H 0$. We first show that

$$\inf_{y\in C}\langle z,\ y-x\rangle<0\qquad\text{for all}\quad x\in\partial_H C\quad\text{and}\quad z\in A_\lambda x. \tag{3.3}$$

Suppose that (3.3) is false, i.e., there exist $x_0\in\partial_H C$ and $z_0\in A_\lambda x_0$ such that

$$\langle z_0,\ y-x_0\rangle\geqq 0\qquad\text{for all}\quad y\in C.$$

Then since $0\in C$, we have that $\langle z_0,\ x_0\rangle\leqq 0$. Suppose that $\langle z_0,\ x_0\rangle=0$. Let y be an arbitrary element of H. Then since $0\in i_H C$, there exists $t>0$ such that $ty\in C$. Then we have that

$$\langle z_0,\ y\rangle=(1/t)\langle z_0,\ ty\rangle\geqq(1/t)\langle z_0,\ x_0\rangle=\langle z_0,\ x_0\rangle=0.$$

Hence from the observation above, we obtain that $\langle z_0,\ y-x_0\rangle\geqq 0$ for all $y\in H$, i.e., $0\in(A_\lambda+\partial I_H)(\partial_H C)$. This contradicts the assumption. Therefore we find that $\langle z_0,\ x_0\rangle<0$. On the other hand, we have, from the assumption, that there exists $w\in A_\lambda 0$ such that $\langle w,\ y\rangle\geqq 0$ for all $y\in C$. Then from the monotonicity of A_λ and the observation above, we find

$$0\leqq\langle z_0-w,\ x_0\rangle=\langle z_0,\ x_0\rangle-\langle w,\ x_0\rangle<0.$$

This is a contradiction. Thus we obtain that (3.3) holds. We now show that (3.2) holds for some $\delta>0$. Suppose that (3.2) does not hold for any $\delta>0$. Then there exist sequences $\{x_i\}\subset\partial_H C$ and $\{z_i\}\subset E^*$ such that $z_i\in A_\lambda x_i$ for each $i\geqq 1$ and

$$\lim_{i\to\infty}\inf_{y\in C}\langle z_i,\ y-x_i\rangle\geqq 0. \tag{3.4}$$

We may suppose that $x_i\to x_0$ weakly in E and $z_i\to z_0$ weakly in E^*. By putting $y=x_0$ in (3.4), we have that $\overline{\lim}\,\langle z_i,\ x_0-x_i\rangle\geqq 0$. Then by applying Lemma 1 in case when $A(t)=A$ for $t\in[0,1]$, we have that $x_i\to x_0$ strongly in E and $z_0\in A_\lambda x_0$. Then (3.4) implies that $\inf_{y\in C}\langle z_0,\ y-x_0\rangle\geqq 0$. While we have by (3.3) that $\inf_{y\in C}\langle z_0,\ y-x_0\rangle<0$. This is a contradiction. Thus we have shown that (3.2) holds.

LEMMA 3. *Let* $p\in E^*$ *and* $n_0\geqq 1$. *Let* $\{p_n\in E^*:\ n\geqq n_0\}$ *be a sequence such that* $\lim_{n\to\infty}p_n=p$ *and* $p_n\in(A_n+\partial I_H)(C)$ *for each* $n\geqq n_0$. *Then* $p\in(A+\partial I_H)(C)$.

Proof. From the assumption, we have that for each $n \geqq n_0$, there exists $(z_n, x_n) \subset G(A)$ such that

$$\left\langle z_n + \frac{1}{n} J x_n - p_n,\ y - x_n \right\rangle \geqq 0 \quad \text{for all} \quad y \in H. \tag{3.5}$$

Since $\{x_n\}$ and $\{z_n\}$ are bounded, we may assume that $x_n \to x_0$ weakly in E and $z_n \to z_0$ weakly in E^*. Then since $p_n \to p$ strongly in E and $(1/n)Jx_n \to 0$ strongly in E^*, (3.5) implies that $\overline{\lim} \langle z_n, x_n \rangle \leqq \langle z_0, x_0 \rangle$. Then from the definition of A, it follows that $\langle z_n, x_n \rangle \to \langle z_0, x_0 \rangle$ and $z_0 \in A x_0$. Then again by (3.5), we have that

$$\langle z_0 - p,\ y - x_0 \rangle \geqq 0 \quad \text{for all} \quad y \in H,$$

i.e., $p \in (A + \partial I_H)(C)$. This completes the proof.

Proof of Theorem 1. By Lemma 3, it is sufficient to show that there exists a positive integer n_0 and a sequence $\{p_n\}_{n \geq n_0} \subset E^*$ such that $\lim_{n \to \infty} p_n = p$ and for each $n \geqq n_0$, $p_n \in (A(t)_n + \partial I_H)(C)$ for all $t \in [0, 1]$. From the assumption, we have that there exists $r > 0$ such that $B(p, r) \cap (A(t) + \partial I_H)(\partial_H C) = \phi$, for all $t \in [0, 1]$, where $B(p, r) \subset E^*$ denotes the open ball about p with radius r. Hence we choose $n_0 \geqq 1$ such that $(1/n_0) J(\partial_H C) \subset B(0, r/2)$. Then we have that $B(p, r/2) \cap (A(t)_n + \partial I_H)(\partial_H C) = \phi$, for all $n \geqq n_0$ and $t \in [0, 1]$. Let x_0 be an element of C such that $p \in (A(0) + \partial I_H) x_0$. We put $p_n = p + (1/n) J(x_0)$, for each $n \geqq n_0$. Then $p_n \in (A(0)_n + \partial I_H)(C)$ for each $n \geqq n_0$. Also we have that for each $n \geqq n_0$, $p_n \notin ((A(t)_n + \partial I_H)(\partial_H C))$ for all $t \in [0, 1]$. We now fix $n \geqq n_0$ and show that $p_n \in (A(t)_n + \partial I_H)(C)$ for all $t \in [0, 1]$. Put $t_0 = \sup \{t \in [0, 1] : p_n \in (A(t)_n + \partial I_H)(C)\}$. Then there exist a sequence $\{t_i\} \subset [0, t_0]$ and a sequence $\{(x_i, z_i)\} \subset E \times E^*$ such that $\lim_{i \to \infty} t_i = t_0$, $z_i \in A(t_i)_n x_i$ for each $i \geqq 1$ and

$$p_n \in A(t_i)_n x_i + \partial I_H x_i \quad \text{for all} \quad i \geqq 1. \tag{3.6}$$

The equation (3.6) can be rewritten as

$$\text{for each } i \geqq 0, \quad \langle z_i - p_n,\ y - x_i \rangle \geqq 0 \quad \text{for all } y \in H. \tag{3.7}$$

Since $\{x_i\}$ and $\{z_i\}$ are bounded, we may assume without any loss of generality that $x_i \to x_0$ weakly in E and $z_i \to z_0$ weakly in E^*. Then from the definition of pseudo-monotone homotopy, we find that $z_0 \in A(t_0)_n x_0$ and $\langle z_i, x_i \rangle \to \langle z_0, x_0 \rangle$. Then again by (3.7), we obtain that $\langle z_0 - p_n, y - x_0 \rangle \geqq 0$ for all $y \in H$, i.e., $p_n \in A(t_0)_n x_0 + \partial I_H x_0$. Thus we have that $p_n \in (A(t_0)_n + \partial I_H)(C)$. Hence we claim that $t_0 = 1$. Suppose that $t_0 < 1$. Since $p_n \in (A(t_0)_n + \partial I_H)(C)$, we have by Lemma 2 that there exists $\delta > 0$ such that

$$\inf_{y \in C} \langle z - p_n,\ y - x \rangle < -\delta \tag{3.8}$$

for all $x \in \partial_H C$ and $z \in A(t_0)_n x$. Then we show that there exists $t \in (t_0, 1]$ such that for some $\delta' > 0$, (3.9) $$\inf_{y\in C}\langle z-p_n, y-x\rangle < -\delta', \qquad \text{for } x\in\partial_H C \text{ and } z\in A(t)_n x.$$

Suppose that (3.9) does not hold for any $\delta'>0$. Then there exist a sequence $\{t_i\}\subset(t_0, 1]$ converging to t_0 and a sequence $\{(x_i, z_i)\}\subset H\times E^*$ such that $x_i\in\partial_H C$, $z_i\in A(t_i)_n x_i$ for each $i\geqq 1$ and

(3.10) $$\lim_{i\to\infty}\inf_{y\in C}\langle z_i-p_n, y-x_i\rangle\geqq 0.$$

Since $\{x_i\}$ and $\{z_i\}$ are bounded, we may assume that $x_i\to x_0$ weakly in E and $z_i\to z_0$ weakly in E^*. Then by putting $y=x_0$ in (3.10), it follows that $z_0\in A(t_0)_n x_0$, $\langle z_i, x_i\rangle\to\langle z_0, x_0\rangle$ and $x_i\to x_0$ strongly in E. Then $x_0\in\partial_H C$. Also we have by (3.10) that

(3.11) $$\inf_{y\in C}\langle z_0-p_n, y-x_0\rangle\geqq 0.$$

Since $x_0\in\partial_H C$, this contradicts (3.8). Thus we obtain that there exists $t\in(t_0, 1]$ such that (3.9) holds for some $\delta'>0$. Then by Lemma 2, we have that $p_n\in(A(t)_n+\partial I_H)(C)$. This contradicts the definition of t_0. Thus we have shown that $t_0=1$, i.e., $p_n\in(A(1)_n+\partial I_H)(C)$.

Let $s\in(0, 1)$. We put $A^s(t)=A(st)$ for each $t\in[0, 1]$. Then $\{A^s(t): t\in[0, 1]\}$ is also a pseudo-monotone homotopy of monotone operators. It is easy to see that our argument above is still valid for $\{A(t): t\in[0, 1]\}$ replaced by $\{A^s(t): t\in[0, 1]\}$. Thus we have that $p_n\in(A^s(1)_n+\partial I_H)(C)$ for each $s\in(0, 1)$. This implies that $p_n\in(A(t)_n+\partial I_H)(C)$ for all $t\in[0, 1]$. This completes the proof.

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