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AN EXTREMAL PROBLEM ASSOCIATED WITH THE SPREAD RELATION II

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Introduction. One of the authors published in 1982 a paper with the same title [6], in which the following result was proved:

THEOREM A. *Let $f(z)$ be meromorphic in the plane of order $\rho\in(0, \infty)$. Further, suppose that $T(r, f)$ varies regularly in the sense of Karamata, i.e.,*

$$(1)\qquad \lim_{r\to\infty}\frac{T(kr, f)}{T(r, f)}=k^{\rho} \qquad (0<k<\infty)$$

holds uniformly for k in any interval $A^{-1}\leqq k\leqq A$, $A>1$. Let $\Lambda(r)$ be a nonnegative function satisfying $\Lambda(r)=o(T(r, f))$ $(r\to\infty)$. Then, if

$$(2)\qquad \delta(\infty, f)>0$$

and

$$(3)\qquad \limsup_{r\to\infty}\ \text{meas}\{\theta\,;\log|f(re^{i\theta})|>\Lambda(r)\}\leqq\frac{4}{\rho}\sin^{-1}\sqrt{\frac{\delta(\infty, f)}{2}}\equiv 2\beta<2\pi\,,$$

there exist a very long set G and a function $L(r)$ varying slowly on $(0, \infty)$ such that $T(r, f)=r^{\rho}L(r)$ $(0<r<\infty)$ and

$$(4)\qquad \frac{T^*(re^{i\theta}, f)}{T(r, f)}\to\cos\rho(\beta-\theta) \qquad (r\to\infty,\ r\in G\,;\ \text{uniformly for } \theta\in[0, \beta]),$$

where $T^*(z)$ denotes the Baernstein characteristic of $f(z)$.

In the present paper we first discuss an improvement of Theorem A. The assumption (1) can be rewritten as $T(r, f)=r^{\rho}L_1(r)$ $(0<r<\infty)$ with a function $L_1(r)$ varying slowly on $(0, \infty)$. Baernstein [3] proved that all the assumptions of Theorem A except (1) imply the existence of a very long set G and a function $L(r)$ varying slowly on G such that $T(r, f)=r^{\rho}L(r)$ $(0<r<\infty)$ and

$$(5)\qquad \frac{N(r, \infty, f)}{T(r, f)}\to\cos\beta\rho \qquad (r\to\infty,\ r\in G)\,.$$

We prove that in Theorem A the assumption (1) is unnecessary. (Therefore the

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above conclusion (5) is contained in (4).) Next, we determine under the assumptions (2) and (3) the asymptotic behavior of $\log|f(z)|$ for values of z whose absolute values lie in a suitable set G_0 having logarithmic density 1 and of the arguments of almost all the zeros and the poles of $f(z)$ in $\{z\,;\,|z|\in G_0\}$. The clue to our investigation is the following fact:

A PRINCIPAL LEMMA. *Let $\rho\in(0,\infty)$ and $L(r)$ be given, where $L(r)$ is a slowly varying function on a very long set G_1 such that $H(r)\equiv r^{\rho}L(r)\neq O(\log r)$ $(r\to\infty)$ is a convex, increasing function of $\log r$. Then there exists a very long set G_2 $(\subset G_1)$ with the property that any increasing unbounded sequence in G_2 is a sequence of Pólya peaks of order ρ of $H(r)$.*

Let $f(z)$ satisfy the assumptions (2) and (3). Then the spread relation [2] implies that $f(z)$ has regular growth and Theorem 2 in [3] yields that $T(r,f)=r^{\rho}L(r)$ with a very long set G_1 and a function $L(r)$ varying slowly on G_1. Using our principal lemma, we find a very long set G_2 $(\subset G_1)$ with the property that any increasing unbounded sequence $\{r_m\}\subset G_2$ is a sequence of Pólya peaks of order ρ of $T(r,f)$. Here we use the spread relation again to obtain

$$\lim_{m\to\infty}\operatorname{meas}\{\theta\,;\,\log|f(r_m e^{i\theta})|>\Lambda(r_m)\}=2\beta\,.$$

Hence $f(z)$ is a meromorphic function of lower order $\rho\in(0,\infty)$ satisfying the hypotheses ES at a sequence $\{r_m\}$ of Pólya peaks of order ρ of $T(r,f)$ (For the definition of the hypotheses ES, see [4, p. 69.]), and so we conclude that all the results in [4] and [5] are applicable to our $f(z)$ with any increasing unbounded sequence $\{r_m\}\subset G_2$.

1. Statement of our theorems.

THEOREM 1. *Suppose $f(z)$ is meromorphic in the plane of order $\rho\in(0,\infty)$. Let $\Lambda(r)$ be a nonnegative function satisfying $\Lambda(r)=o(T(r,f))$ $(r\to\infty)$. Then, if the assumptions (2) and (3) are fulfilled, there exist two very long sets G_1, G_2 $(G_2\subset G_1)$ and a function $L(r)$ varying slowly on G_1 such that $T(r,f)=r^{\rho}L(r)$ $(0<r<\infty)$,*

$$\frac{T^*(re^{i\theta})}{T(r,f)}\to\cos\rho(\beta-\theta)\quad (r\to\infty,\ r\in G_2\,;\ \textit{uniformly for } \theta\in[0,\beta])\,,$$

and

$$\frac{n(r,\infty,f)}{T(r,f)}\to\rho\cos\beta\rho\qquad (r\to\infty,\ r\in G_2)\,.$$

THEOREM 2. *Let the assumptions and notations of Theorem 1 be unchanged. It is then possible to find two sets G_3, J_1 on the positive real axis and a real-valued function $\varphi(u)$ satisfying the following conditions* (i)-(vi).

(i) *G_3 $(\subset G_2)$ is a very long set.*

(ii) *J_1 has density zero.*
(iii) $\log M(u, f)=\{\pi\rho\sin\beta\rho+o(1)\}u^{\rho}L(u)$ $(u\to\infty,\ u\in G_3-J_1)$.
(iv) $|f(ue^{i\varphi(u)})|=M(u, f)$ $(u\in G_3-J_1)$.
(v) $\lim_{\substack{u\to\infty\\ ku,u\in G_3-J_1}}[\varphi(ku)-\varphi(u)]=0$ $(0<k<\infty)$ *holds uniformly for k in any interval* $A^{-1}\leqq k\leqq A,\ A>1$.
(vi) *Let $s>0$ and η $(0<\eta<\beta)$ be given. Denote by $p(u)=p(u;s,\eta,\varphi(u))$ the number of poles of $f(z)$ in the sector*

$$\{te^{i\theta};e^{-s}u\leqq t\leqq e^{s}u,\ \eta\leqq|\theta-\varphi(u)|\leqq\pi\}$$

and by $z(u)=z(u;s,\eta,\varphi(u))$ the number of zeros of $f(z)$ in the sector

$$\{te^{i\theta};e^{-s}u\leqq t\leqq e^{s}u,\ |\theta-\varphi(u)|\leqq\beta-\eta\}.$$

Then

$$p(u)+z(u)=o(u^{\rho}L(u))\qquad(u\to\infty,\ u\in G_3-J_1).$$

THEOREM 3. *Let the assumptions and notations of Theorems* 1 *and* 2 *be unchanged. It is then possible to find two sets G_4 and J_2 on the positive real axis satisfying the following properties* (i)-(iii).
(i) *G_4 $(\subset G_3)$ is a very long set.*
(ii) *J_2 has density zero.*
(iii) *Let η $(0<\eta<(1/2)\min(\beta,\pi-\beta))$ be given. If $u\in G_4-J_1-J_2$, then*

$$\log|f(ue^{i(\theta+\varphi(u))})|\leqq o(T(u, f))$$
$$(u\to\infty;\ \text{uniformly for } \theta,\ \beta+\eta\leqq|\theta|\leqq\pi),$$

and

$$|\log|f(ue^{i(\theta+\varphi(u))})|-\pi\rho\sin\rho(\beta-|\theta|)T(u, f)|=o(T(u, f))$$
$$(u\to\infty;\ \text{uniformly for } \theta,\ \eta\leqq|\theta|\leqq\beta-\eta).$$

THEOREM 4. *Let the assumptions and notations of Theorems* 1 *and* 2 *be unchanged, and assume that $\pi-\beta>\pi/2\rho$. Then it is possible to find four sets G_5, G_6, J_3 and J_4 on the positive real axis satisfying the following properties* (i)-(iv).
(i) *G_5 and G_6 $(G_6\subset G_5\subset G_2)$ are very long sets.*
(ii) *J_3 and J_4 have density zero.*
(iii) *Let η $(0<\eta<(1/2)\min(\beta,\pi-\beta))$ and $s>0$ be given. Denote by $\tilde{z}(u)=\tilde{z}(u;s,\eta,\varphi(u))$ the number of zeros of $f(z)$ in the sector*

$$\{te^{i\theta};e^{-s}u\leqq t\leqq e^{s}u,\ \beta+\eta\leqq|\theta-\varphi(u)|\leqq\pi\}.$$

Then

$$\tilde{z}(u)=o(T(u, f))\qquad(u\to\infty,\ u\in G_5-J_3).$$

(iv) *Let η $(0<\eta<(1/2)\min(\beta,\pi-\beta))$ be given. Then, if $u\in G_6-J_3-J_4$,*

$$\log|f(ue^{i(\theta+\varphi(u))})|\geqq -o(T(u, f))$$
$$(u\to\infty\,;\ \text{uniformly for } \theta,\ \beta+\eta\leqq|\theta|\leqq\pi).$$

THEOREM 5. *Let the assumptions and notations of Theorem 4 be unchanged. Then*

(i) $\dfrac{n(r, 0, f)}{T(r, f)}\to\rho \quad (r\to\infty,\ r\in G_6-J_3-J_4)$.

(ii) *Given* η $(0<\eta<\beta)$, *denote by* $z^+(z)$ *the number of zeros of* $f(z)$ *in the sector*

$$0<|z|\leqq u\,,\quad \varphi(u)+\beta-\eta\leqq\arg z\leqq\varphi(u)+\beta+\eta$$

and denote by $z^-(u)$ *the number of zeros of* $f(z)$ *in the sector*

$$0<|z|\leqq u\,,\quad \varphi(u)-\beta-\eta\leqq\arg z\leqq\varphi(u)-\beta+\eta\,.$$

Then we have for $t\in(0, \sin\eta)$

$$\frac{z^+(u(1+t))-z^+(u(1-t))}{T(u, f)}\to\frac{\rho}{2}\{(1+t)^\rho-(1-t)^\rho\}$$
$$(u\to\infty\,;\ u(1+t),\ u(1-t),\ u\in G_6-J_3-J_4)\,.$$

This relation still holds if z^+ *is replaced by* z^-.

In §§ 2-4 we assume and use the assertion of our principal lemma.

In § 2 we deduce Theorem 1 from Theorems 1 and 2 in [4].

In § 3 we prove Theorem 2 using Theorem 1 and the same reasoning as in the proof of Theorem 3 in [4].

In § 4 we deduce Theorem 3 from Theorem 2 (v) and the first part of Theorem 4 in [5].

Finally, in § 5 we prove our principal lemma.

Remarks. (i) If the hypotheses of Theorem 3 are satisfied and $\pi-\beta\leqq\pi/2\rho$ holds, then the conclusions (iii) and (iv) of Theorem 4 need not be true. (See [5, p. 144].)
(ii) Theorem 4 is proved using Theorem 5 in [5] and the similar arguments as in §§ 2-3, so we omit the proof.
(iii) Theorem 5 is derived from Theorems 1, 3, 4 and the same reasoning as in § 13 in [5], so we omit the proof.

2. Proof of Theorem 1.

Let $f(z)$ satisfy the assumptions (2) and (3). Then as we saw in the introduction $T(r, f)=r^\rho L(r)$ holds with a very long set G_1 and a function $L(r)$ varying slowly on G_1, and further Theorems 1 and 2 in [4] are applicable to $f(z)$ with any increasing unbounded sequence $\{r_m\}\subset G_2$, where G_2 $(\subset G_1)$ is a suitable very long set. Hence we deduce

$$\frac{T^*(re^{i\theta}, f)}{T(r, f)} \to \cos \rho(\beta-\theta) \qquad (r\to\infty,\ r\in\tilde{G}_2\,;\ \text{uniformly for } \theta\in[0, \beta]),$$

and

$$\frac{n(r, \infty, f)}{T(r, f)} \to \rho \cos \beta\rho \qquad (r\to\infty,\ r\in\tilde{G}_2),$$

where $\tilde{G}_2=\bigcup_{m=1}^{\infty}[e^{-1}r_m, er_m]$. A suitable choice of $\{r_m\}\subset G_2$ implies that $G_2\subset\tilde{G}_2$. This completes the proof of Theorem 1.

3. Proof of Theorem 2.

3.1. Construction of G_3 and J_1. We write $G_2=\bigcup_{j=1}^{\infty}[a_j', b_j']$ $(b_j'<a_{j+1}'\,;$ $a_j'\to\infty$, $b_j'\to\infty$, $b_j'/a_j'\to\infty$ as $j\to\infty$). Dropping, if necessary, a finite number of intervals and renumbering the remaining intervals $[a_j', b_j']$ we may assume that $b_j'/a_j'\geqq e$ $(j=1, 2, 3, \cdots)$. Let $\{\varepsilon_n\}_1^{\infty}\downarrow 0$ be given. Choose $\sigma_n\in(0, 1)$ $(n=1, 2, 3, \cdots)$ small enough to imply

$$\rho \cos \beta\rho\Big(8e^{2\gamma+1}+2\gamma \log\frac{1}{\sigma_n}\Big)(e^{2\sigma_n\rho}-e^{-\sigma_n\rho})<\varepsilon_n/5 \qquad (\gamma\equiv\pi/\beta). \tag{3.1}$$

For each σ_n $(n=1, 2, 3, \cdots)$, define a sequence $\{m_j\}_0^{\infty}$ $(0=m_0<m_1<m_2<\cdots\,;\ m_j=m_j(\sigma_n))$ of integers such that $b_j'/a_j'=e^{(m_j-m_{j-1}+\alpha_j)\sigma_n}$ $(\alpha_j=\alpha_j(\sigma_n)\in[0, 1))$, and then $\{(r_n)_m\}_{m=1}^{\infty}$ by

$$\begin{aligned} &(r_n)_{m_{j-1}+1}=a_j' && (j=1, 2, 3, \cdots), \\ &(r_n)_{m+1}/(r_n)_m=e^{\sigma_n} && (m_{j-1}+1\leqq m\leqq m_j-1\,;\ j=1, 2, \cdots). \end{aligned} \tag{3.2}$$

If $f(z)$ satisfies the assumptions (2) and (3), Theorem 1 is valid. Hence we are able to find sequences $\{(R_n)_m'\}_{m=1}^{\infty}$, $\{(R_n)_m''\}_{m=1}^{\infty}$ $(n=1, 2, 3, \cdots)$ such that

$$\{1+(\xi_n)_m\}^{-1}\{t/(r_n)_m\}^{\rho}<T(t)/T((r_n)_m)<\{1+(\xi_n)_m\}\{t/(r_n)_m\}^{\rho}, \tag{3.3}$$

$$\begin{aligned} \{\rho \cos \beta\rho-(\eta_n)_m\}\{t/(r_n)_m\}^{\rho}&<n(t, \infty)/T((r_n)_m) \\ &<\{\rho \cos \beta\rho+(\eta_n)_m\}\{t/(r_n)_m\}^{\rho}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \{\cos \beta\rho-(\zeta_n)_m\}\{t/(r_n)_m\}^{\rho}&<N(t, \infty)/T((r_n)_m) \\ &<\{\cos \beta\rho+(\zeta_n)_m\}\{t/(r_n)_m\}^{\rho} \end{aligned} \tag{3.5}$$

hold for $(R_n)_m'\leqq t\leqq(R_n)_m''$, where

$$(R_n)_m'\to\infty, \quad (r_n)_m/(R_n)_m'\to\infty, \quad (R_n)_m''/(r_n)_m\to\infty,$$

$$(\xi_n)_m(>0)\to 0, \quad (\eta_n)_m(>0)\to 0, \quad (\zeta_n)_m(>0)\to 0$$

as $m\to\infty$. Now, for each ε_n and σ_n $(n=1, 2, 3, \cdots)$ we select a positive integer

$m_{k_{n-1}}$ $(1 \leqq k_1 < k_2 < \cdots < k_n < \cdots)$ such that $m \geqq m_{k_{n-1}}+1$ implies

$$(\xi_n)_m \pi\rho \cos\beta\rho \tan\frac{\beta\rho}{2} + (\eta_n)_m \Big\{4e^\gamma + (e^{2\rho}+1)\log\frac{2e^\gamma}{\gamma} + 4 + (1+e^{2\rho})\log\frac{1}{\sigma_n} + 2\gamma\Big(\frac{e^{2\gamma+\rho}}{\gamma-\rho} + \frac{e^{\gamma-\rho}}{\gamma+\rho}\Big)\frac{1}{e^{2\sigma_n\rho}-1}\Big\} < \varepsilon_n/5\,, \tag{3.6}$$

$$(\eta_n)_m(e^{2\rho}+1)\Big(8e^{2\gamma+1} + 2\gamma\log\frac{1}{\sigma_n}\Big) < \varepsilon_n/5\,, \tag{3.7}$$

$$\{1+(\xi_n)_m\}\,\{((R_n)'_m/(r_n)_m)^{\rho+\gamma} + 2^\rho e^\gamma((r_n)_m/(R_n)''_m)^{\gamma-\rho}\} < \varepsilon_n/5\,, \tag{3.8}$$

and

$$(\zeta_n)_m(1-\cos\beta\rho)(\pi\rho/\sin\pi\rho) < e_n/5\,. \tag{3.9}$$

We then define G_3 by

$$G_3 = \bigcup_{n=1}^{\infty}\Big\{\Big(\bigcup_{m \geqq m_{k_{n-1}}+1}[(r_n)_m,\ (r_n)_m e^{\sigma_n}]\Big) \cap \Big(\bigcup_{j=k_n}^{k_{n+1}-1}[a'_j,\ b'_j]\Big)\Big\}.$$

To construct J_1 we proceed as follows: Let $\{b_j\}$ be the poles of $f(z)$ and put

$$\Sigma_1(u) = \sum_{(r_n)_m e^{-\sigma_n} < |b_j| \leq (r_n)_m e^{2\sigma_n}} H(|b_j|,\ u) \qquad (u>0)\,,$$

where

$$H(t,\ u) = \log\frac{|t|^\gamma + u^\gamma}{||t|^\gamma - u^\gamma|}.$$

Using Cartan's lemma, it is possible to exclude, from the interval $(r_n)_m \leqq u \leqq (r_n)_m e^{\sigma_n}$, an exceptional set $\mathcal{E}_n(m)$ such that

$$\text{meas}\,\mathcal{E}_n(m) < (r_n)_m \sigma_n^2 \tag{3.10}$$

and such that, for $u \in [(r_n)_m,\ (r_n)_m e^{\sigma_n}] - \mathcal{E}_n(m)$,

$$\Sigma_1(u) \leqq \Big(8e^{2\gamma+1} + 2\gamma\log\frac{1}{\sigma_n}\Big)\{n((r_n)_m e^{2\sigma_n},\ \infty) - n((r_n)_m e^{-\sigma_n},\ \infty)\}\,. \tag{3.11}$$

Using $\mathcal{E}_n(m)$, we define J_1 by

$$J_1 = \bigcup_{n=1}^{\infty}\Big\{\Big(\bigcup_{m \geqq m_{k_{n-1}}+1}\mathcal{E}_n(m)\Big) \cap \Big(\bigcup_{j=k_n}^{k_{n+1}-1}[a'_j,\ b'_j]\Big)\Big\}.$$

By the construction of G_3 and J_1, it is easily verified that G_3 $(\subset G_2)$ is a very long set and J_1 has density zero.

3.2. A further consequence of the assumptions (2) **and** (3). Let $f(z)$ be a meromorphic function satisfying the assumptions (2) and (3). Assume that the quantities σ_n, $(r_n)_m$, $(R_n)'_m$, $(R_n)''_m$, $(\xi_n)_m$, $(\eta_n)_m$, $(\zeta_n)_m$ have been selected as in 3.1.. The aim of this section is to show that $u \in [(r_n)_m,\ (r_n)_m e^{\sigma_n}] - \mathcal{E}_n(m)$

$(m \geqq m_{k_n-1}+1)$ implies

$$(3.12) \qquad (Q_n)_m(u) \equiv \sum_{(R_n)'_m<|b_j|\leqq(R_n)''_m} H(|b_j|,\ u) \leqq T(u)\Big(\pi\rho \cos\pi\rho \tan\frac{\beta\rho}{2}+\frac{3}{5}\varepsilon_n\Big),$$

where $\{b_j\}$ denote the poles of $f(z)$. Note that if $\sum_0(u)$, $\sum_2(u)$ are given by

$$\sum\nolimits_0(u)=\sum_{(R_n)'_m<|b_j|\leqq(r_n)_m e^{-\sigma_n}} H(|b_j|,\ u),$$

$$\sum\nolimits_2(u)=\sum_{(r_n)_m e^{2\sigma_n}<|b_j|\leqq(R_n)''_m} H(|b_j|,\ u),$$

then $(Q_n)_m(u)=\sum_0(u)+\sum_1(u)+\sum_2(u)$ holds.

First, we prove that

$$(3.13) \qquad \sum\nolimits_0(u)+\sum\nolimits_2(u) \leqq T(u)\Big(\pi\rho \cos\beta\rho \tan\frac{\beta\rho}{2}+\varepsilon_n/5\Big)$$

$$(u\in[(r_n)_m,\ (r_n)_m e^{\sigma_n}],\ m\geqq m_{k_n-1}+1).$$

From (3.4) we deduce

$$(3.14) \qquad n(t,\ \infty)-\rho\cos\beta\rho T((r_n)_m)(t/(r_n)_m)^\rho$$
$$=(\eta_n)_m(t)T((r_n)_m)(t/(r_n)_m)^\rho=(\tilde{\eta}_n)_m(t)$$

with $|(\eta_n)_m(t)|<(\eta_n)_m$ $((R_n)'_m\leqq t\leqq(R_n)''_m)$. Using Stieltjes integrals and (3.14), we obtain

$$(3.15) \qquad \sum\nolimits_2(u)=\int_{(r_n)_m e^{2\sigma_n}+}^{(R_n)''_m} H(t,\ u)dn(t,\ \infty)$$

$$=\rho^2\cos\beta\rho L((r_n)_m)\int_{(r_n)_m e^{2\sigma_n}+}^{(R_n)''_m} H(t,\ u)t^{\rho-1}dt$$

$$+\int_{(r_n)_m e^{2\sigma_n}+}^{(R_n)''_m} H(t,\ u)d(\tilde{\eta}_n)_m(t).$$

An integration by parts yields

$$(3.16) \qquad \Big|\int_{(r_n)_m e^{2\sigma_n}+}^{(R_n)''_m} H(t,\ u)d(\tilde{\eta}_n)_m(t)\Big|<(\eta_n)_m T((r_n)_m)\Big\{((R_n)''_m/(r_n)_m)^\rho H((R_n)''_m,\ u)$$

$$+e^{2\sigma_n\rho}H((r_n)_m e^{2\sigma_n},\ u)+\int_{(r_n)_m e^{2\sigma_n}}^{(R_n)''_m} (t/(r_n)_m)^\rho\Big|\frac{\partial H}{\partial t}\Big|dt\Big\}.$$

Since

$$\frac{\partial H}{\partial t}=-\frac{2\gamma t^{\gamma-1}u^\gamma}{t^{2\gamma}-u^{2\gamma}} \qquad (t\neq u),$$

we have

$$\Big|\frac{\partial H}{\partial t}\Big|\leqq\frac{2\gamma}{1-e^{-2\sigma_n\gamma}}\cdot\frac{u^\gamma}{t^{\gamma+1}} \qquad (u\leqq te^{-\sigma_n}).$$

Hence

$$(3.17)\qquad \int_{(r_n)_m e^{2\sigma_n+}}^{(R_n)''_m} (t/(r_n)_m)^{\rho}\left|\frac{\partial H}{\partial t}\right| dt$$

$$\leqq \frac{2\gamma u^{\gamma}}{1-e^{-2\sigma_n\gamma}}(r_n)_m^{-\rho}\int_u^{\infty} t^{\rho-\gamma-1}dt \leqq \frac{2\gamma}{\gamma-\rho}\cdot\frac{e^{2\gamma+\rho}}{e^{2\sigma_n\gamma}-1}.$$

Taking into account the obvious estimates

$$0<H((R_n)''_m,\ u)\leqq 4(u/(R_n)''_m)^{\gamma}\qquad (u\leqq (R_n)''_m/2),$$

$$0<H(t,\ u)\leqq \log\frac{1}{\sigma_n}+\log\frac{2e^{\gamma}}{\gamma}\qquad (u\leqq te^{-\sigma_n}),$$

we deduce from (3.16) and (3.17) that

$$\left|\int_{(r_n)_m e^{2\sigma_n+}}^{(R_n)''_m} H(t,\ u)d(\tilde{\eta}_n)_m(t)\right|$$

$$<(\eta_n)_m T(u)\left\{4e^{\gamma}+e^{2\rho}\log\frac{1}{\sigma_n}+e^{2\rho}\log\frac{2e^{\gamma}}{\gamma}+\frac{2\gamma}{\gamma-\rho}\cdot\frac{e^{2\gamma+\rho}}{e^{2\sigma_n\gamma}-1}\right\}.$$

Returning to (3.15), we obtain

$$(3.18)\qquad \Sigma_2(u)\leqq \rho^2\cos\beta\rho L((r_n)_m)\int_u^{\infty}H(t,\ u)t^{\rho-1}dt$$

$$+(\eta_n)_m\left\{4e^{\gamma}+e^{2\rho}\log\frac{1}{\sigma_n}+e^{2\rho}\log\frac{2e^{\gamma}}{\gamma}+\frac{2\gamma}{\gamma-\rho}\cdot\frac{e^{2\gamma+\rho}}{e^{2\sigma_n\gamma}-1}\right\}T(u).$$

In the same way, we have

$$(3.19)\qquad \Sigma_0(u)\leqq \rho^2\cos\beta\rho L((r_n)_m)\int_0^{u}H(t,\ u)t^{\rho-1}dt$$

$$+(\eta_n)_m\left\{4+\log\frac{1}{\sigma_n}+\log\frac{2e^{\gamma}}{\gamma}+\frac{2\gamma}{\gamma+\rho}\cdot\frac{e^{\gamma-\rho}}{e^{2\sigma_n\gamma}-1}\right\}T(u).$$

Since

$$\int_0^{\infty}H(t,\ u)t^{\rho-1}dt=\frac{\pi u^{\rho}}{\rho}\tan(\beta\rho/2),$$

(3.13) follows from (3.18), (3.19), (3.3) and (3.6).

Next, we estimate $\Sigma_1(u)$ for $u\in[(r_n)_m,\ (r_n)_m e^{\sigma_n}]-\mathcal{E}_n(m)\equiv\mathcal{H}_n(m)$. By (3.4)

$$n((r_n)_m e^{2\sigma_n},\ \infty)-n((r_n)_m e^{-\sigma_n},\ \infty)$$

$$<\rho\cos\beta\rho(e^{2\sigma_n\rho}-e^{-\sigma_n\rho})T((r_n)_m)+(\eta_n)_m(e^{2\sigma_n\rho}+e^{-\sigma_n\rho})T((r_n)_m)$$

$$<\rho\cos\beta\rho(e^{2\sigma_n\rho}-e^{-\sigma_n\rho})T(u)+(\eta_n)_m(e^{2\rho}+1)T(u).$$

Then from (3.11), (3.1) and (3.7) we deduce that

$$(3.20)\qquad \Sigma_1(u)<\frac{2}{5}\varepsilon_n T(u)\qquad (u\in\mathcal{H}_n(m)).$$

Thus (3.12) follows from (3.13) and (3.20).

3.3. Completion of proof. If $f(z)$ satisfies (2) and (3), then as we saw in the introduction, $f(z)$ is a meromorphic function of lower order ρ satisfying the hypotheses ES at a sequence $\{r_m\}$ (, where $\{r_m\}$ is any increasing unbounded sequence in G_2,) of Pólya peaks of order ρ of $T(r, f)$. Applying Lemma 15.1 in [4] to $f(z)$, we obtain

$$\pi\rho\sin\beta\rho\leqq\liminf_{\substack{r\to\infty\\ r\in G_2}}\frac{\log M(r, f)}{T(r, f)}. \tag{3.21}$$

On the other hand, using the same reasoning as in [4, § 16], we deduce from (3.8), (3.9) and (3.12) that

$$\log M(u, f)+K_1z(u)+K_2p(u)\leqq\{\pi\rho\sin\beta\rho+\varepsilon_n\}T(u, f) \qquad (u\in\mathcal{H}_n(m),\ m\geqq m_{k_{n-1}}+1), \tag{3.22}$$

where K_1, K_2 are positive constants depending only on s (>0), γ and η (>0). Hence the proofs of (iii) and (vi) follow at once from (3.21) and (3.22). Finally, assertion (v) is derived easily from (vi).

4. Proof of Theorem 3.

We write $G_3=\bigcup_{n=1}^{\infty}[a_n'', b_n'']$ $(b_n''<a_{n+1}''$; $a_n''\to\infty$, $b_n''/a_n''\to\infty$ as $n\to\infty)$. Dropping, if necessary, a finite number of intervals and renumbering the remaining intervals $[a_n'', b_n'']$ we may assume that $b_n''/a_n''\geqq e^2$ $(n=1, 2, 3, \cdots)$. Define a sequence $\{m_j\}_0^\infty$ $(0=m_0<m_1<m_2<\cdots)$ of integers such that $b_n''/a_n''=e^{2(m_n-m_{n-1}+\beta_n)}$ $(\beta_n\in[0, 1))$, and then $\{\tilde{r}_m\}$ by $\tilde{r}_{m_{n-1}+1}=ea_n''$, $r_{m+1}/r_m=e^2$ $(m_{n-1}+1\leqq m\leqq m_n-1)$. Now, we define G_4 by

$$G_4=\bigcup_{m=1}^{\infty}[e^{-1}\tilde{r}_m, e\tilde{r}_m].$$

It is easily verified that G_4 $(\subset G_3)$ is a very long set. Next, choose a sequence $\{u_m\}_1^\infty$ such that $u_m\in[e^{-1}\tilde{r}_m, e\tilde{r}_m]-J_1$ and let $\{\varepsilon_n\}_1^\infty\downarrow 0$ be given. Then by Theorem 4 in [5] we are able to find sets $\mathcal{I}_m(\varepsilon_n)\subset[e^{-1}\tilde{r}_m, e\tilde{r}_m]$ such that

$$\operatorname{meas}\mathcal{I}_m(\varepsilon_n)<\varepsilon_n e^{-1}\tilde{r}_m \tag{4.1}$$

and such that for $u\in[e^{-1}\tilde{r}_m, e\tilde{r}_m]-\mathcal{I}_m(\varepsilon_n)$ $(m\geqq m_{l_{n-1}}+1$; $1\leqq l_1<l_2<l_3<\cdots)$

$$\log|f(ue^{i(\theta+\varphi(u_m))})|\leqq\varepsilon_n T(u) \qquad (\beta-\eta/2\leqq|\theta|\leqq\pi),$$

$$|\log|f(ue^{i(\theta+\varphi(u_m))})|-\pi\rho\sin\rho(\beta-|\theta|)T(u)|\leqq\varepsilon_n T(u) \qquad (\eta/2\leqq|\theta|\leqq\beta-\eta/2).$$

Further by Theorem 2 (v) we may assume that for $u\in[e^{-1}\tilde{r}_m, e\tilde{r}_m]-\mathcal{I}_m(\varepsilon_n)-J_1$ $(m\geqq m_{l_{n-1}}+1)$

$$(4.2)\qquad \begin{cases} \log|f(ue^{i(\theta+\varphi(u))})|\leqq \varepsilon_n T(u) & (\beta-\eta\leqq|\theta|\leqq\pi), \\ |\log|f(ue^{i(\theta+\varphi(u))})|-\pi\rho\sin\rho(\beta-|\theta|)T(u)|\leqq\varepsilon_n T(u) & \\ & (\eta\leqq|\theta|\leqq\beta-\eta). \end{cases}$$

Using (4.1), it is easy to check that the set

$$J_2=\bigcup_{n=1}^{\infty}\left\{\bigcup_{m_{l_{n-1}+1}\leqq m\leqq m_{l_{n+1}-1}}^{\infty}\mathcal{I}_m(\varepsilon_n)\right\},$$

has density zero, and the assertion (ii) follows from (4.2).

5. Proof of the principal lemma.

5.1. Preliminaries.

LEMMA 1. ([7]) *Let $H(r)$ be given by*

$$H(r)=\text{const.}+\int_\alpha^r \psi(t)t^{-1}dt \qquad (r\geqq\alpha>0),$$

where $\psi(t)$ is nonnegative, nondecreasing, and unbounded. Then there exists a function $\phi(t)$ $(t\geqq1)$ satisfying the following conditions (i)-(iv).

(i) *$\phi(t)$ is a continuous function which is continuously differentiable off a discrete set D (where D has no finite accumulation points.).*

(ii) *$\phi(t)$ is strictly increasing and unbounded.*

(iii) $\phi(1)=0$.

(iv) $H_1(r)\equiv\int_1^r\phi(t)t^{-1}dt=H(r)+O(\log r)\quad (r\to\infty)$.

LEMMA 2. *Let ρ $(0<\rho<\infty)$ and $L(r)$ be given, where $L(r)$ is a slowly varying function on a very long set G such that $H(r)\equiv r^\rho L(r)\neq O$ $(\log r)$ is a convex, increasing function of $\log r$. Corresponding to $H(r)$, define $\phi(t)$ $(t\geqq1)$ and $H_1(r)$ $(r\geqq1)$ as in Lemma* 1. *Then*

$$(5.1)\qquad \lambda(r)\equiv\frac{d\log(H_1(r)+1)}{d\log r}=\frac{\phi(r)}{H_1(r)+1}\to\rho \qquad (r\to\infty,\ r\in G).$$

Proof. Put

$$(5.2)\qquad H_1(r)=r^\rho L_1(r).$$

Then $L_1(r)$ is a slowly varying function on G such that $H_1(r)\neq O$ $(\log r)$ is a convex, increasing function of $\log r$. Define $h(r)$ by

$$(5.3)\qquad \lambda(r)=\rho+h(r).$$

By the definition of $\lambda(r)$ and the properties of $\phi(r)$, $\lambda(r)$ is a positive, continuous function for $r>1$, which is continuously differentiable off a discrete set D, where

D has no finite accumulation points. By (5.1), (5.2) and (5.3)

$$H_1(r)+1=r^\rho L_1(r)+1=\exp\left(\int_1^r \lambda(t)t^{-1}dt\right)=r^\rho \exp\left(\int_1^r h(t)t^{-1}dt\right). \tag{5.4}$$

Since $H_1(r)$ is a convex, increasing function of $\log r$, we deduce from (5.3) and (5.4) that

$$(\lambda(r))^2+rh'(r)\geqq 0 \qquad (r\notin D). \tag{5.5}$$

First, we prove $\{h(r)\}^+\equiv\max\{h(r), 0\}\to 0$ $(r\to\infty,\ r\in G)$. Suppose that there exists a sequence $\{r_n\}$ $(\subset G)\uparrow\infty$ such that $h(r_n)=\delta$ for some $\delta>0$. Since $L_1(r)$ is a slowly varying function on G, (5.4) implies

$$\int_r^{\sigma r} h(t)t^{-1}dt\to 0 \qquad (r\to\infty,\ r\in G,\ 0<\sigma<\infty). \tag{5.6}$$

Thus for any fixed $\sigma>1$ there is an $s_n\in(r_n, \sigma r_n)$ such that $h(s_n)=\delta/2$ $(n\geqq n_0(\sigma))$.

Now, for each r_n $(n\geqq n_0)$ we define r'_n by $r'_n=\inf\{s>r_n\,;\,h(s)=\delta/2\}$. By the continuity of $h(r)$, we easily see that $h(r'_n)=\delta/2$ and $h(r)>\delta/2$ $(r_n\leqq r<r'_n)$. It follows from this and (5.6) that

$$r'_n/r_n\to 1 \qquad (n\to\infty). \tag{5.7}$$

Using the mean value theorem to $\lambda(r)$, we deduce from (5.5) and (5.3) that

$$-\delta/2=\lambda(r'_n)-\lambda(r_n)=h(r'_n)-h(r_n)\geqq-\{\lambda(r''_n)\}^2(r''_n)^{-1}(r'_n-r_n) \qquad (r_n<r''_n<r'_n). \tag{5.8}$$

By (5.7) and (5.8), $\lambda(r''_n)\to\infty$ $(n\to\infty)$, which implies

$$h(r''_n)>2\delta \qquad (n\geqq n_1(\delta)). \tag{5.9}$$

(5.9) and the fact that $h(r'_n)=\delta/2$ yield the existence of $u_n\in(r''_n, r'_n)$ satisfying $h(u_n)=\delta$. Here, define $r_n^{(3)}$ by $r_n^{(3)}=\sup\{u<r'_n\,;\,h(u)=\delta\}$. Then it is easily seen that $h(r_n^{(3)})=\delta$ and

$$\delta/2<h(r)<\delta \qquad (r_n^{(3)}<r<r'_n\,;\,n\geqq n_1(\delta)). \tag{5.10}$$

On the other hand, as we stated above, the mean value theorem gives the existence of $r_n^{(4)}\in(r_n^{(3)}, r'_n)$ such that $h(r_n^{(4)})>2\delta$ for $n\geqq n_1$. This contradiction gives

$$\{h(r)\}^+\to 0 \qquad (r\to\infty,\ r\in G). \tag{5.11}$$

Next, we prove

$$\{h(r)\}^-\equiv\max\{-h(r), 0\}\to 0 \qquad (r\to\infty,\ r\in G). \tag{5.12}$$

Suppose that there exists a sequence $\{R_n\}$ $(\in G)\uparrow\infty$ such that $h(R_n)=-\delta'$ for some $\delta'>0$. Using (5.6), we see that $I_n\equiv\{s<R_n\,;\,h(s)=-\delta'/2\}$ is not empty

for $n\geqq n_2(\delta')$. Then, if we put $R_n'=\sup I_n$, $h(R_n')=-\delta'/2$ and $R_n/R_n'\to 1$ $(n\to\infty)$. It follows from these and (5.5) that for some $R_n''\in(R_n', R_n)$

$$\{\lambda(R_n'')\}^2>(\delta'/2)(R_n/R_n'-1)^{-1}\to\infty \qquad (n\to\infty). \tag{5.13}$$

Since $\lambda(r)>0$ $(r>1)$, $\lambda(R_n'')=\rho+h(R_n'')\to\infty$ $(n\to\infty)$ by (5.13). However, the definition of R_n' implies that $h(r)<-\delta'/2$ for $R_n'<r\leqq R_n$. This contradiction proves (5.12). Combining (5.11) and (5.12), we have the desired result.

5.2. Completion of proof. We write $G=\bigcup_{n=1}^{\infty}[a_n, b_n]$ $(b_n<a_{n+1},\ a_n\to\infty,\ b_n/a_n\to\infty)$, and put $a_n'=\lambda_n a_n$, $b_n'=b_n/\lambda_n$, where $\lambda_n=\min(a_n^{\delta_n}, (b_n/a_n)^{\delta_n})$ with a positive sequence $\{\delta_n\}$ satisfying δ_n $(<1/2)\to 0$, $a_n^{\delta_n}\to\infty$, $(b_n/a_n)^{\delta_n}\to\infty$ $(n\to\infty)$. Then $G'=\bigcup_{n=1}^{\infty}[a_n', b_n']$ $(\subset G)$ is a very long set. Now, let $\{r_m\}\subset G'$ be any increasing, unbounded sequence. We prove that $\{r_m\}$ is a sequence of Pólya peaks of order ρ for $H_1(r)+1$. To do this, we follow Bearnstein's procedure in [1, p. 94].

If $h(t)=0$ for all sufficiently large $t\in G$, this assertion is trivial. Otherwise, $\delta(x)=\sup_{\substack{t\geqq x\\ e^t\in G}}|h(e^t)|$ $(h(u)\equiv\lambda(u)-\rho)$ is strictly positive and nonincreasing for $x\geqq 0$. Further, by Lemma 2, $\delta(x)\to 0$ as $x\to\infty$. Define sequences $\{B_m\}$ and $\{b_m\}$ by

$$\log B_m=\int_{\log r_{m-1}}^{\log r_m}\delta(x)^{-1/2}dx,$$

$$\log b_m=-\min\left(\int_{\frac{1}{2}\log r_{m-1}}^{\frac{1}{2}\log r_m}\delta(x)^{-1/2}dx,\ \frac{1}{2}\log r_m\right).$$

It is easily verified that

$$\lim_{m\to\infty}b_m=0,\quad \lim_{m\to\infty}B_m=\lim_{m\to\infty}b_m r_m=+\infty.$$

For each m, define n by $a_n\leqq r_m\leqq b_n$, then $n\to\infty$ as $m\to\infty$. Assume that $b_m r_m\leqq r\leqq B_m r_m$, $a_n\leqq r\leqq b_n$. Then

$$\left|\int_{r_m}^{r}\frac{h(u)}{u}du\right|=\left|\int_{\log r_m}^{\log r}h(e^t)dt\right|\leqq\left|\int_{\log r_m}^{\log r}\delta(t)dt\right|$$

$$\leqq\max\left(\int_{\log r_m}^{\log B_m r_m}\delta(t)dt,\ \int_{\log b_m r_m}^{\log r_m}\delta(t)dt\right)$$

$$\leqq\max((\log B_m)\delta(\log r_m),\ (-\log b_m)\delta(\log b_m r_m))$$

$$\leqq\max\left(\frac{1}{\sqrt{\delta(\log r_m)}}\delta(\log r_m),\ \frac{1}{\sqrt{\delta\left(\frac{1}{2}\log r_m\right)}}\delta\left(\frac{1}{2}\log r_m\right)\right)$$

$$=\sqrt{\delta\left(\frac{1}{2}\log r_m\right)}\equiv\log(1+\varepsilon_m).$$

Hence

$$\frac{H_1(r)+1}{H_1(r_m)+1}=(r/r_m)^{\rho}\exp\left(\int_{r_m}^{r}h(u)u^{-1}du\right)<(1+\varepsilon_m)(r/r_m)^{\rho}$$

$$(b_m r_m \leqq r \leqq B_m r_m, \ a_n \leqq r \leqq b_n),$$

which is the defining inequality for Pólya peaks of order ρ for $H_1(r)+1$. However, since $H(r)=H_1(r)+O(\log r)=(1+o(1))(H_1(r)+1)$ $(r\to\infty)$, $\{r_m\}$ is also a sequence of Pólya peaks of order ρ for $H(r)$. This completes the proof of our principal lemma.

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