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THE AXIOM OF GENERALIZED HYPERSPHERES IN RIEMANNIAN GEOMETRY

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0. Introduction.

A characterization of spaces of constant curvature is the most classical and interesting subject in Riemannian geometry ([1], [2], [3], [9], [10], [11], [12], [13], [14], [15], etc.). Many axioms which characterize spaces of constant curvature have been found out. The axiom of n-planes by Cartan [3] and the axiom of n-spheres by Leung-Nomizu [11] are well-known. In the present note we shall establish the *axiom of generalized hyperspheres* and apply it to geodesic spheres and horospheres to obtain characterizations of spaces of constant curvature.

We prepare the notation for giving our axiom. Let M be a Riemannian manifold of dimension $m\geqq 3$ and N be a hypersurface (at least of class C^2) in M. Suppose that N has a unit normal vector field v. For a characterization of spaces of constant curvature we may assume without loss of generality that $\phi: N\times[0, \varepsilon)\to M$ given by $\phi(q, t):=\exp tv_q$ is diffeomorphic onto its image for some positive ε, because curvature properties are local ones. If $\phi_t(q):=\phi(q, t)$ for any $q\in N$ and for each t, then $N_t:=\phi_t(N)$ is a hypersurface in M for each t. The family $\{N_t; t\in[0, \varepsilon)\}$ will be called the *family of generalized hypersurfaces* associated with ϕ. Set $c_q(t):=\exp tv_q$ for each $q\in N$.

We now introduce the axiom.

Axiom of generalized hyperspheres. For every point $p\in M$ and every $(m-1)$-dimensional subspace T'_p of T_pM, there exists a hypersurface N through p such that $T_pN=T'_p$ and N_t is umbilical at $c_p(t)$ for each $t\in[0, \varepsilon)$.

In this axiom there are many choices of N, since the axiom does not require that N_t is umbilical at $c_q(t)$ for any point $q\neq p$ in N.

Then we shall prove

THEOREM 1. *Let M be a Riemannian manifold of dimension $m\geqq 3$. If M satisfies the axiom of generalized hyperspheres, then M is a space of constant curvature.*

Applying Theorem 1 to geodesic spheres, we shall obtain

COROLLARY 2. *Let M be a Riemannian manifold of dimension $m\geqq 3$. If all*

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small geodesic spheres in M are totally umbilical, then M is a space of constant curvature.

Tachibana-Kashiwada [15] have proved this corollary in the case where M is an Einstein manifold.

COROLLARY 3. *Let M be a Riemannian manifold of dimension* $m \geqq 3$. *If all small concentric geodesic spheres in M are conformal along any geodesic emanating from the common center, then M is a space of constant curvature.*

We shall also have the analogous results for horospheres instead of geodesic spheres. Let M be a simply connected complete Riemannian manifold without conjugate points. Then M is diffeomorphic to $\boldsymbol{R}^m$ where $m := \dim M$, and all geodesics are minimizing. Let $\gamma : (-\infty, \infty) \to M$ be a unit speed geodesic. The Busemann function f_γ on M is given by $f_\gamma(\,.\,) := \lim \{d(\,.\,, \gamma(t)) - t\}$. Set $H_s := f_\gamma^{-1}(s)$ for each $s \in \boldsymbol{R}$, which is called the *horosphere* through $\gamma(s)$ with *central ray* γ. f_γ is in general known to be at least of class C^1. However, under the additional condition on M((for example, nonpositive curvature), f_γ is of class C^2([5], [8]). We say that a unit speed geodesic $\alpha : (-\infty, \infty) \to M$ is an asymptote to γ if $\dot{\alpha}(t) = -\text{grad} f_\gamma(\dot{\alpha}(t))$ for each $t \in \boldsymbol{R}$. There passes a unique asymptote through given point to γ in M.

COROLLARY 4. *Let M be a simply connected complete Riemannian manifold without conjugate points and of dimension* $m \geqq 3$. *If all horospheres in M are of class* C^2 *and totally umbilical, then M is a space of constant curvature.*

COROLLARY 5. *Let M be a simply connected complete Riemannian manifold without conjugate points and of dimension* $m \geqq 3$. *If all horospheres are of class* C^2 *and if all concentric horospheres in M are conformal along any asymptote to the common central ray, then M is a space of constant curvature.*

Of course, our assumption makes no sense if $\dim M = 2$. In that case Green [7] states a characterization of surfaces of negative constant curvature under the additional conditions on horosperes.

Corollary 2 and 4 are direct consequences of Theorem 1. And, under the assumption of Corollary 3 and 5, it will turn out that geodesic spheres and horospheres are totally umbilical (see Lemma 6).

1. Preliminalies.

Our methods of the study will be based on a paper of Eschenburg-O'Sullivan [6]. The following arguments can be seen in [6]. Let M be a Riemannian manifold and let N be a hypersurface (at least of class C^2) in M. Suppose that N has a unit normal vector field v. Let $\phi : N \times [0, \varepsilon) \to M$ be a map given by $\phi(q, t) := \exp tv_q$ for any $(q, t) \in N \times [0, \varepsilon)$. For each t let $\phi_t : N \to M$ be a map given by $\phi_t(q) := \phi(q, t)$ for any $q \in N$. Suppose that $N_t := \phi_t(N)$ is a hypersurface in M for each t. Define a unit vector field V on $\phi(N \times [0, \varepsilon))$ by setting $V(\phi(q, t)) := \dot{c}_q(t)$, where $c_q : [0, \varepsilon) \to M$ is the geodesic with $c_q(t) := \exp tv_q$. Obviously, $V | N_t$ is a unit normal vector field on N_t for each t. Fix a point $p \in N$ and let $c := c_p$. For each t, if $D(t) := \phi_{t*} \cdot P_t^{-1}$ where P_t is the parallel translation along c from p to $c(t)$, then $D(t)$ is an isomorphism of $T_{c(t)}N_t$ onto itself. For a vector $x \in T_pN$, if $X(t) := P_t x$ for any t, then $DX(t) = \phi_{t*}(x)$ is a Jacobi vector field along c. Hence

$$D''X + R(DX, \dot{c})c = 0.$$

Thus, if $Y := DX$,

$$(D''D^{-1})Y + R(Y, \dot{c})c = 0. \tag{*}$$

Also, if $A(t)$ is the second fundamental form of N_t relative to $V | N_t$ for each t,

$$D'X(t) = \nabla_{\dot{c}(t)} DX(t) = \nabla_{DX(t)} V(\phi(p, t)) = -A(t)(DX(t)).$$

Hence, if $U(t) := D'D^{-1}(t)$, we have

$$U(t) = -A(t). \tag{2*}$$

If we differentiate U covariantly and substitute into (*),

$$U'Y + U^2Y + R(Y, \dot{c})c = 0.$$

Therefore, if $R_{c(t)}$ is a linear map of $T_{c(t)}N_t$ into itself given by $R_{c(t)}(y) := R(y, \dot{c})\dot{c}$ for any vector $y \in T_{c(t)}N_t$, then

$$U' + U^2 + R_c = 0. \tag{3*}$$

2. Proofs and Lemma.

In this section we shall give the proof of Theorem 1 and Lemma 6. Corollary 2 and 4 are direct applications of Theorem 1, so we may omit the proofs of them. Corollary 3 and 5 will be also reduced to Corollary 2 and 4 by Lemma 6, so it suffices to prove Lemma 6.

Proof of Theorem 1.

From Schur's theorem (see [4], p. 16) it is sufficient to prove that the sectional curvature depends only on the point $p \in M$. Let $x \in T_pM$ be an arbitrary unit vector and let T'_p be the subspace of T_pM orthogonal to x. Take a hypersurface N and a variation ϕ along N as in the axiom of generalized hyperspheres. Set $c(t) := \phi(p, t) = \exp tv_p$. Since the submanifold N_t is umbilical at $c(t)$, $U(t) = \lambda(t)I$ for some function λ along c(see (2*)). From (3*), $R_c = (-\lambda' - \lambda^2)I$. Hence, if $n := m-1$, then

$$\mathrm{Ric}(\dot{c}, \dot{c}) = n(-\lambda' - \lambda^2).$$

Thus we have

$$R_c=(1/n)\operatorname{Ric}(\dot{c},\ \dot{c})I.$$

In particular,

$$R(y,\ x)x=(1/n)\operatorname{Ric}(x,\ x)y$$

for any vector $y\in T'_p$. If $y\in T_pM$ is a unit vector orthogonal to x,

$$\operatorname{Ric}(x,\ x)=n\langle R(y,\ x)x,\ y\rangle=n\langle R(x,\ y)y,\ x\rangle=\operatorname{Ric}(y,\ y). \tag{4*}$$

This implies that the Ricci curvature at p is independent of the direction. Again, by the identity (4*), the sectional curvature depends sonly on the point of M. This completes the proof.

Now we prove the following.

LEMMA 6. *Let N be a hypersurface in a Riemannian manifold M and let ϕ be a normal geodesic variation along N. If $\phi_t: N\rightarrow N_t$ is conformal for each t, then N_t is totally umbilical for each t.*

Proof. Let $p\in N$ and $c(t):=\exp tv_p$ for any t. Let $D(t):=\phi_{t*}\cdot P_t^{-1}$ as in Section 1. The assumption implies that there exists a positive function ρ on $N\times[0,\ \varepsilon)$ such that $\langle\phi_{t*}(x),\ \phi_{t*}(y)\rangle=\rho(q,\ t)\langle x,\ y\rangle$ for any $q\in N$ and for any vectors x and $y\in T_qN$. If $X(t):=P_tx$ and $Y(t):=P_ty$ for each t and vectors x and $y\in T_pN$, then

$$\langle DX(t),\ DY(t)\rangle=\rho(p,\ t)\langle x,\ y\rangle.$$

If we differentiate both side with respect to t,

$$\langle D'X(t),\ DY(t)\rangle+\langle DX(t),\ D'Y(t)\rangle=\rho'(p,\ t)\langle x,\ y\rangle$$

for each t. Therefore, since $U:=D'D^{-1}$ is symmetric,

$$2\langle UDX(t),\ DY(t)\rangle=\rho'(p,\ t)\langle x,\ y\rangle=(\rho'(p,\ t)/\rho(p,\ t))\langle DX(t),\ DY(t)\rangle$$

for each t. Since $D(t)$ is invertible for each t,

$$U(t)z=(\rho'(p,\ t)/2\rho(p,\ t))z$$

for every vector $z\in T_{c(t)}N_t$. Therefore N_t is totally umbilical for each t, because $U(t)$ is the second fundamental form of N_t relative to $-\dot{c}(t)$ (see (2*)).

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