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THE CONNECTION BETWEEN THE SYMMETRIC SPACE $E_6/SO(10)\cdot SO(2)$ AND PROJECTIVE PLANES

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Introduction.

We are interested in symmetric spaces of the three types $E\mathrm{III}$, $E\mathrm{VI}$ and $E\mathrm{VIII}$ on the same line as projective planes, since we want to understand all exceptional Lie groups geometrically and systematically. The first aim in this paper is to clarify the structure of maximal flat tori in a compact symmetric space Π of the type $E\mathrm{III}$ which is constructed in the set of projections associated with involutive automorphisms of the compact simple Lie algebra of the type E_6 (Theorem 3.6). Next we write down the roots of the symmetric space Π and also give a relation between the roots and the isotropy groups of two points in Π. In Section 5 two objects, points and lines, are introduced into Π, and this space is studied from the viewpoint of projective geometry. Finally it is showed that Π is a projective plane in the wider sense (Theorem 5.10).

1. Preliminaries.

Let $\mathfrak{A}$ be a composition algebra over the real field $\boldsymbol{R}$ and let a, b, c be elements in $\mathfrak{A}$. If a conjugation $-: a\rightarrow\bar{a}$ is usually defined in $\mathfrak{A}$, we have a symmetric inner product $(a, b)=1/2(a\bar{b}+\overline{a\bar{b}})$. If a commutator and an associator are defined by $[a, b]=ab-ba$ and $(a, b, c)=(ab)c-a(bc)$ respectively, any inner derivation of $\mathfrak{A}$ can be generated by $D_{a,b}$, where $D_{a,b}(c)=[[a, b], c]-3(a, b, c)$.

Let $\mathfrak{A}^{(1)}\otimes M^3\otimes\mathfrak{A}^{(2)}$ denote an tensor product over $\boldsymbol{R}$ composed of one 3×3 matrix algebra M^3 with coefficients in $\boldsymbol{R}$ and two composition algebras $\mathfrak{A}^{(i)}$. If the confusion does not occur, we write aXu instead of $a\otimes X\otimes u$, where $a\in\mathfrak{A}^{(1)}$, $u\in\mathfrak{A}^{(2)}$ and $X\in M^3$. In this vector space a product can be defined by $xy=abXYuv$ for $x=aXu$ and $y=bYv$. Furthermore an involution and a trace Tr can be introduced by $aXu\rightarrow\bar{a}X^T\bar{u}$ and $Tr(aXu)=a\,\mathrm{tr}(X)Iu$ respectively, where $T: X\rightarrow X^T$ is the transposed operator of matrix, $\mathrm{tr}(X)=1/3(x_{11}+x_{22}+x_{33})$ for $X=(x_{ij})\in M^3$, and I is the 3×3 unit matrix.

Let $\mathfrak{M}$ denote a vector space over $\boldsymbol{R}$ which is generated by all elements in $\mathfrak{A}^{(1)}\otimes M^3\otimes\mathfrak{A}^{(2)}$ with the trace Tr being 0 and the skew-symmetric form with respect to the involution $aXu\rightarrow\bar{a}X^T\bar{u}$. Let $L(\mathfrak{A}^{(1)}, M^3, \mathfrak{A}^{(2)})$ be the vector space

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$Der\,\mathfrak{A}^{(1)}\oplus\mathfrak{M}\oplus Der\,\mathfrak{A}^{(2)}$ (direct sum) over $\boldsymbol{R}$, where $Der\,\mathfrak{A}^{(i)}$ is the Lie algebra of inner derivations of $\mathfrak{A}^{(i)}$. In this space we define an anti-commutative product $[\ ,\ ]$ in the following way (cf. [1], Section 3):

(1) $[D^{(i)}, D^{(j)}]=\begin{cases}\text{the Lie product of } Der\,\mathfrak{A}^{(i)} & (i=j)\\ 0 & (i\neq j)\end{cases}$

(2) $[D^{(1)}, aXu]=(D^{(1)}a)Xu$ and $[D^{(2)}, aXu]=aX(D^{(2)}u)$,

(3) For $x=aXu$ and $y=bYv$,

$$[x, y]=(X, Y)(u, v)D_{a,b}+(xy-yx-Tr(xy-yx))+(X, Y)(a, b)D_{u,v},$$

where $D^{(i)}\in Der\,\mathfrak{A}^{(i)}$ and $(X, Y)=\mathrm{tr}(XY)$. Then $L(\mathfrak{A}^{(1)}, M^3, \mathfrak{A}^{(2)})$ becomes a real Lie algebra by this product. If $\mathfrak{A}^{(1)}$ is the Cayley algebra $\mathfrak{C}$ (over $\boldsymbol{R}$) with the non-split type, it is a compact simple Lie algebra of the type F_4, E_6, E_7 or E_8 according as $\mathfrak{A}^{(2)}$ is $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Q}$ or $\mathfrak{C}$, where $\boldsymbol{C}$ and $\boldsymbol{Q}$ are the fields of complex and quaternion numbers with the non-split types respectively. We note that $Der\,\boldsymbol{C}=\{0\}$ and the Killing form B of $L(\mathfrak{C}, M^3, \boldsymbol{C})$ is given by $B(D_1+aXu, D_2+bYv)=3B_0(D_1, D_2)+144(a, b)(X, Y)(u, v)$, where B_0 is the Killing form of $Der\,\mathfrak{C}$.

For the remaining sections, a basis of $\mathfrak{C}$ is given explicitly:

a basis: $e_0, e_1, \cdots, e_7$;

rules of product:

$$e_1e_2=e_3,\quad e_1e_4=e_5,\quad e_6e_7=e_1,\quad e_2e_5=e_7,$$

$$e_3e_4=e_7,\quad e_3e_5=e_6,\quad e_6e_4=e_2,$$

$e_ie_j=-e_je_i$ $(i, j\geqq 1$ and $i\neq j)$, $e_ie_i=-e_0$ $(i\geqq 1)$, e_0 is the unit element,

a conjugate operator $-$: $e_0\to e_0$, $e_i\to -e_i$ $(1\leqq i\leqq 7)$.

Then we can realize $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$ as subalgebras in $\mathfrak{C}$ generated by $\{e_0\}$, $\{e_0, e_1\}$ and $\{e_0, e_1, e_2, e_3\}$ respectively.

2. Construction of a symmetric space Π.

Let $\mathfrak{G}$ be a real semi-simple Lie algebra with a Lie product $[\ ,\ ]$. Let $\mathfrak{X}$ be a subset in $\mathfrak{G}$ consisting of all elements x which satisfy an identity $(\mathrm{ad}\,x)((\mathrm{ad}\,x)^2+1)((\mathrm{ad}\,x)^2+4)=0$, where $\mathrm{ad}\,x$ is the adjoint representation of x. Then, for each $x\in\mathfrak{X}$, the eigenspaces for $\mathrm{ad}\,x$ can be obtained in $\mathfrak{G}$:

$$\mathfrak{G}_0(x)=\{z\in\mathfrak{G}: (\mathrm{ad}\,x)z=0\},$$

$$\mathfrak{G}_i(x)=\{z\in\mathfrak{G}: (\mathrm{ad}\,x)^2z=-i^2z\},\qquad i=1, 2.$$

Moreover three transformations $\{P_i(x)\}$ of $\mathfrak{G}$ can be defined by $P_0(x)=1+5/4(\mathrm{ad}\,x)^2+1/4(\mathrm{ad}\,x)^4$, $P_1(x)=-4/3(\mathrm{ad}\,x)^2-1/3(\mathrm{ad}\,x)^4$ and $P_2(x)=1/12(\mathrm{ad}\,x)^2+1/12$

$(\operatorname{ad} x)^4$, where 1 is the identity transformation of $\mathfrak{G}$. We assume that $\mathfrak{X} \neq \{0\}$ and $\mathfrak{G}_1(x) \neq \{0\}$.

By direct calculations we have the two following lemmas.

LEMMA 2.1. *Each transformation $P_i(x)$ is a projection of $\mathfrak{G}$ onto $\mathfrak{G}_i(x)$. These satisfy $P_i(x)P_j(x)=0$ $(i \neq j)$ and $P_0(x)+P_1(x)+P_2(x)=1$.*

LEMMA 2.2. *$\mathfrak{G}$ has a direct sum decomposition $\mathfrak{G}=\mathfrak{G}_0(x)\oplus\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$, and $(\mathfrak{G}_0(x)\oplus\mathfrak{G}_2(x))\oplus\mathfrak{G}_1(x)$ is a Cartan decomposition of $\mathfrak{G}$ with respect to an involutive automorphism $1-2P_1(x)$ $(=\exp \pi(\operatorname{ad} x))$.*

EXAMPLE. When $\mathfrak{G}$ is the compact simple Lie algebra $L(\mathfrak{C}, M^3, \boldsymbol{C})$ with the type E_6, we take $K_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ in $\mathfrak{M}\cap\mathfrak{X}$ and then the eigenspaces $\{\mathfrak{G}_i(K_1)\}$ can be given by

dimension

$$\mathfrak{G}_0(K_1): \quad Der\,\mathfrak{C} \quad \oplus \begin{pmatrix} 2a & 0 & 0 \\ 0 & -a & b \\ 0 & -b & -a \end{pmatrix} \qquad 14+16=30,$$

$$\mathfrak{G}_1(K_1): \begin{pmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & 0 \\ -b_2 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & a_1 & a_2 \\ a_1 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix} \qquad 16+16=32,$$

$$\mathfrak{G}_2(K_1): \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix} \qquad 8+8=16,$$

where a, a_1, a_2 (resp. b, b_1, b_2) are linear combinations of $e_0\otimes e_1$ and $e_i\otimes e_0$ (resp. $e_0\otimes e_0$ and $e_i\otimes e_1$), $i=1, 2, \cdots, 7$.

PROPOSITION 2.3. *If $\mathfrak{G}$ is a semi-simple Lie algebra, the connected component of any element x in $\mathfrak{X}$ is a reductive homogeneous space and the tangent space at x is $\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$.*

Proof (cf. [5]). The tangent space at x can be considered a subspace in $\mathfrak{G}$ (as a vector space) which consists of all elements w satisfying

$$\lim_{t\to 0}\frac{1}{t}\{\Phi(x+tw)-\Phi(x)\}=0 \qquad (t\in\boldsymbol{R}), \tag{1}$$

where $\Phi(z)=(\operatorname{ad} z)^5+5(\operatorname{ad} z)^3+4(\operatorname{ad} z)$ for $z\in\mathfrak{G}$. Note that $\Phi(x)=0$ for $x\in\mathfrak{X}$.

First it is shown that any element w in $\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$ is a tangent vector at

x. Since $\operatorname{ad} x$ is a non-singular transformation on $\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$, there exists an element $-z\in\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$ such that $[x, -z]=w$. Hence, for $t\in\boldsymbol{R}$,

$$\lim_{t\to 0}\frac{1}{t}\{\Phi(x+tw)-\Phi(x)\}=\lim_{t\to 0}\frac{1}{t}\Phi((\exp t(\operatorname{ad} z))x)=0.$$

Next we show that any tangent vector $w\in\mathfrak{G}$ at x is contained in $\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$. (1) can be rewritten as, for $\tilde{w}=\operatorname{ad} w$ and $\tilde{x}=\operatorname{ad} x$,

$$\tilde{w}\tilde{x}^4+\tilde{x}\tilde{w}\tilde{x}^3+\tilde{x}^2\tilde{w}\tilde{x}^2+\tilde{x}^3\tilde{w}\tilde{x}+\tilde{x}^4\tilde{w}+5\tilde{w}\tilde{x}^2+5\tilde{x}\tilde{w}\tilde{x}+5\tilde{x}^2\tilde{w}+4\tilde{w}=0. \tag{2}$$

Since this equation is linear for w and any element in $\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$ satisfies (2), we may assume $w\in\mathfrak{G}_0(x)$. Then $\tilde{x}\tilde{w}=\tilde{w}\tilde{x}$ holds by $[x, w]=0$ and so (2) becomes $\tilde{w}(5\tilde{x}^4+15\tilde{x}^2+4)=0$. This implies $-6[w, y]=0$ for any $y\in\mathfrak{G}_1(x)$, i.e., $[w, \mathfrak{G}_1(x)]=\{0\}$. Since $[\mathfrak{G}_1(x), \mathfrak{G}_1(x)]=\mathfrak{G}_0(x)\oplus\mathfrak{G}_2(x)$ by Lemma 2.2 and the semi-simplicity of $\mathfrak{G}$, we obtain $[w, \mathfrak{G}]=\{0\}$. Hence $w=0$.

Let G be the adjoint group of $\mathfrak{G}$ and let H be the isotropy group at $x\in\mathfrak{X}$. Then, the connected component of x becomes a reductive homogeneous space G/H with a pseudo-Riemannian structure defined by the Killing form of $\mathfrak{G}$ (cf. [7], p. 343). In fact, $\mathfrak{G}$ (resp. $\mathfrak{G}_0(x)$) is the Lie algebra of G (resp. H) and $\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$ is the tangent space at x to $\mathfrak{X}$. Furthermore, since $g(\operatorname{ad} w)g^{-1}=\operatorname{ad} gw$ and $g(\operatorname{ad} x)g^{-1}=\operatorname{ad} x$ for any $g\in H$ and any tangent vector w at x, we can see, from (2), that H makes the tangent space at x invariant.

Remark. If $\mathfrak{G}=L(\mathfrak{C}, M^3, \mathfrak{A}^{(2)})$ and $K_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}$, the dimension of the connected component of K_1 is 30, 48, 84 or 156 according as $\mathfrak{G}$ is a compact simple Lie algebra with the type F_4, E_6, E_7 or E_8.

Hereafter we write $P(x)$ for simplicity instead of $P_1(x)$. The action of the adjoint group G on the set of all endomorphisms h of $\mathfrak{G}$ is defined by $g\cdot h=ghg^{-1}$ for $g\in G$. When we assume that $\mathfrak{G}=L(\mathfrak{C}, M^3, \mathfrak{A}^{(2)})$, let Π be the orbit of $P(K_1)$ by G, i.e., $\Pi=\{g\cdot P(K_1);\ g\in G\}$. Then the tangent space at $P(K_1)$ to Π can be given by $\mathfrak{G}_1(K_1)$ and the Lie algebra of the isotropy group at $P(K_1)$ can be also given by $\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1)$. These facts are proved, making use of $g\cdot P(K_1)=P(gK_1)$, in the proof of the following.

PROPOSITION 2.4. *Π is a compact connected (globally) symmetric space. Its dimension is* 16, 32, 64 *or* 128 *according as $\mathfrak{G}$ is a compact simple Lie algebra with the type F_4, E_6, E_7 or E_8. Moreover Π is locally diffeomorphic to $F_4/Spin(9)$, $E_6/SO(10)\cdot SO(2)$, $E_7/SO(12)\cdot SO(3)$ or $E_8/SO(16)$.*

Proof. Let $\mathfrak{U}$ be the Lie algebra of the isotropy group at $P(K_1)$, i.e., $\mathfrak{U}=\{z\in\mathfrak{G} : (\exp t(\operatorname{ad} z))\cdot P(K_1)=P(K_1)$ for all $t\in\boldsymbol{R}\}$. First we show $\mathfrak{U}=\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1)$. Since $1-2P(K_1)$ is an (involutive) automorphism of $\mathfrak{G}$ as in Lemma 2.2, it holds that $(1-2P(K_1))(\exp t(\operatorname{ad} z))(1-2P(K_1))^{-1}=\exp t(\operatorname{ad}(1-2P(K_1))z)$ for $z\in\mathfrak{G}$ and $t\in\boldsymbol{R}$. If $z\in\mathfrak{U}$, this identity and the simplicity of $\mathfrak{G}$ imply $z=(1-2P(K_1))z$ and so $P(K_1)z=0$. This means $z\in\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1)$. The converse can be also seen easily.

Next we introduce a G-invariant Riemannian structure into Π by the Killing form of $\mathfrak{G}$. Then G is exactly the connected component (of the identity map) in the isometry group of Π. The action of G on Π is given by $h: P(gK_1)\rightarrow h\cdot P(gK_1)$ for $h\in G$ and $P(gK_1)\in\Pi$. Each point $P(gK_1)$ in Π has an involutive isometry $1-2P(gK_1)$ which makes $P(gK_1)$ itself an isolated fixed point. In this way Π becomes a compact connected (globally) symmetric space. The types of symmetric spaces can be determined by direct calculations.

3. Maximal flat tori in Π.

In the sections 3, 4 and 5, we assume that $\mathfrak{G}$ is the compact simple Lie algebra (over $\boldsymbol{R}$) with the type E_6, i.e., $\mathfrak{G}=L(\mathfrak{C}, M^3, \boldsymbol{C})$. Then

$$K_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix},\quad K_2=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\quad K_3=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

are elements in the manifold $\mathfrak{X}\subset\mathfrak{G}$, where we neglect the unit elements e_0 in the composition algebras $\boldsymbol{C}$ and $\mathfrak{C}$.

The matrix representation of the projection $P((\exp t(\operatorname{ad} K_2))K_1)$, $t\in\boldsymbol{R}$, is given here by $P_1(x)=-4/3(\operatorname{ad} x)^2-1/3(\operatorname{ad} x)^4$ for $x\in\mathfrak{X}$, making use of $(\exp t(\operatorname{ad} K_2))K_1 =(\cos t)K_1-(\sin t)K_3$: We remark that $L(\mathfrak{C}, M^3, \boldsymbol{C})=Der\,\mathfrak{C}\oplus\mathfrak{M}$ and the set of all elements (of $\mathfrak{G}$) written in (2), (3) and (4) below becomes a basis of $\mathfrak{M}$;

(1) on $Der\,\mathfrak{C}$, the form is the 0 matrix,

(2) on the each subspace consisting of $e_iK_1e_j$, $e_iK_2e_j$ and $e_iK_3e_j$ ($i, j=0$ or $i, j\geqq 1$), the form is

$$\begin{pmatrix}\sin^2 t & 0 & 1/2\sin 2t\\ 0 & 1 & 0\\ 1/2\sin 2t & 0 & \cos^2 t\end{pmatrix},$$

(3) on the each subspace consisting of e_iI_1, e_iI_2, e_iF_1, e_iF_2 and e_iF_3 ($i\geqq 1$), the form is

$$\begin{pmatrix}1/2\sin^2 2t & 1/2\sin^2 2t & 0 & 1/2\sin 4t & 0\\ 1/2\sin^2 2t & 1/2\sin^2 2t & 0 & 1/2\sin 4t & 0\\ 0 & 0 & \sin^2 t & 0 & -1/2\sin 2t\\ 1/4\sin 4t & 1/4\sin 4t & 0 & \cos^2 2t & 0\\ 0 & 0 & -1/2\sin 2t & 0 & \cos^2 t\end{pmatrix}$$

where $I_1=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$, $I_2=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$, $F_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$, $F_2=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$ and $F_3=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$,

(4) on the subspace consisting of I_1e_1, I_2e_1, F_1e_1, F_2e_1 and F_3e_1, the form is the same as that in (3).

LEMMA 3.1. *The curve $P((\exp t(\mathrm{ad}\,K_2))K_1)$, $t\in\boldsymbol{R}$, in Π is a simply closed geodesic with the initial point $P(K_1)$ and the tangent vector K_2. The period is π and the length is $4\sqrt{6}\pi$. In this geodesic, the midpoint $P((\exp \pi/2(\mathrm{ad}\,K_2))K_1)$ $(=P(K_3))$ is the only point which commutes with $P(K_1)$.*

Proof. We calculate only the length l of this geodesic $r(t)=P((\exp t(\mathrm{ad}\,K_2))K_1)$, $t\in\boldsymbol{R}$. The remainning assertion can be derived easily by the above matrix representation of $r(t)$. Let B be the Killing from of $\mathfrak{G}$, then $-B$ gives an inner product on each tangent space to Π by the definition of Rimannian structure on Π. Since $r(t)$ has the tangent vector K_2 at every point, we have

$$l=\int_0^{\pi}(-B(\dot{r}(t),\ \dot{r}(t)))^{1/2}dt=\int_0^{\pi}(-B(K_2,\ K_2))^{1/2}dt=(-144\,\mathrm{tr}\,K_2K_2)^{1/2}\pi=4\sqrt{6}\,\pi.$$

In the tangent space $\mathfrak{G}_1(K_1)$ at $P(K_1)$ to Π, let $\mathfrak{T}$ be the subspace spanned by two vectors K_2 and $e_1K_2e_1$. Since the symmetric space Π has the rank 2, $\mathfrak{T}$ is a maximal abelian subspace. We shall study in detail the maximal flat torus T in Π associated with $\mathfrak{T}$.

LEMMA 3.2. *The tangent vectors K_2 and $e_1K_2e_1$ are transitive by the adjoint group G of $\mathfrak{G}$.*

Proof. Put $\alpha_1=\exp 3\pi/2(\mathrm{ad}\,e_1K_1e_1)$ and $\alpha_2=\exp \pi/2(\mathrm{ad}\,K_1)$, then $\alpha_1\alpha_2(K_2)=\alpha_1(K_3)=e_1K_2e_1$.

In the torus T, two simply closed geodesics $r(t)$ and $h(s)$ are defined by $r(t)=P((\exp t(\mathrm{ad}\,K_2))K_1)$ and $h(s)=P((\exp s(\mathrm{ad}\,e_1K_2e_1))K_1)$, $0\leqq t,\ s<\pi$. Note that $\alpha_1\alpha_2\cdot r(t)=h(t)$.

LEMMA 3.3. *The geodesics $r(t)$ and $h(s)$ meet at only two point with $t=s=0$ and $t=s=\pi/2$.*

Proof. From the two relations $r(t)K_1=(\sin^2 t)K_1+1/2(\sin 2t)K_3$ and $h(s)K_1=(\sin^2 s)K_1+1/2(\sin 2s)e_1K_3e_1$, we can see easily $r(t)\neq h(s)$ as projections of $\mathfrak{G}$ except for the two cases of $t=s=0$ and $t=s=\pi/2$. In the former case, $r(0)$ and $h(0)$ are equal to $P(K_1)$. In the latter case, $r(\pi/2)=P(-K_3)=P(K_3)$ and $h(\pi/2)=P(e_1K_3e_1)$ hold. Since $\alpha_1\alpha_2$ in Lemma 3.2 preserves the Killing form B of $\mathfrak{G}$, it makes the 0- and 1-eigenspaces of $P(K_3)$ invariant by $\mathfrak{G}_1(K_3)=\mathfrak{G}_1(e_1K_3e_1)$ and $B(\mathfrak{G}_0(x)\oplus\mathfrak{G}_2(x),\ \mathfrak{G}_1(x))=\{0\}$ for $x\in\mathfrak{X}\subset\mathfrak{G}$. This concludes that $P(K_3)=P(e_1K_3e_1)$.

In the torus T we shall find all points commuting with $P(K_1)$. Put $\alpha(t)=\exp t(n_0(\mathrm{ad}\, K_2)+n_1(\mathrm{ad}\, e_1K_2e_1))$ for n_0, $n_1\in\boldsymbol{R}$. Moreover, put $\Theta_i(t)=\exp t(\mathrm{ad}\, e_iK_2e_i)$ and $t_i=tn_i$ $(i=0, 1)$, then $\alpha(t)$ can be rewritten as $\alpha(t)=\theta_0(t_0)\theta_1(t_1)$. The commutativity $P(K_1)P(\alpha(t)K_1)=P(\alpha(t)K_1)P(K_1)$ implies that $P(K_1)P(\alpha(t)K_1)z=0$ for $z\in\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1)$. We use this fact in the next proof.

PROPOSITION 3.4. *In the maximal flat torus T, the number of points commuting with $P(K_1)$ is exactly three except for itself.*

Proof. By Lemma 3.1, it is sufficient to consider the points $P(\alpha(t)K_1)$, restricted to $0\leqq t_0$, $t_1<\pi$, as elements in T. If $P(\alpha(t)K_1)$ and $P(K_1)$ are commutative, it can be shown by the above remark that

$$0=P(K_1)P(\alpha(t)K_1)K_1=P(K_1)\alpha(t)P(K_1)\alpha(t)^{-1}K_1$$

$$=1/2(\sin 2t_0\cos 2t_1)K_3+1/2(\cos 2t_0\sin 2t_1)e_1K_3e_1.$$

Hence there are two possible cases (i) $\sin 2t_i=0$ $(i=0, 1)$ and (ii) $\cos 2t_i=0$ $(i=0, 1)$. We decompose (i) into the four cases. (1) If $t_0=t_1=0$, $P(\alpha K_1)=P(K_1)$ holds, where the parameter t in $\alpha(t)$ is omitted. (2) If $t_0=\pi/2$ and $t_1=0$, $P(\alpha K_1)=P(K_3)$ is obtained. (3) If $t_0=0$ and $t_1=\pi/2$, $P(\alpha K_1)=P(e_1K_3e_1)=P(K_3)$ holds as in the proof of Lemma 3.3. (4) If $t_0=t_1=\pi/2$, we have $P(\alpha K_1)=P(K_1)$, in fact $P(\alpha K_1)=P(\theta_1\theta_0K_1)=\theta_1P(\theta_0K_1)\theta_1^{-1}=\theta_1P(\theta_1K_1)\theta_1^{-1}$ (by (2) and (3))$=P(\theta_1^2K_1)=P(K_1)$. Next (ii) is also decomposed into the four cases: (1) $t_0=t_1=\pi/4$, (2) $t_0=\pi/4$, $t_1=3\pi/4$, (3) $t_0=3\pi/4$, $t_1=\pi/4$ and (4) $t_0=t_1=3\pi/4$. Put $N_i=P(\alpha K_1)$, $i=1, 2, 3, 4$, corresponding to the each case. Then $N_1=N_4$ and $N_2=N_3$ hold from (2), (3), (4) in (i).

Since N_1, N_2 and $P(K_3)$ are midpoints in simply closed geodesics with the initial point $P(K_1)$, the geodesic symmetry $1-2P(K_1)$ makes them fixed (cf. [2]). This means that these points are commutative with $P(K_1)$. At last we show that the four points $P(K_1)$, $P(K_3)$, N_1 and N_2 are different from each other. That $P(K_1)\neq P(K_3)$ is given by $P(K_1)K_1=0$ and $P(K_3)K_1=K_1$. That $P(K_1)\neq N_i$ $(i=1, 2)$ is obtained from $P(K_1)e_iF_2=e_iF_2$ and $N_i(e_iF_2)=0$. It follows that $P(K_3)\neq N_i$. By $P(K_1)\neq P(K_3)$, we can see $N_1\neq N_2$.

LEMMA 3.5. *The points N_1 and N_2 are transitive by an involutive isometry of Π which makes $P(K_1)$ fixed.*

Proof. Put $\beta(t)=\exp t(\mathrm{ad}\, D_{e_3,e_5})$, $t\in\boldsymbol{R}$, then $\beta(t)$ is an automorphism of $\mathfrak{C}$. Especially $\beta(\pi/2)$ is involutive and its eigenvalue is 1 (resp. -1) on the subspace $\{e_0, e_3, e_5, e_6\}$ (resp. $\{e_1, e_2, e_4, e_7\}$). We extend $\beta(t)$ $(=\beta)$ to an automorphism of $\mathfrak{G}$ $(=Der\,\mathfrak{C}\oplus\mathfrak{M})$ by

$$D_{a,b}+uXv\longrightarrow D_{\beta a,\beta b}+(\beta u)Xv.$$

The same notation as $\beta(t)$ is given for this extended automorphism. It is easy that $\beta(t)\cdot P(K_1)=P(\beta(t)K_1)=P(K_1)$. Furthermore it holds that $\beta(\pi/2)\cdot N_1=$

$P(\beta(\pi/2)\theta_1(\pi/4)\theta_0(\pi/4)K_1) = P(1/2\beta(\pi/2)(K_1 - e_1K_1e_1 - K_3 - e_1K_3e_1)) = P(1/2(K_1 + e_1K_1e_1 - K_3 + e_1K_3e_1)) = P(1/2(-K_1 - e_1K_1e_1 + K_3 - e_1K_3e_1)) = P(\theta_1(3\pi/4)\theta_0(\pi/4)K_1) = N_2$.

Let $\boldsymbol{Z}$ be the ring of integers. Put $\mathfrak{E}_\varepsilon = \{(t_0, t_1) \in \boldsymbol{R}^2 : t_0 = (\varepsilon/2+n)\pi$ and $t_1 = (\varepsilon/2+m)\pi$ for $n, m \in \boldsymbol{Z}\}$, where $\varepsilon = 0$ or 1. Since $\exp(\mathrm{ad}\,\mathfrak{T})$ acts transitively on the torus T, we have the following theorem by the same method as the proof of Proposition 3.4 again.

THEOREM 3.6. *In the maximal flat torus T, two points $P(\theta_0(t_0)\theta_1(t_1)K_1)$ and $P(\theta_0(s_0)\theta_1(s_1)K_1)$ are identical if and only if they satisfy $(t_0 - s_0, t_1 - s_1) \in \mathfrak{E}_0 \cup \mathfrak{E}_1$.*

Remark. Let $d(\ ,\ )$ denote the distance between two points in Π. Then it can be obtained as in Lemma 3.1 that $d(P(K_1), P(K_3)) = 2\sqrt{6}\pi$ and $d(P(K_i), N_j) = 2\sqrt{3}\pi$ for $i = 1, 3$ and $j = 1, 2$.

4. The roots of the symmetric space Π.

The Lie algebra $\mathfrak{G}$ has a direct sum decomposition $\mathfrak{G} = (\mathfrak{G}_0(K_1) \oplus \mathfrak{G}_2(K_1)) \oplus \mathfrak{G}_1(K_1)$ by Lemma 2.2. Then, as in Proposition 2.4, the Lie triple system $\mathfrak{G}_1(K_1)$ is the tangent space at $P(K_1)$ to Π and the Lie subalgebra $\mathfrak{G}_0(K_1) \oplus \mathfrak{G}_2(K_1)$ is the Lie algebra $\mathfrak{U}$ of the isotropy group U at $P(K_1)$. The maximal flat torus T in Π has the tangent space $\mathfrak{T}$ at $P(K_1)$. $\mathfrak{T}$ is spanned by two tangent vectors K_2 and $e_1K_2e_1$ and it is a maximal abelian subspace of $\mathfrak{G}_1(K_1)$.

We give here a Cartan subalgebra $\mathfrak{H}$, containing the abelian space $\mathfrak{T}$, by

$$\mathfrak{H} = \{D_{e_2, e_3}, D_{e_4, e_5}, (I_1 - I_2)e_1, e_1(I_1 - I_2), K_2, e_1K_2e_1\},$$

where the unit elements e_0 are omitted again. Let $\{\lambda\}$ be the set of roots in the root space decomposition with respect to $\mathfrak{H}$. Then, by restricting $\{\lambda\}$ to the subspace $\mathfrak{T}$ in $\mathfrak{H}$, we can have the restricted roots for the symmetric space Π. In fact, the positive roots for the operation $\mathrm{ad}(aK_2 + be_1K_2e_1)$, $a, b \in \boldsymbol{C}$, are $-(a-b)\boldsymbol{i}$, $-2(a-b)\boldsymbol{i}$, $-(a+b)\boldsymbol{i}$, $-2(a+b)\boldsymbol{i}$, $-2a\boldsymbol{i}$ and $-2b\boldsymbol{i}$, where $\boldsymbol{C}$ is a field of coefficients and contains $\boldsymbol{i}$ with $\boldsymbol{i}^2 = -1$. The multiplicities of the restricted roots are 8, 1, 8, 1, 6 and 6 respectively. The simple roots are $-(a-b)\boldsymbol{i}$ and $-2b\boldsymbol{i}$. Then two vectors x_1 and x_2 in $\mathfrak{T}$ corresponded to them with respect to the Killing form B of $\mathfrak{G}$ are given by $x_1 = \boldsymbol{i}/96K_2 - \boldsymbol{i}/96e_1K_2e_1$ and $x_2 = \boldsymbol{i}/48e_1K_2e_1$. Consequently, we have the same Dynkin diagram for the symmetric space Π as that in Helgason's [4] (p. 534) from the fact $B(x_1, x_1) = 1/48$, $B(x_2, x_2) = 1/24$ and $B(x_1, x_2) = -1/48$.

If we make the correspondence of any vector $aK_2 + be_1K_2e_1$ in $\mathfrak{T}$ with the point $(a, b) \in \boldsymbol{R}^2$, the diagram $\mathfrak{S}$ of the pair (G, U), i.e., the set $\{x \in \mathfrak{T} : \lambda(x) \in \pi\boldsymbol{i}\boldsymbol{Z}$ for some root λ of $\Pi\}$, is given by the straight lines in $\boldsymbol{R}^2$ (cf. [4], p. 295) such that, for any $n \in \boldsymbol{Z}$, $a-b = n\pi$, $a-b = n\pi/2$, $a+b = n\pi$, $a+b = n\pi/2$, $a = n\pi/2$ and $b = n\pi/2$. Hence the singular set S in the torus T can be determined by $S = P((\exp(\mathrm{ad}\,\mathfrak{S}))K_1)$. Moreover, since we can define the bijective correspondence:

$(a, b) \to P((\exp \operatorname{ad}(aK_2+be_1K_2e_1))K_1)$ by Theorem 3.6 between the set $\{(a, b) \in \boldsymbol{R}^2 : 0 \leqq a < \pi, 0 \leqq b < \pi/2\}$ and the torus T, the singular set S can be decomposed into the seven geodesics, i.e., $S = \cup S_i$, where (1) $S_1 = P((\exp t(\operatorname{ad} K_2))K_1)$, $0 \leqq t < \pi$, (2) $S_2 = P((\exp t(\operatorname{ad} e_1K_2e_1))K_1)$, $0 \leqq t < \pi/2$, (3) $S_3 = P((\exp t(\operatorname{ad} e_1K_2e_1))K_3)$, $0 \leqq t < \pi/2$, (4) $S_4 = P((\exp t \operatorname{ad}(K_2+e_1K_2e_1))K_1)$, $0 \leqq t < \pi/2$, (5) $S_5 = P((\exp t \operatorname{ad}(K_2+e_1K_2e_1))K_3)$, $0 \leqq t < \pi/2$, (6) $S_6 = P((\exp t \operatorname{ad}(-K_2+e_1K_2e_1))K_3)$, $0 \leqq t < \pi/2$ and (7) $S_7 = P((\exp \operatorname{ad}((\pi - t)K_2 + te_1K_2e_1))K_1)$, $0 < t < \pi/2$.

In the Lie subalgebra $\mathfrak{U}$ of $\mathfrak{G}$, three subsets are defined by (1) for any $Q \in \Pi$, $\mathfrak{U}(Q) = \{x \in \mathfrak{U} : (\exp(\operatorname{ad} x)) \cdot Q = Q\}$, (2) $\mathfrak{U}(\mathfrak{T}) = \{x \in \mathfrak{U} : [x, \mathfrak{T}] = \{0\}\}$, and (3) for each root λ of Π, $\mathfrak{U}_\lambda = \{x \in \mathfrak{U} : [y, [y, x]] = \lambda(y)^2 x$ for any $y \in \mathfrak{T}\}$. Moreover put $S_\lambda = \{Q \in T : Q = P((\exp(\operatorname{ad} y))K_1)$ with $\lambda(y) \in \pi i \boldsymbol{Z}$ for some $y \in \mathfrak{T}\}$. Then we can state the following lemma by the useful identity (cf. [6], p. 64)

$$\mathfrak{U}(Q) = \mathfrak{U}(\mathfrak{T}) \oplus \Sigma \mathfrak{U}_\lambda,$$

where $Q \in T$ and the sum Σ runs over the positive roots $\{\lambda\}$ such that $Q \in S_\lambda$. Note that the dimension of $\mathfrak{U}(\mathfrak{T})$ is 16 and that of $\mathfrak{U}_\lambda$ is equal to the multiplicity of λ.

LEMMA 4.1. *For each point $Q \in T$, the Lie subalgebra $\mathfrak{U}(Q)$ can be determined as follows:*

(1) $\mathfrak{U}(P(K_1)) = \mathfrak{U}$, *because all roots pass through* $P(K_1)$,
(2) $\mathfrak{U}(P(K_3)) = \mathfrak{U}(\mathfrak{T}) \oplus \mathfrak{U}_{-2(a-b)i} \oplus \mathfrak{U}_{-2(a+b)i} \oplus \mathfrak{U}_{-2ai} \oplus \mathfrak{U}_{-2bi}$,
(3) $\mathfrak{U}(N_1) = \mathfrak{U}(\mathfrak{T}) \oplus \mathfrak{U}_{-2(a-b)i} \oplus \mathfrak{U}_{-2(a+b)i} \oplus \mathfrak{U}_{-(a-b)i}$,
(4) $\mathfrak{U}(N_3) = \mathfrak{U}(\mathfrak{T}) \oplus \mathfrak{U}_{-2(a-b)i} \oplus \mathfrak{U}_{-2(a+b)i} \oplus \mathfrak{U}_{-(a+b)i}$,
(5) $\mathfrak{U}(Q) = \mathfrak{U}(\mathfrak{T}) \oplus \mathfrak{U}_{-2bi}$ *for* $Q \in S_1$ *but* $Q \neq P(K_1), P(K_3)$,
(6) $\mathfrak{U}(Q) = \mathfrak{U}(\mathfrak{T}) \oplus \mathfrak{U}_{-2ai}$ *for* $Q \in S_2 \cup S_3$ *but* $Q \neq P(K_1), P(K_3)$,
(7) $\mathfrak{U}(Q) = \mathfrak{U}(\mathfrak{T}) \oplus \mathfrak{U}_{-(a-b)i} \oplus \mathfrak{U}_{-2(a-b)i}$ *for* $Q \in S_4$ *but* $Q \neq P(K_1), N_1$,
(8) $\mathfrak{U}(Q) = \mathfrak{U}(\mathfrak{T}) \oplus \mathfrak{U}_{-2(a-b)i}$ *for* $Q \in S_5$ *but* $Q \neq P(K_3), N_3$,
(9) $\mathfrak{U}(Q) = \mathfrak{U}(\mathfrak{T}) \oplus \mathfrak{U}_{-2(a+b)i}$ *for* $Q \in S_6$ *but* $Q \neq P(K_3), N_1$,
(10) $\mathfrak{U}(Q) = \mathfrak{U}(\mathfrak{T}) \oplus \mathfrak{U}_{-(a+b)i} \oplus \mathfrak{U}_{-2(a+b)i}$ *for* $Q \in S_7$ *but* $Q \neq N_3$,
(11) $\mathfrak{U}(Q) = \mathfrak{U}(\mathfrak{T})$ *for* $Q \in T - S$, *i.e., each regular point* Q.

Remark. (i) The dimension of these subalgebras are: (1) 46, (2) 30, (3) 26, (4) 26, (5) 22, (6) 22, (7) 25, (8) 17, (9) 17, (10) 25 and (11) 16. (ii) For each $\mathfrak{U}(Q)$ in (5), (6), (8), (9) or (11), it is a subalgebra also in $\mathfrak{U}(P(K_3))$. (iii) $\mathfrak{U}$ has the one-dimensional center $\{(2I_1+I_2)e_1\}$. (iv) The bases of $\mathfrak{U}_{-2(a+b)i}$ and $\mathfrak{U}_{-2(a-b)i}$ can be given by $e_1I_3 - I_3e_1$ and $e_1I_3 + I_3e_1$ respectively, where $I_3 = -I_1 - I_2$. The basis of $\mathfrak{U}_{-(a-b)i}$ consists of eight elements $K_1 - e_1K_1e_1$, $F_1e_1 + e_1F_1$ and $e_iK_1e_1 - e_1e_iF_1$ $(i = 2, 3, \cdots, 7)$.

5. The connection between Π and projective planes.

Two geometrical objects, points and lines, are introduced into the symmetric space Π and the connection between Π and projective planes is studied there.

The aim in this section is to give an affirmative answer to a Freudenthal's conjecture (cf. [3], p. 175) which asks the existence of one line passing through two general points in Π.

By the isotropy group U at $P(K_1)$, we can make two orbits in Π : the one contains $P(K_3)$ and the other contains N_1 and N_3 ($=N_2$) (see Lemma 3.5). Since $d(P(K_1), P(K_3))=2\sqrt{6}\pi$ and $d(P(K_1), N_1)=2\sqrt{3}\pi$, they are mutually disjoint sets. Moreover, they are compact connected totally geodesic submanifolds (i.e., globally symmetric spaces) by Lemma 2.1 in [2] because both $P(K_3)$ and N_1 are midpoints in simply closed geodesics with the initial point $P(K_1)$. By the transitivity of U on the set of maximal flat tori in Π passing through $P(K_1)$, we can see the following.

LEMMA 5.1. *Any point in Π commuting with $P(K_1)$ is contained in the orbits of $P(K_3)$ or N_1, and the converse is also true.*

Let $L(P(K_1))$ mean the orbit of $P(K_3)$ by U. The geodesic symmetry at any point Q in $L(P(K_1))$ is given again by the restriction of $1-2Q$ to this space because this involutive isometry makes $P(K_1)$ fixed, i.e., it does $L(P(K_1))$ invariant. The Lie algebra of the connected component in the isometry group of $L(P(K_1))$ can be given by $(\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1))-\{z\}(\cong so(10))$, where $z=(2I_1+I_2)e_1$. In fact, the isotropy group U does not act on $L(P(K_1))$ effectively and $\exp t(\mathrm{ad}\, z)$, $t\in\boldsymbol{R}$, is the identity transformation on $L(P(K_1))$. Hence, the Lie algebra of the isotropy group at $P(K_3)$ for U also becomes $(\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1))\cap(\mathfrak{G}_0(K_3)\oplus\mathfrak{G}_2(K_3))-\{z\}$, i.e., $(Der\,\mathfrak{C}\oplus\{e_iI_1\}\oplus\{e_iI_2\})\oplus\{I_2e_1\}(\cong so(8)\oplus so(2))$, $i=1, 2, \cdots, 7$. From these arguments, we have the following.

PROPOSITION 5.2. *The orbit $L(P(K_1))$ is a compact connected symmetric space. It is locally diffeomorphic to $SO(10)/SO(8)\cdot SO(2)$ with the dimension 16.*

LEMMA 5.3. $(1-2P(K_1))(1-2P(K_2))(1-2P(K_3))=1$.

This is a direct consequence by an easy calculation. We note that the involutive isometries $\{1-2P(K_i)\}$ are commutative with each other.

PROPOSITION 5.4. *In the orbit $L(P(K_1))$, $P(K_2)$ is the only point such that it commutes with $P(K_3)$ and has the distance $2\sqrt{6}\pi$ from $P(K_3)$.*

Proof. Since $\alpha\cdot P(K_1)=P(K_1)$ and $\alpha\cdot P(K_3)=P(K_2)$ hold for an isometry $\alpha=\exp\pi/2(\mathrm{ad}\,K_1)$ of Π, $P(K_2)$ is contained in $L(P(K_1))$. And, it commutes with $P(K_3)$ from the same reason as in Lemma 3.1 and the distance $2\sqrt{6}\pi$ is also obtained similarly.

From the fact that the symmetric space $L(P(K_1))$ has the rank 2 by Proposition 5.2, we can get a 2-dimensional maximal flat torus V (in this space) which contains $P(K_2)$ and $P(K_3)$. Then, for any point Q in $L(P(K_1))$, there exists an isometry $\beta\in U$, by the transitivity of the isotropy subgroup at $P(K_3)$

in U on the maximal flat tori of $L(P(K_1))$ passing through $P(K_3)$, such that $\beta\cdot V$ contains $P(K_3)$ and Q. If Q commutes with $P(K_3)$ and has the distance $d(Q, P(K_3))=2\sqrt{6}\pi$, $\beta\cdot P(K_2)=Q$ holds by Proposition 3.4 and by the remark in Theorem 3.6. On the other hand, since β makes $P(K_1)$ and $P(K_3)$ fixed, we can also have $\beta\cdot P(K_2)=P(K_2)$ by Lemma 5.3. Therefore this concludes that $Q=P(K_2)$.

Remark. (1) In the orbit $L(P(K_1))$ there is a 12-dimensional compact connected symmetric space with the same type as $SO(8)/SO(6)\cdot SO(2)$. This consists of all points which commute with $P(K_3)$ and have the distance $2\sqrt{3}\pi$ from $P(K_3)$. (2) The orbit of N_1 by U becomes a compact connected symmetric space with the same type as $SO(10)/SU(5)\cdot SO(2)$.

Now we generalize the notation $L(P(K_1))$ to an arbitrary point P in Π. Let $L(P)$ denote the set of all points in Π which commute with P and have the distance $2\sqrt{6}\pi$ from P. We call this set $L(P)$ *a line* associated with P and call P *a point* again in the sense of projective geometry. And the incidence is defined by the relation of inclusion. Any line is diffeomorphic to $L(P(K_1))$ as a manifold. Let Π^L denote the set of all lines in Π.

LEMMA 5.5. *The correspondence $L: \Pi\to\Pi^L$ is bijective.*

Proof. By the transitivity of isometries of Π, it is sufficient to show that $L(P(K_1))=L(Q)$ implies $P(K_1)=Q$. For any Q in Π, there exists an isometry α in U such that $\alpha\cdot Q$ is contained in the torus T (see Theorem 3.6). Since α makes $P(K_1)$ fixed and $L(P(K_1))=L(\alpha\cdot Q)$ holds, we can see that $\alpha\cdot Q$ commutes with all points in $L(P(K_1))$. Hence $\alpha\cdot Q=P(K_1)$ by Proposition 3.4, since $d(\alpha\cdot Q, P(K_3))=2\sqrt{6}\pi$ and $\alpha\cdot Q$ and $P(K_3)$ are commutative. Therefore, it follows that $Q=\alpha^{-1}\cdot P(K_1)=P(K_1)$.

PROPOSITION 5.6. *The correspondence L gives the duality between Π and Π^L.*

Proof. If $L(P)$ is a line containing a point Q, we get easily $P\in L(Q)$ because P commutes with Q and has the distance $d(P, Q)=2\sqrt{6}\pi$. The converse is also true.

LEMMA 5.7. *For two distinct points there exists at least one line passing through them.*

Proof. By the transitivity, we may assume that one point is $P(K_1)$ and the other is an arbitrary point Q. Since $T\subset L(P(K_2))$ and there is an isometry α in U such that $\alpha\cdot Q\in T$, $L(\alpha^{-1}\cdot P(K_2))$ is a line containing $P(K_1)$ and Q.

From now on we shall study the number of lines passing through two distinct points. In the next lemmas, let $P(K_1)$ be the base point in Π and we consider only the maximal flat tori containing this point.

If V is an arbitrary maximal flat torus in Π containing $P(K_1)$ and Q, the tangent space $\mathfrak{V}$ at $P(K_1)$ to V can be given by the set $\{y\in\mathfrak{G}_1(K_1): (\exp t(\mathrm{ad}\, y))\cdot$

$P(K_1)\in V$ for any $t\in\boldsymbol{R}\}$. We remark that if $Q=P(x)$, $x\in\mathfrak{X}$, the subspace $\mathfrak{G}_1(x)$ in $\mathfrak{G}$ can be regarded as the tangent space at Q as before. Then we can assert that:

LEMMA 5.8. *$\mathfrak{V}\subset\mathfrak{G}_1(K_1)\cap\mathfrak{G}_1(x)$ holds as a subset in $\mathfrak{G}$.*

Proof. Let $r(t)=(\exp t(\operatorname{ad} y))\cdot P(K_1)$, $y\in\mathfrak{V}$, be a geodesic with a tangent vector y such that $r(t_1)=Q$ for some $t_1\in\boldsymbol{R}$. Then $\mathfrak{G}_1(x)=(\exp t_1(\operatorname{ad} y))\mathfrak{G}_1(K_1)$ holds. Hence we can get $\mathfrak{V}\subset\mathfrak{G}_1(x)$ by $\mathfrak{V}\subset\mathfrak{G}_1(K_1)$ and $[y, \mathfrak{V}]=\{0\}$. This means that any element in $\mathfrak{V}$ can be regarded as a tangent vector at Q again.

LEMMA 5.9. *For two maximal flat tori V_1 and V_2 containing $P(K_1)$ and Q, there exists $z\in\mathfrak{U}(Q)$ such that* $(\exp(\operatorname{ad} z))\cdot V_1=V_2$.

Proof. Select $y_i\in\mathfrak{V}_i$, $i=1, 2$, such that the closures of the sets $\{(\exp t_i(\operatorname{ad} y_i))\cdot P(K_1)\}$, for any $t_i\in\boldsymbol{R}$, become V_i respectively. Define a continuous function f of $\exp(\operatorname{ad}\mathfrak{U}(Q))$ into $\boldsymbol{R}$ by $f(h)=B(h(y_1), y_2)$, where B is the Killing form of $\mathfrak{G}$. Since the group $\exp(\operatorname{ad}\mathfrak{U}(Q))$ is the connected component in a subgroup in U which makes Q fixed, it is a compact set. Hence f has a extremal value at some $h_0\in\exp(\operatorname{ad}\mathfrak{U}(Q))$ and we obtain

$$\left\{\frac{d}{dt}B((\exp t(\operatorname{ad} y))h_0(y_1),\ y_2)\right\}_{t=0}=0$$

for any $y\in\mathfrak{U}(Q)$. It follows that $0=B([y, h_0(y_1)], y_2)=B(y, [h_0(y_1), y_2])$. This means that $[h_0(y_1), y_2]$ is contained in the orthogonal complement $\mathfrak{U}(Q)^\perp$ for $\mathfrak{U}(Q)$. On the other hand, we have $[h_0(y_1), y_2]\in\mathfrak{U}(Q)$ because it holds that $\mathfrak{U}(Q)=(\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1))\cap(\mathfrak{G}_0(x)\oplus\mathfrak{G}_2(x))$ by Lemma 2.2 and also $h_0(y_1)$, $y_2\in\mathfrak{G}_1(K_1)\cap\mathfrak{G}_1(x)$ by Lemma 5.8, where $Q=P(x)$ is assumed. Hence we can show $[h_0(y_1), y_2]\in\mathfrak{U}(Q)\cap\mathfrak{U}(Q)^\perp=\{0\}$ by the nondegeneracy of the Killing form B. This concludes that $h_0(y_1)\in\mathfrak{V}_2$ and, therefore, $h_0\cdot V_1=V_2$.

DEFINITION. (1) Two distinct points in Π are said to be in the general position if any simply closed geodesic with the minimal length does not contain both of them. If not so, they are said to be in the singular position. (2) Two distinct lines $L(P)$ and $L(Q)$ in Π are said to be in the general (resp. singular) position if P and Q are in the general (resp. singular) position.

THEOREM 5.10. *Π is a projective plane in the wider sense, that is, Π satisfies the following properties*:

(1) *For two distinct points there exists only one line passing through them if the points are in the general position. If in the singular position, the set of lines passing through the points forms a complex projective space in the usual sense as a manifold.*

(2) *The correspondence L asserts the duality of* (1) *for two distinct lines.*

Proof. We prove only (1) for two distinct points P and Q. (2) is a direct consequence from Proposition 5.6 and (1). Let two lines $L(Q_1)$ and $L(Q_2)$ pass through the points. By transitivity, it may be assumed that $P=P(K_1)$, $Q_2=P(K_2)$ and $Q\in T$, where T is the torus in Theorem 3.6 and so $T\subset L(P(K_2))$. Then, since there exists a maximal flat torus V in $L(Q_1)$ which contains $P(K_1)$ and Q, we have an element $z\in\mathfrak{U}(Q)$ by Lemma 5.9 such that $(\exp(\mathrm{ad}\,z))\cdot T=V$. Under the notations in Lemma 4.1, simply closed geodesics in T with the initial point $P(K_1)$ and the minimal length $4\sqrt{3}\pi$ are only S_4 and S_7. If Q is in $T-(S_4\cup S_7)$, the isometry $\exp(\mathrm{ad}\,z)$ makes $P(K_3)$ fixed by the Remark (ii) in Lemma 4.1. This implies $P(K_3)\in V\subset L(Q_1)$. Hence Q_1 commutes with $P(K_1)$ and $P(K_3)$, and it also has the distance $d(Q_1, P(K_i))=2\sqrt{6}\pi$, $i=1, 3$. We can say, therefore, $Q_1=P(K_2)$ from Proposition 5.4. If Q is in $S_4\cup S_7$, many lines passing through $P(K_1)$ and Q can be found. We study in the case of $Q\in S_4$ because the other case is similar. By (2) in Lemma 4.1, the isometries $\exp(\mathrm{ad}\,\mathfrak{U}_{-(a-b)i})$ move $P(K_3)$ and, hence, do also $P(K_2)$. Put $\Omega=\{(\exp(\mathrm{ad}\,y))\cdot P(K_2)\}$, for all $y\in\mathfrak{U}_{-(a-b)i}$. Then, since $L(\Omega)$ gives all lines passing through $P(K_1)$ and Q, the following lemma completes our proof.

LEMMA 5.11. *Ω is the complex projective space with the complex dimension* 4.

Proof. Note that Ω depends only on the geodesic S_4 but does not on each point $Q\in S_4$. Since $\mathfrak{U}_{-(a-b)i}$ is the tangent space at $P(K_2)$ to Ω and it is a Lie triple system in the tangent space $\mathfrak{G}_1(K_2)$ at $P(K_2)$ to Π, Ω is a compact connected (globally) symmetric space and the type of Ω can be determined from (3), (7) and Remark (iv) in Lemma 4.1. Consequently Ω is diffeomorphic to $SU(5)/SU(4)\cdot SO(2)$ because this type has only one kind of local isometry class.

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