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ON THE ELASTIC CLOSED PLANE CURVES

BY HIROSHI YANAMOTO

§1. Introduction.

With respect to the total curvature of a closed curve C of class C^2 in a 3-dimensional Euclidean space E^3, we have the classical Fenchel inequality ([3] in 1929)

$$\int_C k(s)ds \geqq 2\pi\,, \tag{1.1}$$

where s denotes the arc length parameter of C and $k(s)$ the curvature of C. If a closed curve C is knotted in E^3, then the Fary inequality

$$\int_C k(s)ds \geqq 4\pi \tag{1.2}$$

holds good (cf. Fary [2] and J. Milnor [5]).

If a closed curve C is regarded as an elastic rod, then the bending energy $E(C)$ of the deflected curve C from $k=0$ is given by (cf. [4], [8])

$$E(C)=\frac{1}{2}\int_C k^2(s)ds\,. \tag{1.3}$$

For any real number t, we get

$$0 \leqq \int_C (k(s)-t)^2 ds = \int_C k^2(s)ds - 2t\int_C k(s)ds + t^2\int_C ds\,.$$

Then, from (1.1) we obtain

$$E(C)=\frac{1}{2}\int_C k^2(s)ds \geqq 2\pi^2/L\,, \tag{1.4}$$

where L is the length of the closed curve C. The equality holds good if and only if C is a circle of radius $L/2\pi$ in the plane.

Concerning the inequality (1.4), I. Bives ([1], p. 283) showed the following:

Let M be a circle of radius r, isometrically immersed into E^N. If k denotes the curvature function, then

$$\int_N k^2(s)ds \geqq 2\pi/r \tag{1.5}$$

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with equality iff M is embedded as a circle.

The pourpose of this note is to study the variational problem of the functional

$$E(C)=\frac{1}{2}\int_C k^2(s)ds$$

under the condition $\int_C ds=L=\text{constant}$. The equilibrium states of the elastic curves are the stationary points of the bending energy $E(C)$ with $L=\text{constant}$. If the second variation of E evaluated at some equilibrium state is positive definite, then the equilibrium state is called stable.

Thus we have the following questions (Bernoulli's problem).

(Q_1) Find the closed curves on the plane E^2 for which the functional E is stationary under the constant length.

(Q_2) Investigate the elastic stability for these stationary curves.

We study the following classical Euler's theorem

THEOREM A. *If* $E(C)=\frac{1}{2}\int_C k^2(s)ds$ *is critical for a closed plane curve with* $L=\text{constant}$, *then the curve* C *is either the plane circle* C_n (cf. Fig. 1) *with the radius* $L/2\pi n$ *or the curve* D_m (cf. Fig. 2) *which is congruent ot*

$$\text{(1.6)}\qquad \begin{cases} x(s)=\dfrac{2p}{\sqrt{R}}\cos\phi\,, \\ y(s)=\dfrac{1}{\sqrt{R}}\displaystyle\int_{-\pi/2}^{\phi}\left(2\sqrt{1-p^2\sin^2\phi}-\frac{1}{\sqrt{1-p^2\sin^2\phi}}\right)d\phi\,, \end{cases}$$

where ϕ *varies from* $-\pi/2$ *to* $3\pi/2$ *and* R, p^2 *are given by*

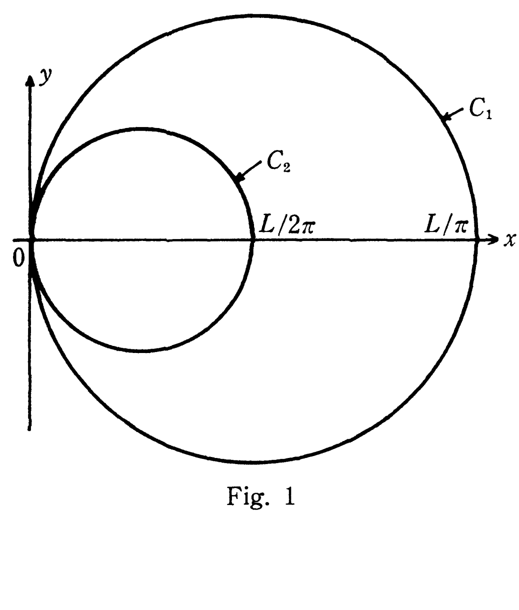

Fig. 1

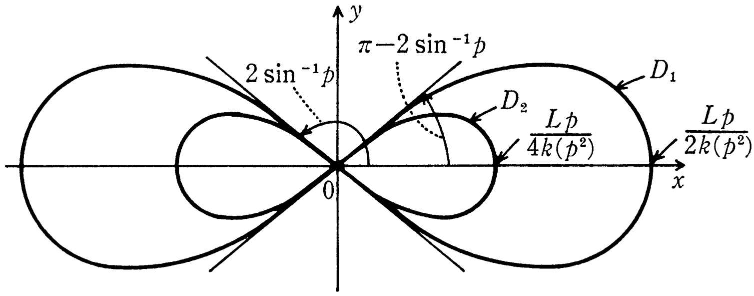

Fig. 2

$$\sqrt{R}=\frac{4m}{L}\int_0^{\pi/2}\frac{1}{\sqrt{1-p^2\sin^2\phi}}\,d\phi\left(=\frac{4m}{L}K(p^2)\right), \tag{1.7}$$

$$p^2\,;\,2\int_0^{\pi/2}\sqrt{1-p^2\sin^2\phi}\,d\phi=\int_0^{\pi/2}\frac{1}{\sqrt{1-p^2\sin^2\phi}}\,d\phi\,. \tag{1.8}$$

p^2 and $\sin^{-1}p$ are in the intervals;

$$0.82<p^2<0.83 \quad \textit{and} \quad 1.13\,\mathrm{Rad}<\sin^{-1}p<1.15\,\mathrm{Rad}$$

For C_n and D_m, the critical values are as follows:

$$E(C_n)=2\pi^2n^2/L\,,\qquad E(D_m)=16\,m^2(2p^2-1)K^2(p^2)/L$$

and

$$E(C_1)=2\pi^2/L<E(D_1)=16(2p^2-1)K^2(p^2)/L<E(C_2)=8\pi^2/L\,.$$

§ 2. Critical closed plane curves.

Let $C:[0,\ L]\ni s\rightarrow(x(s),\ y(s))\in E^2$ be a C^2 plane curve with arc length parameter s. Then the tangent vector $(dx/ds,\ dy/ds)$ to the curve is of unit length and satisfies the Frenet equation

$$\frac{d^2x}{ds^2}=-k(s)\frac{dy}{ds}\,,\qquad \frac{d^2y}{ds^2}=k(s)\frac{dx}{ds}\,. \tag{2.1}$$

If $\theta(s)$ is the angle between the tangent $(dx/ds,\ dy/ds)$ and the positive x-axis, the curvature function $k(s)$ is given by

$$k(s)=\frac{d\theta}{ds}\,. \tag{2.2}$$

Assuming that $(x(0),\ y(0))=0$ (the origin in E^2), $(x(s),\ y(s))$ is written by

$$x(s)=\int_0^s\cos\theta(s)ds\,,\qquad y(s)=\int_0^s\sin\theta(s)ds\,. \tag{2.3}$$

Necessary and sufficient conditions for this curve C to be closed are

(a) $k(s)$ is periodic with period deviding L,
(b) $\theta(L)-\theta(0)$ is $2\pi n$ (n=the rotation index of C),
(c) $x(L)=y(L)=0$.

Now we consider the variational problem with respect to $E(C)=\frac{1}{2}\int_C k^2(s)ds$ witn L=constant. For an arbitrary variation C_ε of C such that

$$C_\varepsilon : \theta_\varepsilon(s)=\theta(s)+\eta_\varepsilon(s) \qquad (\eta_\varepsilon(0)=\eta_\varepsilon(L)=0)\,, \tag{2.4}$$

we get

$$E(C_\varepsilon)=\frac{1}{2}\int_{C_\varepsilon}\left(k(s)+\frac{\partial\eta_\varepsilon(s)}{\partial s}\right)^2 ds\,.$$

Putting $\eta_\varepsilon(s)=\varepsilon\eta(s)+\varepsilon^2 h(s)+[\varepsilon^3]$, we see that

$$E(C_\varepsilon)=\frac{1}{2}\int_0^L\left[k^2(s)+2\varepsilon\frac{d\theta}{ds}\frac{d\eta}{ds}+\varepsilon^2\left\{\left(\frac{d\eta}{ds}\right)^2+2\frac{d\theta}{ds}\frac{dh}{ds}\right\}\right]ds+[\varepsilon^3]\,,$$

$$\int_{C_\varepsilon}\cos(\theta(s)+\eta_\varepsilon(s))ds=-\varepsilon\int_0^L\eta(s)\sin\theta(s)ds+[\varepsilon^2]\,,$$

$$\int_{C_\varepsilon}\sin(\theta(s)+\eta_\varepsilon(s))ds=\varepsilon\int_0^L\eta(s)\cos\theta(s)ds+[\varepsilon^2]\,.$$

If C is critical, then we have

$$0=\left.\frac{dE(C_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}=\int_0^L\frac{d\theta}{ds}\frac{d\eta}{ds}ds=-\int_0^L\frac{d^2\theta}{ds^2}\eta(s)ds\,,$$

for any $\eta(s)$ satisfying

$$-\int_0^L\eta(s)\sin\theta(s)ds=\int_0^L\eta(s)\cos\theta(s)ds=0\,.$$

From this there exist two constants λ and μ such that

$$-\lambda\sin\theta+\mu\cos\theta=\frac{d^2\theta}{ds^2}\,.$$

That is,

$$-R\sin(\theta-\alpha)=\frac{d^2\theta}{ds^2} \tag{2.5}$$

holds good, where we have put

$$R=\sqrt{\lambda^2+\mu^2}$$

and

$$\alpha : R\cos\alpha=\lambda\,, \qquad R\sin\alpha=\mu\,.$$

In the case of $R=0$, we have

$$k(s)=\frac{d\theta}{ds}=\frac{2\pi}{L}n \qquad (n=1, 2, 3, \cdots).$$

This means geometrically that the closed plane curve C is a circle C_n of radius $L/(2\pi n)$. The rotation index of C_n is $n\,(n=1, 2, 3, \cdots)$.

In the case of $R>0$, multiplying (2.5) by $d\theta/ds$, we have

$$d+R\cos(\theta-\alpha)=\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2, \tag{2.6}$$

where d is a constant of integration.

In the case of $-R<d<R$, putting $p=[(d+R)/2R]^{1/2}$, we obtain

$$2R\left\{p^2-\sin^2\left(\frac{\theta-\alpha}{2}\right)\right\}=\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 \tag{2.7}$$

and hence

$$k(s)=\frac{d\theta}{ds}=\pm 2\sqrt{R}\sqrt{p^2-\sin^2\left(\frac{\theta-\alpha}{2}\right)}, \tag{2.7$'$}$$

where p and θ satisfy the following:

$$0<p<1, \qquad -p\leqq\sin\left(\frac{\theta-\alpha}{2}\right)\leqq p.$$

We put

$$\theta(0)=\alpha-2\sin^{-1}p \qquad (0<\sin^{-1}p<\pi/2). \tag{2.8}$$

Then, taking account of the conditions (a) and (b), (2.7)$'$ may be written in the form

$$s=\frac{1}{2\sqrt{R}}\int_{\alpha-2\sin^{-1}p}^{\theta}\frac{1}{\sqrt{p^2-\sin^2\left(\frac{\theta-\alpha}{2}\right)}}\,d\theta \tag{2.9}$$

where s and θ run on

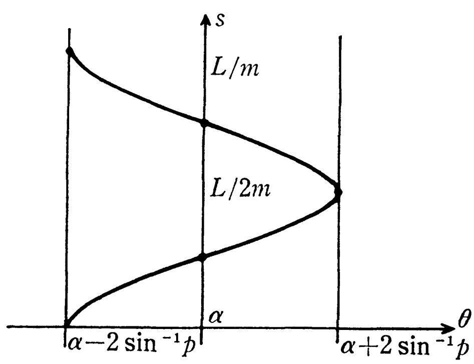

Fig. 3

(2.10) $$0 \leqq s \leqq \frac{L}{2m}, \qquad \alpha - 2\sin^{-1}p \leqq \theta \leqq \alpha + 2\sin^{-1}p .$$

The integration of (2.9) can be simplified by using

(2.11) $$\sin\left(\frac{\theta-\alpha}{2}\right) = p\sin\phi ,$$

(2.11)′ $$\frac{\theta-\alpha}{2} = \sin^{-1}(p\sin\phi) .$$

It is seen from (2.11)′ that when θ varies from $\alpha-2\sin^{-1}p$ to $\alpha+2\sin^{-1}p$ the quantity ϕ varies from $-\pi/2$ to $\pi/2$.

On the other hand, from (2.7) and (2.11), we obtain

(2.12) $$\left(\frac{d\theta}{ds}\right)^2\left\{\left(\frac{d\phi}{ds}\right)^2 - R(1-p^2\sin^2\phi)\right\} = 0 ,$$

from which

(2.13) $$\frac{d\phi}{ds} = \sqrt{R}\sqrt{1-p^2\sin^2\phi} \qquad (\sin^2\phi \neq 1) .$$

Hereby, we have

(2.14) $$s = \frac{1}{\sqrt{R}}\int_{-\pi/2}^{\phi} \frac{1}{\sqrt{1-p^2\sin^2\phi}} d\phi ,$$

where s and ϕ run on $0 \leqq s \leqq L/m$ and $-\pi/2 \leqq \phi \leqq 3\pi/2$. Putting

(2.15) $$K(p^2) = \int_0^{\pi/2} \frac{1}{\sqrt{1-p^2\sin^2\phi}} d\phi \qquad (0<p<1) ,$$

we obtain

(2.16) $$L/m = \frac{4}{\sqrt{R}} K(p^2) .$$

$K(p^2)$ is known as the complete elliptic integral of the first kind. Next we must check whether the condition (c) is satisfied or not for the curve given by

(2.17) $$\begin{cases} \theta = \alpha + 2\sin^{-1}(p\sin\phi) , & -\pi/2 \leqq \phi \leqq 3\pi/2 , \\ s = \dfrac{1}{\sqrt{R}}\displaystyle\int_{-\pi/2}^{\phi} \frac{1}{\sqrt{1-p^2\sin^2\phi}} d\phi . & \end{cases}$$

By (2.17), we have

$$\int_0^L \cos\theta(s)ds = m\int_0^{L/m} \cos\theta(s)ds$$

$$= \frac{m}{\sqrt{R}}\int_{-\pi/2}^{3\pi/2} \frac{\cos\{\alpha+2\sin^{-1}(p\sin\phi)\}}{\sqrt{1-p^2\sin^2\phi}} d\phi ,$$

and one for $\sin\theta(s)$. In view of

$$\cos\{\alpha+2\sin^{-1}(p\sin\phi)\}$$
$$=(\cos\alpha)(1-2p^2\sin^2\phi)-2p\sin\alpha\sin\phi\sqrt{1-p^2\sin^2\phi}\ ,$$

etc., we get

$$\int_0^L\cos\theta(s)ds=\frac{m\cos\alpha}{\sqrt{R}}\int_{-\pi/2}^{3\pi/2}\frac{1-2p^2\sin^2\phi}{\sqrt{1-p^2\sin^2\phi}}d\phi\,,$$
$$\int_0^L\sin\theta(s)ds=\frac{m\sin\alpha}{\sqrt{R}}\int_{-\pi/2}^{3\pi/2}\frac{1-2p^2\sin^2\phi}{\sqrt{1-p^2\sin^2\phi}}d\phi\,.$$

Therefore, the condition (c) is equivalent to

$$2\int_0^{\pi/2}\sqrt{1-p^2\sin^2\phi}\,d\phi=\int_0^{\pi/2}\frac{1}{\sqrt{1-p^2\sin^2\phi}}d\phi\,.$$

Putting

$$E(p^2)=\int_0^{\pi/2}\sqrt{1-p^2\sin^2\phi}\,d\phi\,, \tag{2.18}$$

we have

$$2E(p^2)=K(p^2)\,,\qquad p^2=\frac{d+R}{2R}\,. \tag{2.19}$$

$E(p^2)$ is known as the complete elliptic integral of the second kind. Using the Iwanami Math. dictionary ([7], second edition, p. 974, Fig. 12), we get

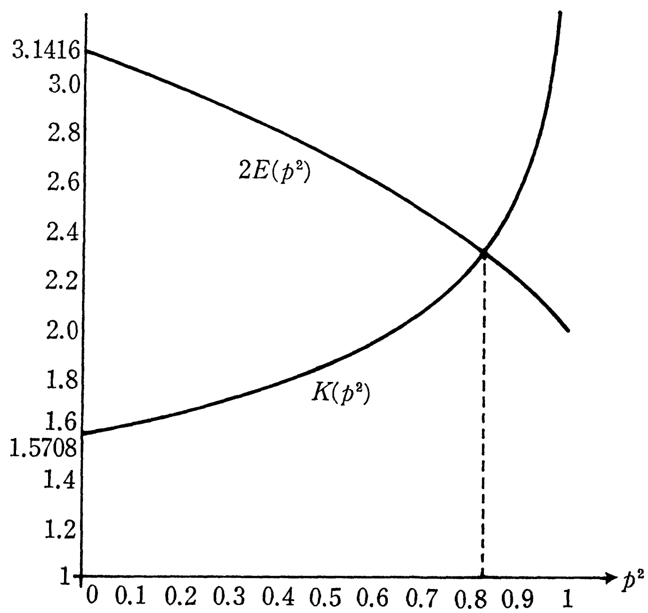

Fig. 4

We see that there is the constant p^2 in the interval

$$0.82<p^2<0.83 .$$

Hence we have

$$129°29'19''<2\sin^{-1}p<131°46'49'' .$$

We see that $(x(s), y(s))$ is given by

$$\begin{cases} x(s)=\dfrac{1}{\sqrt{R}}\left\{2p\sin\alpha\cos\phi+\cos\alpha\displaystyle\int_{-\pi/2}^{\phi}\left(2\sqrt{1-p^2\sin^2\phi}-\dfrac{1}{\sqrt{1-p^2\sin^2\phi}}\right)d\phi\right\}, \\ y(s)=\dfrac{1}{\sqrt{R}}\left\{-2p\cos\alpha\sin\phi+\sin\alpha\displaystyle\int_{-\pi/2}^{\phi}\left(2\sqrt{1-p^2\sin^2\phi}-\dfrac{1}{\sqrt{1-p^2\sin^2\phi}}\right)d\phi\right\}, \end{cases}$$

from which we get the following table.

s	0	$\cdots$	$L/(4m)$	$\cdots$	$L/(2m)$	$\cdots$	$3L/(4m)$	$\cdots$	L/m
ϕ	$-\pi/2$		0		$\pi/2$		π		$3\pi/2$
θ	$\alpha-2\sin^{-1}p$		α		$\alpha+2\sin^{-1}p$		α		$\alpha-2\sin^{-1}p$
$k(s)$	0	$+$	$2\sqrt{R}p$	$+$	0	$-$	$-2\sqrt{R}p$	$-$	0
$x(s)$	0		$2p\sin\alpha/\sqrt{R}$		0		$-2p\sin\alpha/\sqrt{R}$		0
$y(s)$	0		$-2p\cos\alpha/\sqrt{R}$		0		$2p\cos\alpha/\sqrt{R}$		0

The closed plane curve C may be drawn as follows (cf. Fig. 5).

Remark 1. In particular, for $\alpha=\pi/2$, we get

$$\begin{cases} x(s)=\dfrac{2p}{\sqrt{R}}\cos\phi , \\ y(s)=\dfrac{1}{\sqrt{R}}\displaystyle\int_{-\pi/2}^{\phi}\left(2\sqrt{1-p^2\sin^2\phi}-\dfrac{1}{\sqrt{1-p^2\sin^2\phi}}\right)d\phi . \end{cases}$$

Remark 2. For D_m, we get

$$\begin{aligned} E(D_m)&=\frac{1}{2}\int_0^L(d\theta/ds)^2ds=\frac{m}{2}\int_{-\pi/2}^{3\pi/2}\frac{d\phi}{ds}\left(\frac{d\theta}{d\phi}\right)^2d\phi \\ &=8m\sqrt{R}\int_0^{\pi/2}\left\{\sqrt{1-p^2\sin^2\phi}-\frac{1-p^2}{\sqrt{1-p^2\sin^2\phi}}\right\}d\phi \\ &=\frac{16m^2}{L}(2p^2-1)K^2(p^2) . \end{aligned}$$

For instance we have

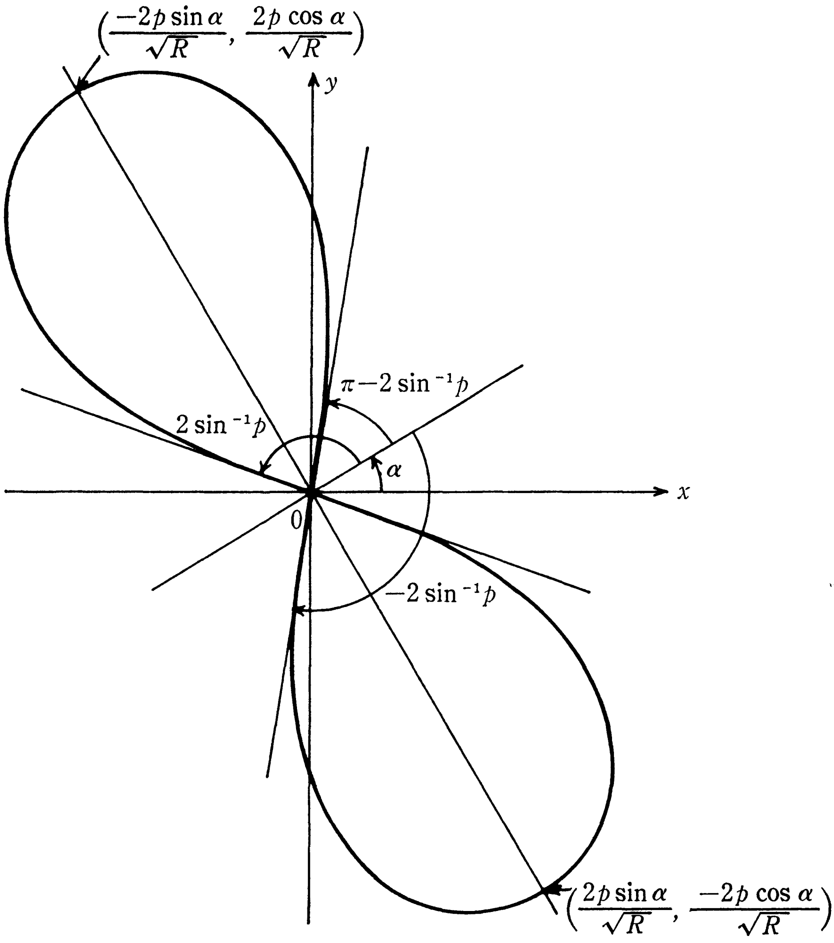

Fig. 5

$$E(C_1)=2\pi^2/L\fallingdotseq 19.74/L<E(D_1)=\frac{16}{L}(2p^2-1)K^2(p^2)\fallingdotseq 54.42/L$$

$$<E(C_2)=\frac{8\pi^2}{L}\fallingdotseq 78.96/L\,.$$

Let us now turn to the case $d=R$ or $d>R$. In the case of $d=R$, by (2.7) and $\theta(0)=\alpha$ we get

$$k(s)=d\theta/ds=2\sqrt{R}\cos\left(\frac{\theta-\alpha}{2}\right),$$

from which

$$s=\int_\alpha^\theta \frac{1}{2\sqrt{R}\cos\left(\frac{\theta-\alpha}{2}\right)}d\theta=\frac{1}{\sqrt{R}}\log\left|\tan\left(\frac{\pi}{4}+\frac{\theta-\alpha}{4}\right)\right|.$$

Thus we obtain

$$e^{\sqrt{R}s}=\tan\left(\frac{\pi}{4}+\frac{\theta-\alpha}{4}\right).$$

It requires the infinite arc length to obtain $\theta=\pi+\alpha$. Therefore, the case of $d=R$ does not occur.

In the case of $d>R$, putting

(2.20) $$q=\sqrt{2R/(d+R)} \qquad (0<q<1),$$

we get

(2.21) $$k(s)=d\theta/ds=\sqrt{2(d+R)}\sqrt{1-q^2\sin^2\left(\frac{\theta-\alpha}{2}\right)}.$$

Suppose now that $\theta(0)=\alpha$. Then we have the following (cf. Fig. 6):

(2.22) $$s=\frac{1}{\sqrt{2(d+R)}}\int_\alpha^\theta \frac{d\theta}{\sqrt{1-q^2\sin^2\left(\frac{\theta-\alpha}{2}\right)}}\left(=\int_\alpha^\theta \frac{d\theta}{\sqrt{2\{d+R\cos(\theta-\alpha)\}}}\right).$$

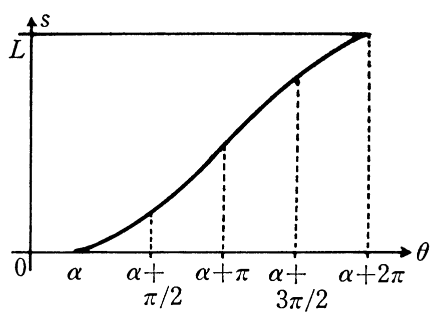

Fig. 6

Hence we obtain

(2.23) $$L=\sqrt{\frac{8}{d+R}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-q^2\sin^2\phi}}\left(=\sqrt{\frac{8}{d+R}}K(q^2)\right).$$

Let us now check whether the condition (c) is satisfied or not. By means of

$$\int_0^L \cos\theta(s)ds=\frac{2}{\sqrt{2(d+R)}}\int_0^\pi \frac{\cos(\alpha+2\phi)}{\sqrt{1-q^2\sin^2\phi}}d\phi,$$

$$\int_0^L \sin\theta(s)ds=\frac{2}{\sqrt{2(d+R)}}\int_0^\pi \frac{\sin(\alpha+2\phi)}{\sqrt{1-q^2\sin^2\phi}}d\phi,$$

the condition (c) is equivalent to

$$\int_0^{\pi} \frac{\cos 2\phi}{\sqrt{1-q^2 \sin^2 \phi}} d\phi = 0, \qquad \int_0^{\pi} \frac{\sin 2\phi}{\sqrt{1-q^2 \sin^2 \phi}} d\phi = 0. \tag{2.24}$$

The first equation reduces to

$$2\int_0^{\pi/2} \sqrt{1-q^2 \sin^2 \phi}\, d\phi = (2-q^2)\int_0^{\pi/2} \frac{1}{\sqrt{1-q^2 \sin^2 \phi}} d\phi,$$

that is,

$$2E(q^2) = (2-q^2)K(q^2), \qquad 0<q^2<1.$$

However, we can verify that $2E(q^2)<(2-q^2)K(q^2)$. In fact, $(2-q^2)K(q^2)-2E(q^2) = \pi q^4/16+(q^6)$ for $q^2 \fallingdotseq 0$, and we get the following figure (cf. Fig. 7).

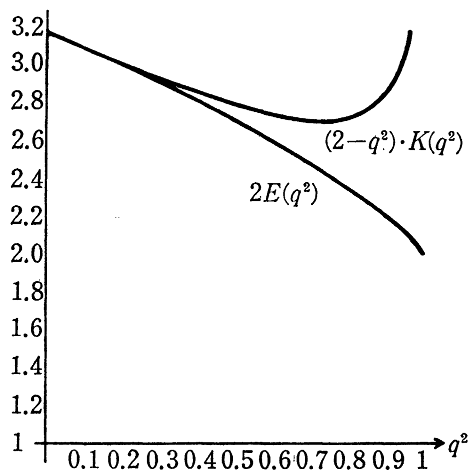

Fig. 7

Hence the curve C given by (2.22) does not satisfy the condition (c), that is, the curve is not closed.

Summarizing the results obtained above, we get the theorem A in the introduction.

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