Y. MAEDA
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POINTWISE CONVERGENCE OF THE PRODUCT INTEGRAL FOR A CERTAIN INTEGRAL TRANSFORMATION ASSOCIATED WITH A RIEMANNIAN METRIC

BY YOSHIAKI MAEDA

§ 0. Introduction.

In [13] Inoue-Maeda present a rigorous meaning to the convergence of the path integral in a non-compact curved space. Though comparing with Feynman's original idea, they considered the case where $i\hbar^{-1}$ is replaced by $-\lambda$ ($\lambda>0$). Namely, they considered a certain integral transformation associated with a given Lagrangian function of the form; $L(x, \dot{x})=g_{ij}(x)\dot{x}_i\dot{x}_j+V(x)$, where $G=(g_{ij}(x))$ defines a Riemannian metric and $V(x)$ is a smooth function with the compact support, and showed the convergence of its product integral in the *topology of the uniform operator norm*.

The purpose of this paper is to continue the above work as follows; First, we extend the above integral transformations to those which acts on sections of a general vector bundle (Cf. (0.1)). Also, we construct fundamental solutions for parabolic systems geometrically. Here, we shall deal with the case $V=0$, only for simplicity. The second aim is to show the convergence of the product integral of the integral transformation in a refined topology (*pointwise convergence of the kernel function*).

We suspect that these observation for the convergence of the product integral may have interesting applications, and here we can derive the asymptotic behavior of the fundamental solution for a parabolic system defined on the non-compact manifold in terms of geometrical invariants.

Let (M, g) be a smooth, complete m-dimensional Riemannian manifold and let E be a vector bundle over M with a linear connection D. Suppose that E is furnished with an inner product $\langle\,,\rangle_x$ at each fibre E_x, $x\in M$, preserved by D. Using the connection D, we can consider the parallel translation along the minimal geodesic γ_c from y to x, which maps an element of E_y to that of E_x. We denote it by $P(x, y)$ (Cf. § 2).

Denote by $C_0(E)$ the set of all continuous sections of E with compact support and by $C^\infty(E)$ that of all smooth sections of E. Put $C_0^\infty(E)=C_0(E)\cap C^\infty(E)$.

For $\xi\in C_0^\infty(E)$, we define the L^2-norm as

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$$\|\xi\|_{L^2(E)}=\left[\int_M\langle\xi(x),\ \xi(x)\rangle_x d\mu_g(x)\right]^{1/2}$$

where $d\mu_g(x)$ denotes the canonical measure defined by the Riemannian metric g. We denote by $L^2(E)$ the Hilbert space of sections ξ of E such that $\|\xi\|_{L^2(E)}<+\infty$.

Now, consider the following integral transformation in $L^2(E)$ with parameters $0\leqq s<t$, and $\lambda>0$ (Cf. [7], [12], [13] and [16]),

$$H(\lambda\,;t,\ s)\xi(x)=(2\pi\lambda^{-1})^{-m/2}\int_M\rho(t,\ s\,;x,\ y)[\exp(-\lambda S(t,\ s\,;x,\ y))] \times P(x,\ y)\xi(y)d\mu_g(y)\,, \tag{0.1}$$

for $\xi\in C_0^\infty(E)$: Here $S(t,\ s\,;x,\ y)=d^2(x,\ y)/(2(t-s))$, where $d(x,\ y)$ is the distance function and $\rho(t,\ s\,;x,\ y)$ is defined by

$$\rho(t,\ s\,;x,\ y)=|\det[-\partial_x\partial_y S(t,\ s\,;x,\ y)]/\mu_g(x)\mu_g(y)|^{1/2}, \tag{0.2}$$

where $\mu_g(x)=[\det(g_{ij}(x))]^{1/2}$ ((0.2) is assumed to be well-defined here. In fact, it is guaranteed under the assumption (A.0) which is stated later.) (Cf. [7] and [16]).

The kernel function of $H(\lambda\,;t,\ s)$ will be denoted by $H(\lambda\,;t,\ s\,;x,\ y)$ which may be considered as a section on $E\boxtimes E^*$; the vector bundle over $M\times M$ whose fibre at $(x,\ y)\in M\times M$ is given by the tensor product $E_x\boxtimes E_y^*$.

We consider the product integral for the above operator (0.1). Namely, let σ_N be the N-equal subdivision of the interval $[0,\ t]$ for given $t>0$ and any positive integer N,

$$\sigma_N: 0=t_0<t_1<\cdots<t_{N-1}<t_N=t\,,\qquad t_j=(j/N)t\,.$$

We set

$$H(\lambda\,;\sigma_N|t)=H(\lambda\,;t,\ t_{N-1})H(\lambda\,;t_{N-1},\ t_{N-2})\cdots H(\lambda\,;t_1,\ 0)\,, \tag{0.3}$$

and denote by $H(\lambda\,;\sigma_N|t\,;x,\ y)$ the kernel function of (0.3).

In order to state our results, we introduce the following assumptions:

(A.0) $(M,\ g)$ is a connected, simply connected, complete Riemannian manifold and has non-positive sectional curvature.

(A.1) There exists a positive constant k_1 such that for any multi-index $\alpha=(\alpha_1,\ \cdots,\ \alpha_m)$, $0\leqq|\alpha|\leqq 3$ and $x\in M$,

$$|\nabla^\alpha R_{ijk}{}^h(x)|_x\leqq k_1\,, \tag{0.4}$$

where $|\ \ |_x$ is the norm at x defined by g and ∇ and $R_{ijk}{}^h$ are the Riemannian connection and the curvature tensor defined by g respectively.

(A.2) There exists a positive constant k_2 such that the curvature 2-form Ω of D satisfies

$$|D^\alpha\Omega(x)|_x\leqq k_2 \qquad \text{for}\quad 0\leqq|\alpha|\leqq 3,\quad \text{and}\quad x\in M, \tag{0.5}$$

where $|\ |_x$ is the norm at x defined by $\langle\ ,\ \rangle_x$.

Remark. Combining the Riemannian metric $g_{ij}(x)$ and the inner product $\langle\ ,\ \rangle_x$, we can define the norms for the sections of the tensor product bundles of the tangent bundle TM, the cotangent bundle T^*M, E and E^*. Also, here we extend naturally the action of D and ∇ to tensor fields with values in E (or E^*). Including these, we denote these by the same letters $|\ |$, D and ∇ unless there occurs no confusion.

We can state the main theorem of this paper.

MAIN THEOREM. *Let (M, g) be a m-dimensional Riemannian manifold and let E be a vector bundle over M satisfying* (A.0)-(A.2). *Fix $T>0$ arbitraily. Then, the limit*

$$\boldsymbol{H}(\lambda\,;t\,;x,\ y)=\lim_{N\to\infty} H(\lambda\,;\sigma_N|t\,;x,\ y) \tag{0.6}$$

converges uniformly on $M\times M$ in the norm defined by $E\boxtimes E^*$ (Cf. §2) *for any $0<t<T$. Moreover, $\boldsymbol{H}(\lambda\,;t\,;x,\ y)$ gives a fundamental solution of the following parabolic equation*:

$$\begin{cases} [(\partial/\partial t)-\lambda^{-1}\mathcal{H}_x]\boldsymbol{H}(\lambda\,;t\,;x,\ y)=0\,, \\ \lim_{t\to 0+}\boldsymbol{H}(\lambda\,;t\,;x,\ y)=\delta_y(x)\otimes Id_y \end{cases} \tag{0.7}$$

where

$$\mathcal{H}_x=(1/2)\Delta_x^D-(1/12)Scal_g(x)\,, \qquad \Delta^D=-D^*D\,, \tag{0.8}$$

D^* is the adjoint operator of D with respect to the inner product on $L^2(E)$ and $Scal_g(x)$ is the scalar curvature.

On the other hand, in the course of the proof of the main theorem, we can get the asymptotic behavior of $\boldsymbol{H}(\lambda\,;t,\ s\,;x,\ y)$ as $t\to 0+$, which is a partial extension of the results in Molchanov [15] who treated the case where $E=M\times\boldsymbol{R}$.

COROLLARY. *Under the same assumptions as in the main theorem, the fundamental solution $\boldsymbol{H}(\lambda\,;t,\ s\,;x,\ y)$ of* (0.7) *satisfies, for any $0<\varepsilon<1/2$,*

$$\begin{aligned} &|\boldsymbol{H}(\lambda\,;t\,;x,\ y)-(2\pi\lambda^{-1}t)^{-m/2}\rho(x,\ y)[\exp(-\lambda(d^2(x,\ y)/2t))]P(x,\ y)|_{(x,y)} \\ &\qquad\leqq\gamma' t^{-m/2+3/2}[\exp(-\lambda\varepsilon^{**}d^2(x,\ y)/2t)]\,, \end{aligned} \tag{0.9}$$

for any $x,\ y\in M$, with some positive constant γ', where $\varepsilon^{**}=1-2\varepsilon$, $\rho(x,\ y)=|\det_g(dExp_x^{-1})_y|^{1/2}$ and Exp_x is the exponential mapping defined by g (Cf. §2).

Remark. By a technical reason ε^{**} appears in the above inequality, but it seems necessary for general cases.

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§ 1. Outline of the proof of Main theorem and related remarks.

In this section, we state the plan to prove the main theorem in the introduction. First, in § 3, we show the following basic properties of $H(\lambda; t, s)$ defined by (0.1).

PROPOSITION 1.1. *Assume* (A.0)-(A.2). *On fixing* $T>0$ *arbitrarily, the following properties hold for* $0 \leqq s<t<T$:

(a) *The integral transformation* $H(\lambda; t, s)$ *defines a bounded linear operator in* $L^2(E)$.

(b) $$\lim_{t\to s} \|H(\lambda; t, s)\xi-\xi\|_{L^2(E)}=0,$$

Let $\mathscr{B}(L^2(E))$ be the set of all bounded linear operators on $L^2(E)$ and we introduce the topology by the operator norm in it. Now, we study the convergence of the product integral of $H(\lambda; t, 0)$ in $\mathscr{B}(L^2E))$. So, we prove the following in § 4-5, which is one of the key results:

THEOREM 1.2. *Under the same assumptions as in Proposition* 1.1, *the following properties hold*:

(a) *There exists a positive constant* $C_0=C_0(\lambda; T)$ *such that*

$$\begin{aligned}(1.1)\qquad &\|H(\lambda; t+t', s)\xi-H(\lambda; t+t', t')H(\lambda; t', s)\xi\|_{L^2(E)}\\ &\leqq C_0[(t+t'-s)^{3/2}-t^{3/2}+(t'-s)^{3/2}]\|\xi\|_{L^2(E)}\end{aligned}$$

for any $\xi\in L^2(E)$ *and* $0\leqq s<t'<t<t+t'<T$.

(b) *There exists a limit* $\boldsymbol{H}(\lambda; t)=\lim_{N\to\infty}H(\lambda; t, t_{N-1})\cdots H(\lambda; t_1, 0)$, $t_j=(j/N)t$, $j=1, \cdots, N-1$, *in* $\mathscr{B}(L^2(E))$ *for any* $t>0$. *Therefore*, $\{\boldsymbol{H}(\lambda; t)\}_{t\geqq 0}$ *with* $\boldsymbol{H}(\lambda; 0)$ $=$*the identity operator, forms a* C^0 *semi-group in* $L^2(E)$.

(c) *The infinitesimal generator* $\lambda^{-1}\mathscr{H}$ *of* $\boldsymbol{H}(\lambda; t)$ *is given by*

$$\begin{aligned}(1.2)\qquad \lambda^{-1}(\mathscr{H}_x\xi)(x)&=[(\partial/\partial t)\boldsymbol{H}(\lambda; t)\xi(x)|_{t=0}]\\ &=\lambda^{-1}[(1/2)\Delta_x^D-(1/12)Scal_g(x)]\xi(x).\end{aligned}$$

Theorem 1.2 shows that the product integration of (0.6) determines a fundamental solution of the heat type equation (0.8) in the distribution sense. To show the regularity, we construct a kernel function by another method (so-called Levi's method) which is rather standard in the theory of partial differential equation (Cf. Friedman [9]). Using this estimate, we prove the main theorem stated in § 0. Namely, we show in § 7 the following:

THEOREM 1.3. *Under the same assumptions as in Proposition* 1.1, *we can construct a fundamental solution* $\boldsymbol{H}(\lambda; t)$ *with the following estimate: For any* $0<\varepsilon<1/4$, *there exists a positive constant* $\gamma=\gamma(\lambda; T, \varepsilon)$ *which dose not depend on* σ_N *such that*

$$(1.3) \qquad |\boldsymbol{H}(\lambda; t; x, y)-H(\lambda; \sigma_N|t; x, y)|_{(x,y)} \leqq \gamma t^{-(m-3)/2} N^{-1/2}[\exp(-\lambda\varepsilon^{(4)} d^2(x, y)/2t))]$$

where $\varepsilon^{(4)}=1-4\varepsilon$ *and* $\boldsymbol{H}(\lambda; t; x, y)$ *is the kernel function of* $\boldsymbol{H}(\lambda; t)$.

Remark 1. We cannot prove the convergence of $H(\lambda; \sigma_N|t; x, y)$ without constructing the fundamental solution by Levi's method. This may be still an interesting problem.

For the sake of our computations, we shall introduce the local coordinate expression. Given $\bar{x}\in M$, let U be a local coordinate neighborhood of $\bar{x}$ with the coordinate $(x^1, \cdots, x^m)$ such that $E|_U$ is trivialized as $E|_U\cong U\times\boldsymbol{F}$, where $\boldsymbol{F}$ is the standard fibre of E. Taking a frame field $\{e_a(x)\}$ of $E|_U$ (i.e. $e_a(x)$ depends smoothly on $x\in U$ and $\{e_a(x)\}$ forms a basis on $\boldsymbol{F}$ for any $x\in U$).

Denote by $\Gamma_{jb}^a(x)$ the component of $D_j=D_{(\partial/\partial x^j)}$. Then, for each $\xi\in C^\infty(E)$, its covariant derivative D_j can be expressed by

$$(1.4) \qquad (D_j\xi)^a(x)=\partial_j\xi^a(x)+\Gamma_{jb}^a(x)\xi^b(x).$$

Also, for any $\phi\in\Omega^1(E)$, a E-valued 1-form, expressed by $\phi(x)=\phi_i(x)dx^i$ with $\phi_i(x)=\phi_i{}^a(x)e_a(x)$, we have

$$(1.5) \qquad (D_j\phi)_i{}^a(x)=\partial_j\phi_i{}^a(x)-\left\{ {k \atop ji} \right\}(x)\phi_k{}^a(x)+\Gamma_{jb}^a(x)\phi_i{}^b(x),$$

where $\left\{ {k \atop ji} \right\}(x)$ is the Christoffel symbol of g. Moreover, the local coordinate expression of the covariant derivatives for any tensor field with values in E is obtained similarly. Using these notations, Δ^D can be expressed as

$$(\Delta^D\xi)^a(x)=g^{ij}(x)[\delta_b^a\partial_i+\Gamma_{ib}^a(x)][\delta_c^b\partial_j+\Gamma_{jc}^b(x)]\xi^c(x),$$

for any $\xi\in C^\infty(E)$.

Finally, we give some remarks about Main Theorem.

Remark 2. (i) *Trivial bundle*, $E=M\times\boldsymbol{R}$ (or $M\times\boldsymbol{C}$). A section of the trivial bundle can be identified with a function on M and $C(E)\cong C(M)$. Taking the trivial connection, i.e. $P(x, y)=id.$, we get a integral transformation acting for functions on M, which is considered in [12]. So, in this case the limit

$$(1.4) \qquad \lim_{N\to\infty}\int_M\cdots\int_M H(\lambda; t, t_{N-1}; x, z_{N-1})\cdots H(\lambda; t_1, 0; z_1, y)d\mu_g(z_{N-1})\cdots d\mu_g(z_1)$$

exists as a function on $M\times M$ for fixed t, $0<t<T$, under the assumptions (A.0)-

(A.1). We may denote its limit by

$$\int_{C_{t,0;x,y}}[\exp(-\lambda S(\gamma))]\mathcal{D}_F(\gamma) \qquad \text{(Cf. Feynman [8])},$$

where $C_{t,0;x,y}$ is the path space from y at $t=0$ to x at $t=t$, $S(\gamma)$ is the classical action along the path $C_{t,0;x,y}$ and $\mathcal{D}_F(\gamma)$ is the 'notorious' Feynman measure on $C_{t,0;x,y}$.

(ii) *The bundle of p-forms,* $E=\Lambda^p T^*M$. In this bundle, we can induce the inner product $\langle\,,\rangle_x$ and the connection D cannonically by g. Namely, for $\xi(x)=\xi_{i_1\cdots i_p}(x)dx^{i_1}\wedge\cdots\wedge dx^{i_p}$, $\eta(x)=\eta_{i_1\cdots i_p}(x)dx^{i_1}\wedge\cdots\wedge dx^{i_p}\in C(E)$, we define the inner product and the covariant derivative by

$$\langle\xi(x),\ \eta(x)\rangle_x=g^{i_1j_1}(x)\cdots g^{i_pj_p}(x)\xi_{i_1\cdots i_p}(x)\eta_{j_1\cdots j_p}(x)\,, \tag{1.6}$$

and

$$(D_j\xi)_{i_1\cdots i_p}(x)=\partial_j\xi_{i_1\cdots i_p}(x)-\left\{{k \atop ji_1}\right\}(x)\xi_{ki_2\cdots i_p}(x)-\cdots-\left\{{k \atop ji_p}\right\}(x)\xi_{i_1\cdots i_{p-1}k}(x)\,. \tag{1.7}$$

Then, we get the operator $\mathcal{H}_x=-(1/2)\Delta_L-(1/12)Scal_g(x)$: Here Δ_L is the rough Laplacian defined by g (Cf. [14]), and it is given by

$$\begin{aligned}(\Delta_L\xi)_{i_1\cdots i_p}(x)=(\Delta_H\xi)_{i_1\cdots i_p}(x)&+\sum_{r=1}^{p}R_{i_r}{}^{j}(x)\xi_{i_1\cdots j\cdots i_p}(x)\\&+\sum_{r>s}R_{i_ri_s}{}^{jk}(x)\xi_{i_1\cdots j\cdots k\cdots i_p}(x)\,,\end{aligned} \tag{1.8}$$

where $R_{ij}(x)$ and $R_{ijk}{}^h(x)$ are the component of the Ricci tensor and the curvature tensor of g respectively, and Δ_H denotes the Hodge-de Rham operator.

(iii) As a generalization of (ii), one may construct the fundamental solution of the parabolic equation whose infinitesimal generator is the following:

(a) The Lichnerowicz Laplacian acting on tensor fields.

(b) The spinorial Laplacian of Lichnerowicz when M admits a spinorial structure.

§ 2. Classical action and parallel translation.

Throughout this paper, notations and definitions concerning the differential geometry will be refered to [3] and [13].

We recall a geodesic, i.e. a curve $\gamma(\tau)$ which satisfies the following differential equation,

$$\frac{\delta^2\gamma^i(\tau)}{\delta\tau^2}\equiv\frac{d^2\gamma^i(\tau)}{d\tau^2}+\left\{{i \atop jk}\right\}(\gamma(\tau))\frac{d\gamma^j(\tau)}{d\tau}\cdot\frac{d\gamma^k(\tau)}{d\tau}=0\,, \tag{2.1}$$

where $\delta/\delta\tau$ denotes the covariant derivative along the curve $\gamma(\tau)$. Given $x\in M$, we define a mapping Exp_x from the tangent space T_xM into M by $Exp_x\tau X=\gamma(\tau)$, where $\gamma(\tau)$ satisfies (2.1) with the initial conditions $\gamma(0)=x$, $\dot\gamma(0)=X\in T_xM$.

By the assumption (A.0), Exp_x gives a diffeomorphism from T_xM onto M.

For each $X \in T_xM$, identifying $T_X(T_xM)$ with T_xM, we may induce naturally the scalar product in $T_X(T_xM)$. We denote by $(dExp_x)_X$ the differential mapping of Exp_x at X. Define also the function $\theta(x, y)$ on $M \times M$ by $\theta(x, y) = |\det_g(dExp_x)_X|$ (Cf. [3]). Then, the function $\rho(t, s; x, y)$ defined by (0.2) can be written as

$$\rho(t, s; x, y) = (t-s)^{-m/2}\rho(x, y), \tag{2.2}$$

where $\rho(x, y) = \theta(x, y)^{-1/2}$.

To give the estimate of $\rho(x, y)$, recall a Jacobi field $J(\tau)$ along the geodesic $\gamma(\tau)$. By (A.0)-(A.1) and the Rauch comparison theorem we get the following (Cf. [4]): Let $J(\tau)$ be a Jacobi field along geodesic $\gamma(\tau) = Exp_x\tau\omega$, $|\omega| = 1$ with initial conditions $J(0) = 0$, $\dot{J}(0) \neq 0$. Then, there exists a positive constant k_3 independent of x such that for any $y \in M$,

$$r|\dot{J}(0)|_x \leqq |J(r)|_y \leqq (\sinh k_3 r / k_3 r)|\dot{J}(0)|_x \tag{2.3}$$

where $r = d(x, y)$. In particular, we have

$$|\rho(x, y)| \leqq 1. \tag{2.4}$$

Denote by SM and S_xM the unit sphere bundle over M and the fibre of SM at $x \in M$ respectively. Using the Jacobi equation and (A.1), we have

LEMMA 2.1. *Assume that* (A.0)-(A.1) *hold. Given any* $x, y \in M$, *there exists a positive constant* k_4 *such that*

$$|\dot{J}(r)|_y \leqq (k_4 \exp(k_4 r))|\dot{J}(0)|_x. \tag{2.5}$$

Moreover, we have

$$|\rho_r(x, y)| \leqq k_4 \exp(k_4 r), \qquad r = d(x, y), \tag{2.6}$$

where $\rho_r(x, y) = (d/dr)\rho(x, Exp_x r\omega)$, $Exp_x r\omega = y$, $\omega \in S_xM$.

Now, we give an estimate of the higher order derivatives of the functions $\rho(x, y)$, which will be proved in Appendix.

PROPOSITION 2.2. *Assume that* (M, g) *satisfies* (A.0)-(A.1). *Then, there exists a positive constant* k_5 *such that for any* $x, y \in M$,

$$|\nabla_y^\alpha \rho(x, y)| \leqq k_5 \exp(k_5 r), \qquad r = d(x, y), \quad 0 \leqq |\alpha| \leqq 3. \tag{2.7}$$

Next, we recall the parallel transformation of a section of the vector bundle E by the connection D. Given a curve $\gamma(\tau)$ on M such that $\gamma(s) = y$, $\gamma(t) = x$, $s < t$, and $\xi(y) \in E_y$, define $\tilde{\xi}(\tau) \in E_{\gamma(\tau)}$ by

$$\frac{\delta}{\delta\tau}\tilde{\xi}(\tau)=D_{\dot{\gamma}(\tau)}\tilde{\xi}(\tau)=0\,, \qquad \tilde{\xi}(0)=\xi(y)\,. \tag{2.8}$$

We write $\tilde{\xi}(t)$ by $P_s^t(D, \gamma)\xi(y)$. Since (2.8) is a first order differential equation, the solution of (2.8) exists uniquely for any given curve $\gamma(\tau)$. In particular, if $\gamma_c(\tau)$ be a classical path which satisfies (2.1) with $\gamma_c(0)=y$, $\gamma_c(t)=x$, then we have

$$P_s^t(D, \gamma_c)\xi(y)=P_0^1(D, \tilde{\gamma}_c)\xi(y)\,,$$

where $\tilde{\gamma}_c(\tilde{\tau})=\gamma_c(s+\tilde{\tau}(t-s))$, for any $x\in M$ and any $y\in M$. Moreover, we write $P_0^1(D, \tilde{\gamma}_c)$ by $P(x, y)$ for simplicity.

Since each vector spaces E_x and E_y^* are equipped with the inner products, we can induce the inner product on $E_x\boxtimes E_y^*$, which will be denoted by $\langle\ ,\ \rangle_{(x,y)}$. That is, given $\xi(x, y)=\xi(x)\boxtimes\xi^*(y)$, $\eta(x, y)=\eta(x)\boxtimes\eta^*(y)$, we put

$$\langle\xi(x, y), \eta(x, y)\rangle_{(x,y)}=\langle\xi(x), \eta(x)\rangle_x\cdot\langle\xi^*(y), \eta^*(y)\rangle_y\,. \tag{2.9}$$

Also, we denote the norm on $E_x\boxtimes E_y^*$ by

$$|\xi(x, y)|_{(x,y)}=\langle\xi(x, y), \xi(x, y)\rangle_{(x,y)}^{1/2}\,. \tag{2.10}$$

We extend ∇ and D to a tensor fields with the values in E, and denote also them by the same letter.

The following is obvious from the definition (2.8) (Cf. [2]):

LEMMA 2.3. *For any x, $y\in M$, we get*
(i) *$P(x, y)$ is a smooth (local) section on $E\boxtimes E^*$.*
(ii) *$P(x, x)=Id.$, the identity operator on E_x.*
(iii) $\langle\nabla_x d^2(x, y), D_x P(x, Y)\rangle_x=0$.

LEMMA 2.4. *Assume* (A.0)-(A.2). *Given $\xi\in C^\infty(E)$, we get*

$$\|P(x, y)\xi(y)\|_x=\|\xi(y)\|_y\,, \tag{2.11}$$

for any x, $y\in M$.

Proof. Consider (2.8). Then, we get

$$\begin{aligned}\frac{d}{d\tau}\langle\tilde{\xi}(\tau), \tilde{\xi}(\tau)\rangle_{\gamma(\tau)}&=2\langle D_{\dot{\gamma}(\tau)}\tilde{\xi}(\tau), \tilde{\xi}(\tau)\rangle_{\gamma(\tau)}\\&=0\,.\end{aligned} \tag{2.12}$$

Using (2.12) and (A.2), we get (2.11).

The following properties are useful throughout our computations:

LEMMA 2.5. *$P(x, y)$ satisfies the following.*

$$D_x P(x, y)_{|x=y}=0\,, \quad \Delta_x^D P(x, y)_{|x=y}=0\,. \tag{2.13}$$

(2.14) $$D_y P(x, y)_{|y=x}=0\,, \quad \Delta_y^D P(x, y)_{|y=x}=0\,.$$

Proof. Let $\{e_a(y)\}_{a=1}^p$ be an orthonormal basis at E_y, where $\dim E_y=p$. Extend $\{e_a\}$ to a local frame field so that they are parallel. Take a normal coorinate $(y^1, \cdots, y^m)$ at y and denote by Γ_{ja}^b the coefficients of D. By putting $\tilde{\xi}(\tau)=\tilde{\xi}^a(\tau)e_a(\gamma_c(\tau))$, (2.8) can be written as

(2.15) $$\frac{d\tilde{\xi}^a(\tau)}{d\tau}+\Gamma_{jb}^a(\gamma_c(\tau))\frac{d\gamma^j(\tau)}{d\tau}\tilde{\xi}^b(\tau)=0\,.$$

So, using the Taylor expansion, we can write $\tilde{\xi}(\tau)$ by

(2.16) $$\tilde{\xi}^a(\tau)=\tilde{\xi}^a(0)+(\tilde{\xi}^a)'(0)\tau+(1/2)(\tilde{\xi}^a)''(0)\tau^2+O(\tau^3)\,,$$

where $(\tilde{\xi}^a)'(0)=(d/d\tau)\tilde{\xi}^a(0)$, etc. Differentiating (2.15) successively with respect to τ, we get

(2.17) $$(\tilde{\xi}^a)'(0)=-\Gamma_{jb}^a(y)Y^j\xi^b(y)$$

(2.18) $$(\tilde{\xi}^a)''(0)=-[\partial_l\Gamma_{jb}^a(y)Y^lY^j+\Gamma_{jc}^a(y)\Gamma_{lb}^c(y)Y^jY^l]\xi^b(y)\,,$$

where $Exp_yY=x$, because $\left\{{i \atop jk}\right\}(y)=0$. Substituting (2.17) and (2.18) into (2.16) and putting $\tau=1$, we have

(2.19) $$\begin{aligned}\tilde{\xi}^a(1)&=P_b^a(x, y)\xi^b(y)\\&=[\delta_b^a-\Gamma_{jb}^a(y)Y^j-(1/2)[\partial_l\Gamma_{jb}^a(y)YY^j\\&\qquad+\Gamma_{jc}^a(y)\Gamma_{lb}^c(y)YY^j]+O(Y^3)]\xi^b(y)\,,\end{aligned}$$

where $P_b^a(x, y)$ is the component of $P(x, y)$ with respect to $\{e_a\}$. Recall

(2.20) $$\begin{aligned}D_{x,k}D_{x,j}P_b^a(x, y)&=\partial_k\partial_jP_b^a(x, y)+\partial_k\Gamma_{jc}^a(Y)P_b^c(x, y)\\&\quad+\Gamma_{jc}^a(Y)\partial_kP_b^c(x, y)+\Gamma_{kc}^a(Y)\partial_jP_b^c(x, y)\\&=-(1/2)[\partial_k\Gamma_{jb}^a(y)+\partial_j\Gamma_{kb}^a(y)+2\Gamma_{kc}^a(y)\Gamma_{jb}^c(y)]\\&\quad+\partial_k\Gamma_{jb}^a(y)+\Gamma_{kc}^a(y)\Gamma_{jb}^a(y)+O(Y)\,,\end{aligned}$$

which proves the second equality of (2.13). For (2.14), remark that for any $x, z\in M$, we have

(2.22) $$P(x, z)P(z, x)=P(x, x)\,.$$

Differentiating (2.22) covariantly and using Lemma 2.3 (ii) and (2.13), we obtain (2.14).

Lastly, we get an estimate for the higher order derivatives of $P(x, y)$, which will be shown in Appendix.

PROPOSITION 2.6. *Assume that* (M, g) *satisfies* (A.0)-(A.2). *Then, there exists a positive constant* k_4' *such that for any* $x, y\in M$ *and for* $0\leqq|\alpha|\leqq 3$,

$$\text{(2.13)}\qquad \begin{cases} |(D_x)^\alpha P(x, y)|_{(x,y)}\leqq k_4'\exp(k_4'r)\,, \\ |(D_y)^\alpha P(x, y)|_{(x,y)}\leqq k_4'\exp(k_4'r)\,, \qquad r=d(x, y)\,. \end{cases}$$

§ 3. Basic properties of $H(\lambda; t, s)$.

Recall the operator $H(\lambda; t, s)$ in (0.1). Using the notations as in § 2, it can be written as follows:

$$\text{(3.1)}\qquad H(\lambda; t, s)\xi(x)=(2\pi\lambda^{-1})^{-m/2}\int_M \rho(t, s; x, y)[\exp(-\lambda(d^2(x, y)/2(t-s)))] \times P(x, y)\xi(y)d\mu_g(y)$$

for $\xi\in C_0(E)$.

In this section, we shall give some properties of (3.1). Using (2.4) and (2.11) and copying the proof of Lemma 2.1 [13], we get the following, which implies the part of (a) in Proposition 1.1:

PROPOSITION 3.1. *Assume that* (M, g) *satisfies* (A.0)-(A.2) *and fix* $T>0$ *arbitrarily. Then, the operator* $H(\lambda; t, s)$ *is stable, that is, there exists a positive constants* $C_1=C_1(\lambda; T)$ *such that*

$$\text{(3.2)}\qquad \|H(\lambda; t, s)\xi(x)\|_{L^2(E)}\leqq(\exp(C_1(t-s)))\|\xi\|_{L^2(E)}$$

for $0\leqq s<t<T$ *and* $\xi\in C_0^\infty(E)$.

Next, we study the behavior of $H(\lambda; t, s)$ as $t\downarrow s$. Namely, we have the following which is the part (b) in Proposition 1.1:

PROPOSITION 3.2. *Under the same assumptions as in Proposition* 3.1, *we have for any* $0\leqq s<t<T$,

$$\text{(3.3)}\qquad \lim_{t\downarrow s}\|H(\lambda; t, s)\xi-\xi\|_{L^2(E)}=0\,,$$

for any $\xi\in C_0^\infty(E)$. *Therefore, for fixed* $s\geqq 0$, *putting* $H(\lambda; s, s)=$*the identity transformation, we have the mapping from* $t\in[s, T)$ *to* $H(\lambda; t, s)\in L^2(E)$, *strongly continuous in* t *for each* $\xi\in L^2(E)$. *Also, the similar statement in* s *as above holds.*

Proof. By Proposition 3.1, it is sufficient to prove (3.3) for each $\xi\in C_0^\infty(E)$. We define a cut off function $\chi\in C_0^\infty(M)$ as $\chi(x)\equiv 1$ if $d(x, \operatorname{supp}\xi)\leqq 2$ and $\equiv 0$ if $d(x, \operatorname{supp}\xi)\geqq 3$. We show the following:

$$\text{(3.4)}\qquad \lim_{t\downarrow s}\|H_1(\lambda; t, s)\xi-\xi\|_{L^2(E)}=0\,,$$

(3.5) $$\lim_{t\downarrow s}\|H_2(\lambda;t,s)\xi\|_{L^2(E)}=0,$$

where $H_1(\lambda;t,s)=\chi(x)H(\lambda;t,s)\xi(x)$ and $H_2(\lambda;t,s)=(1-\chi(x)H(\lambda;t,s)\xi(x)$.

For proving (3.4), putting $y=Exp_x r\omega$, $\omega\in S_xM$, we get

$$\tilde{\xi}(x,y)=\xi(x)+\tilde{\xi}_1(x;r\omega),\quad \rho(x,y)=1+\rho_1(x;r\omega),$$

where $\tilde{\xi}(x,y)=P(x,y)\xi(y)\in E_x$ and

$$\tilde{\xi}_1(x;r\omega)=\int_0^r D_{\gamma_c(\tau)}\tilde{\xi}(x,\gamma_c(\tau))d\tau$$

$$\rho_1(x;r\omega)=\int_0^r(d/d\tau)\rho(x,\gamma_c(\tau))d\tau,$$

where $\gamma_c(\tau)=Exp_x\tau\omega$. Using Lemma 2.1 and Proposition 2.6, we get the following, which is similar to that in Proposition 2.2, [13]:

(3.6) $$|H_1(\lambda;t,s)\xi(x)-\xi(x)|_x\leqq C_2\chi(x)\operatorname{vol}(S^{m-1})(t-s)^{1/2}\sup_{x\in M}[|D\xi|_x+|\xi|_x]\times\int_0^\infty r^m\exp(-[\lambda r^2/2(t-s)-k_8(t-s)r])dr,$$

with some positive constants C_2 and k_8. Therefore, by integrating (3.6), there exists a positive constant $C_2'=C_2'(\lambda;T)$ depending on the support of ξ such that

(3.7) $$\|H_1(\lambda;t,s)\xi-\xi\|_{L^2(E)}\leqq C_2'(t-s)^{1/2}\sup_{x\in M}[|D\xi|_x+|\xi|_x],$$

which gives (3.4).

Also, following the same way as in Proposition 2.2, [13] and using Lemma 2.1, Proposition 2.2 and Proposition 2.6, we get (3.5).

For later use, we give some properties of the kernel function $H(\lambda;t,s;x,y)$ of (3.1), which is proved analogously as in Proposition 3.2.

LEMMA 3.3. *Assume* (A.0)-(A.2). *Let $\xi(\tau,y)$ be a continuous, bounded mapping from $[s,t]\times M$ to E such that for any fixed τ, $\xi(\tau,\cdot)\in C(E)$, and put*

(3.8) $$\xi(t,\tau;x)=\int_M H(\lambda;t,\tau;x,y)\#_y\xi(\tau,y)d\mu_g(y),$$

where $0\leqq s<\tau<t<T$, and $\#_y$ is the inner product between E_y and E_y^*. Then, the following properties hold:

(3.10) $$(\partial/\partial t)\xi(t,\tau;x)=\int_M(\partial/\partial t)H(\lambda;t,\tau;x,y)\#_y\xi(\tau,y)d\mu_g(y),$$

(3.11) $$(D_x)^\alpha\xi(t,\tau;x)=\int_M(D_x)^\alpha H(\lambda;t,\tau;x,y)\#_y\xi(\tau,y)d\mu_g(y),\quad 0\leqq|\alpha|\leqq 2,$$

and

(3.12) $$\lim_{\substack{t>\tau\geq s,\\ t\to s}} \xi(t, \tau; x)=\xi(s, x)$$

Similarly, let $\xi^*(\tau, y)$ be a continuous, bounded mapping from $[s, t]\times M$ to E^* such that for each fixed τ, $\xi^*(\tau, \cdot)\in C(E^*)$. We get

LEMMA 3.4. *Under the same assumptions as in Lemma* 3.3, *we have*

(3.13) $$\lim_{\substack{s\leq\tau<t,\\ t\to s}} \int_M \xi^*(\tau, x)^{\#}_x H(\lambda; t, s; x, y)d\mu_g(x)=\xi^*(t, x).$$

§4. Convergence of the product integral in the operator norm.

In this section we shall show the parts (a) and (b) in Theorem 1.2. Take $T>0$ arbitrarily and fix it.

By the direct computation using the Hamilton-Jacobi equation for $S(t, s; x, y)$, the continuity equation for $\rho(t, s; x, y)$ (Cf. Lemma 1.1 and Lemma 1.5 in [13]), and Lemma 2.3 (iii), we have the following properties for the kernel function $H(\lambda; t, s; x, y)$:

(4.1) $$\begin{aligned}&[(\partial/\partial s)+(2\lambda)^{-1}\Delta^D_y]H(\lambda; t, s; x, y)\\ &\quad=(2\lambda)^{-1}(2\pi\lambda^{-1})^{-m/2}e^{-\lambda S}[2\langle\nabla_y\rho, D_yP\rangle+\Delta_y\rho P+\rho\Delta^D_yP]\end{aligned}$$

and

(4.2) $$\begin{aligned}&[(\partial/\partial t)-(2\lambda)^{-1}\Delta^D_x]H(\lambda; t, s; x, y)\\ &\quad=-(2\lambda)^{-1}(2\pi\lambda^{-1})^{-m/2}e^{-\lambda S}[2\langle\nabla_x\rho, D_xP\rangle+\Delta_x\rho P+\rho\Delta^D_xP].\end{aligned}$$

For $\xi\in C^\infty_0(E)$ and $0\leq s<t<t'<T$, we may write

(4.3) $$\begin{aligned}&H(\lambda; t+t', s)\xi(x)-H(\lambda; t+t', t')H(\lambda; t', s)\xi(x)\\ &\quad=\int_M H(\lambda; t, t', s; x, y)^{\#}_y\xi(y)d\mu_g(y),\end{aligned}$$

where

$$\begin{aligned}&H(\lambda; t, t', s; x, y)\\ &\quad=H(\lambda; t+t', s; x, y)-\int_M H(\lambda; t+t', t'; x, z)^{\#}_z H(\lambda; t'x; z, y)d\mu_g(z).\end{aligned}$$

Since $H(\lambda; t, s; x, y)$ has a singularity at $t=s$, we define, for positive ε,

(4.4) $$\begin{aligned}&H^\varepsilon(\lambda; t, t', s; x, y)\\ &\quad=\int_{s+\varepsilon}^t \frac{d}{d\sigma}\int_M H(\lambda; t+t', \sigma; x, z)^{\#}_z H(\lambda; \sigma, s; z, y)d\mu_g(z),\end{aligned}$$

which satisfies $\lim_{\varepsilon\downarrow 0} H^\varepsilon(\lambda; t, t', s; x, y)=H(\lambda; t, t', s; x, y)$ for any (t, t', s, x, y), $x\neq y$. Exchanging the differentiation and the integral in (4.4), we have

$$H^{\varepsilon}(\lambda\,;t,\,t',\,s\,;x,\,y) = -\int_{s+\varepsilon}^{t}[2\pi\lambda^{-1}(t+t'-\sigma)]^{-m/2}[2\pi\lambda^{-1}(\sigma-s)]^{-m/2}(2\lambda)^{-1} \times\int_{M}\sum_{i=1}^{3}h_i(\lambda\,;t,\,t',\,s,\,\sigma\,;x,\,y,\,z)d\mu_g(z)d\sigma\,, \tag{4.5}$$

where

$$h_1(\lambda\,;t,\,t',\,s,\,\sigma\,;x,\,y,\,z) =[\Delta_z\rho(x,\,z)-\Delta_z\rho(y,\,z)][\exp(-\lambda[S(t+t',\,\sigma\,;x,\,z)+S(\sigma,\,x\,;z,\,y)])] \times P(x,\,z)_z^{\#}P(z,\,y)\,, \tag{4.6}$$

$$h_2(\lambda\,;t,\,t',\,s\,;x,\,y,\,z) =[\exp(-\lambda[S(t+t',\,\sigma\,;x,\,z)+S(\sigma,\,s\,;z,\,y)])] \times[\langle\nabla_z\rho(x,\,z),\,D_zP(x,\,z)\rangle_{z}{}_z^{\#}P(z,\,y)-P(x,\,z)_z^{\#}\langle\nabla_z(y,\,z),\,D_zP(y,\,z)\rangle_z] \tag{4.7}$$

and

$$h_3(\lambda\,;t,\,t',\,s,\,\sigma\,;x,\,y,\,z) =[\exp(-\lambda[S(t+t',\,\sigma\,;x,\,z)+S(\sigma,\,s\,;z,\,y)])]\rho(x,\,z)\rho(z,\,y) \times[\Delta_zP(x,\,z)_z^{\#}P(z,\,y)-P(x,\,z)_z^{\#}\Delta_zP(z,\,y)]\,. \tag{4.8}$$

Consider each terms $h_i(\lambda\,;t,\,t',\,s,\,\sigma\,;x,\,y,\,z)$, $i=1,\,2,\,3$, in (4.6-8). Remark that there exists a positive constant k_8 such that

$$|\Delta_z\rho(x,\,z)-\Delta_z\rho(x,\,z)_{|z=x}|\leqq\int_0^r|\nabla_{\dot\gamma_c(\tau)}\Delta_z\rho(x,\,\gamma_c(\tau))|\,d\tau \leqq k_8d(x,\,z)\exp(k_8d(x,\,z))\,, \tag{4.9}$$

and

$$|\Delta_z\rho(x,\,z)_{|z=x}-\Delta_z\rho(y,\,z)_{|z=y}|\leqq k_8d(x,\,y)\exp(k_8d(x,\,y)) \tag{4.10}$$

because of Proposition 2.2. By (4.9), (4.10) and using Proposition 2.6, we have with some constant $k_9>0$,

$$|h_1(\lambda\,;t,\,t',\,s,\,\sigma\,;x,\,y,\,z)|_{(x,y)} \leqq 2k_9[d(x,\,z)\exp(k_8d(x,\,z))+d(y,\,z)\exp(k_8d(y,\,z))] \times\exp(-\lambda[S(t+t',\,\sigma\,;x,\,z)+S(\sigma,\,s\,;z,\,y)])\,. \tag{4.11}$$

By a similar computation as in [13] using Proposition 2.2 and 2.6, there exists a constant $C_3=C_3(\lambda\,;T)>0$ such that for any $i=1,\,2,\,3$,

$$\int_M\int_M[2\pi\lambda^{-1}(t+t'-\sigma)]^{-m/2}[2\pi\lambda^{-1}(\sigma-s)]^{-m/2} \times|h_i(\lambda\,;t,\,t',\,s,\,\sigma\,;x,\,y,\,z)|_{(x,y)}d\mu_g(z)d\mu_g(y) \leqq C_3[(t+t'-\sigma)^{1/2}+(\sigma-s)^{1/2}]\,, \tag{4.12}$$

and

$$\int_M \int_M [2\pi\lambda^{-1}(t+t'-\sigma)]^{-m/2}[2\pi\lambda^{-1}(\sigma-s)]^{-m/2} \times |h_i(\lambda; t, t', s, \sigma; x, y, z)|_{(x,y)} d\mu_g(z) d\mu_g(x) \leqq C_3[(t+t'-\sigma)^{1/2}+(\sigma-s)^{1/2}]. \tag{4.13}$$

Taking a limit as $\varepsilon\to 0+$, and doing the similar computations as in [13], § 3, we have

PROPOSITION 4.1. *For any t, t', s, $0\leqq s<t'<t<t+t'<T$ and $\xi\in L^2(E)$, the following inequality holds:*

$$\|H(\lambda; t+t', s)\xi - H(\lambda; t+t', t')H(\lambda; t', s)\xi\|_{L^2(E)} \leqq C_0[(t+t'-s)^{3/2}-t^{3/2}+(t'-s)^{3/2}]\|\xi\|_{L^2(E)}, \tag{4.14}$$

with some constant $C_0=C_0(\lambda; T)$.

We devide a closed interval $[0, t]$, $0<t\leqq T$, into subintervals, i.e.

$$\sigma_N: 0=t_0<t_1<\cdots<t_{N-1}<t_N=t, \quad t_j=(j/N)t, \qquad j=0, 1, \cdots, N. \tag{4.15}$$

And we define the operator

$$H(\lambda; \sigma_N|t)=H(\lambda; t, t_{N-1})\cdots H(\lambda; t_1, 0). \tag{4.16}$$

Combining with Proposition 4.1 and the similar argument as in Lemma 4.6, [13], we have the following, which is the part (b) in Theorem 1.2 (Cf. [6]).

PROPOSITION 4.2. *Assume that (M, g) satisfies* (A.0)-(A.2). *Then, $\{H(\lambda; \sigma_N|t)\}$ forms a Cauchy sequence in $\mathcal{B}(L^2(E))$. Therefore, there exists a C^0 semi-group $\boldsymbol{H}(\lambda; t)$, $t>0$, in $L^2(E)$ such that for any $t>0$,*

$$\lim_{N\to\infty} \|\boldsymbol{H}(\lambda; t)-H(\lambda; \sigma_N|t)\|_{\mathcal{B}(L^2(E))}=0, \tag{4.17}$$

Moreover, there exists a positive constant $C_0'=C_0'(\ ; T)$ such that

$$\|\boldsymbol{H}(\lambda; t)-H(\lambda; \sigma_N|t)\|_{\mathcal{B}(L^2(E))}\leqq C_0' t N^{-1/2}(\exp(C_3 t^{1/2})). \tag{4.18}$$

Remark. By a slight modification of the above, we can generalize Proposition 4.2 for arbitrary subdivision of $[0, t]$ (Cf. [13] and [10]).

§ 5. Computation of the infinitesimal generator.

To finish the proof of Theorem 1.2, we only compute the infinitesimal generator of $\boldsymbol{H}(\lambda; t)$. Namely, we get

PROPOSITION 5.1. *Assume that* (M, g) *satisfies* (A.0)-(A.2). *Then,*

$$(\partial/\partial t)\boldsymbol{H}(\lambda; t)\xi(x)=\lambda^{-1}[(1/2)\Delta_x^D-(1/12)Scal_g(x)]\boldsymbol{H}(\lambda; t)\xi(x) \tag{5.1}$$

for $\xi\in C_0^\infty(E)$.

To prove the above proposition, we remark the following, which proved as same as in Lemma 4.2, [13],

LEMMA 5.2. *Given* $\xi\in C_0^\infty(E)$, *we have*

$$(\partial/\partial t)\boldsymbol{H}(\lambda; t)\xi(x)_{|t=0}=(\partial/\partial t)H(\lambda; t, 0)\xi(x)_{|t=0}. \tag{5.2}$$

Since $\boldsymbol{H}(\lambda; t)$ is a C^0 semi-group, it is sufficient for proving Proposition 5.1 to show the following:

PROPOSITION 5.3. *Under the same assumptions as in Proposition* 5.1, *we have for any* $\xi\in C_0^\infty(E)$ *and* $x\in M$,

$$H(\lambda; t, 0)\xi(x)-\xi(x)=t\lambda^{-1}\mathcal{H}_x\xi(x)+tG(t; \xi), \tag{5.3}$$

where $\mathcal{H}_x\xi(x)=[(1/2)\Delta_x^D-(1/12)Scal_g(x)]\xi(x)$ *and* $G(t; \xi)$ *satisfies*

$$\lim_{t\to 0}\|G(t; \xi)\|_{L^2(E)}=0. \tag{5.4}$$

Proof. Recall (4.2). By using the integration by parts, we get

$$\begin{aligned}(\partial/\partial t)H(\lambda; t, 0)\xi(x)&-(\lambda^{-1}/2)H(\lambda; t, 0)\Delta_x^D\xi(x)\\ &=\lambda^{-1}(2\pi\lambda^{-1}t)^{-m/2}\int_M[\exp(-\lambda S(t, 0; x, y))]Q(x, y)\xi(y)d\mu_g(y),\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}Q(x, y)=&-(1/2)\Delta_y\rho(x, y)P(x, y)+\langle\nabla_y\rho(x, y), D_yP(x, y)\rangle_y\\ &+(1/2)\rho(x, y)\Delta_y^DP(x, y).\end{aligned} \tag{5.6}$$

Noticing that $\Delta_y\rho(x, y)_{|y=x}=(1/6)Scal_g(x)$ (Cf. [3], [13]), we have

$$\begin{aligned}\lambda\tilde{G}(t; \xi)=&(\partial/\partial t)H(\lambda; t, 0)\xi(x)-(1/2)H(\lambda; t, 0)\Delta^D\xi(x)\\ &+(1/12)Scal_g(x)\xi(x),\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}\tilde{G}(t; \xi)=&(1/2)[H(\lambda; t, 0)-I]\xi(x)\\ &-(1/2)(2\pi\lambda^{-1}t)^{-m/2}\int_M[\exp(-\lambda S(t, 0; x, y))]\\ &\times[q_1(x, y)+q_2(x, y)]\xi(y)d\mu_g(y),\end{aligned} \tag{5.8}$$

$$(5.9)\qquad \begin{cases} q_1(x, y)=[\Delta_y\rho(x, y)-\Delta_y\rho(x, y)_{|x=y}]P(x, y), \\ q_2(x, y)=2\langle\nabla_y\rho(x, y), D_xP(x, y)\rangle+\rho(x, y)\Delta_yP(x, y) \end{cases}$$

Using Proposition 3.2, we have, for some constant $C_4=C_4(\lambda; T)>0$,

$$(5.10)\qquad \begin{aligned} &\|\boldsymbol{H}(\lambda; t, 0)\Delta^D\xi(x)-\Delta^D\xi(x)\|_{L^2(E)} \\ &\leqq C_4t^{1/2}\sup_{x\in M}[|\nabla\xi|_x+|\xi|_x]+O(t; \xi), \end{aligned}$$

and

$$(5.11)\qquad \lim_{t\to 0+} O(t; \xi)=0.$$

Also, by Proposition 2.2-2.6, we have

$$(5.12)\qquad |q_i(x, y)|_{(x,y)}\leqq k_9\exp(k_9d(x, y)), \qquad i=1, 2.$$

for any $x, y\in M$ with some constant $k_9>0$. Then, we get for some constant $C_5=C_5(\lambda; T)$,

$$(5.13)\qquad \|G(t; \xi)\|_{L^2(E)}\leqq[C_5t^{1/2}\exp(k_9t^{1/2})]\|\xi\|_{L^2(E)}.$$

Remarking $H(\lambda; t, 0)\xi(x)-\xi(x)=\int_0^t(d/d\sigma)H(\lambda; \sigma, 0)\xi(x)d\sigma$, we have the desired results.

Proposition 5.3 gives the part (c) in Theorem 1.2. By §§4-5, we finish to prove Theorem 1.2 completely.

Now, for a later use in §6, we prepare the following properties: Let $\xi(\tau, y)$ be a mapping from $[s, t]\times M$ into E which satisfies

(i) $\xi(\tau, \cdot)\in C(E)$ for each fixed $\tau\in[s, t]$.

(ii) $\xi(\tau, y)$ is Hölder continuous in $[s, T)\times M$.

(iii) Given any closed interval $[s_1, t_1]\subset[s, t]$, $\xi(\tau, y)$ is bounded on $[s_1, t_1]\times M$.

(iv) For any $t\in[s, T)$, $\int_s^t d\tau\int_M|\xi(\tau, y)|_y d\mu_g(y)<+\infty$.

PROPOSITION 5.4. *Assume that (M, g) satisfies* (A.0)-(A.2). *Let $\xi(\tau, y)$ be as above. Put $\xi(t, \tau; x)$ and $\Xi(t, x)$ by*

$$\xi(t, \tau; x)=\int_M H(\lambda; t, \tau; x, y)\xi(\tau; y)d\mu_g(y),$$

$$\Xi(t, x)=\int_s^t\xi(t, \tau; x)d\tau,$$

Then, there exists a positive constant C_6 depending only the closed interval $[s_1, t_1]$ such that

$$(5.14)\qquad |(\partial/\partial t)\xi(t, \tau; x)|_x\leqq C_6(t-\tau)^{-(1-\gamma/2)}, \qquad s_1\leqq\tau<t<t_1,$$

where γ is the Hölder exponent of ξ at (t, x). Also, in (5.14) *the same inequalities replacing $\partial/\partial t$ by D_x and Δ_x^D hold. Moreover, we have*

$$\mathcal{H}_x \Xi(t, x)=\int_s^t \mathcal{H}_x \xi(t, \tau ; x) d\tau , \tag{5.15}$$

and

$$(\partial/\partial t)\xi(t, x)=\xi(t, s)+\int_s^t (\partial/\partial t)\xi(t, \tau ; x) d\tau . \tag{5.16}$$

Proof. Given any $(t, x)\in[s, T)\times M$, let γ be the Hölder exponent of ξ at this point. Take a closed interval $[s_1, t_1]$ such that $s<s_1<t_1<T$. Then, there exists a positive constant C_6' and δ, $0<\delta<1$, such that $s_1<t-\delta$ and if $|t-\tau|<\delta$ and $d(x, y)<\delta$, then

$$|P(x, y)\xi(\tau, y)-\xi(t, x)|_x \leqq C_6'(|t-\tau|^\gamma+(d(x, y))^\gamma) \tag{5.17}$$

and if $t-\delta<\tau<t<t'$, then

$$\int_M (\partial/\partial t')H(\lambda ; t', \tau ; x, y)\xi(\tau, y) d\mu_g(y)=-I_1-I_2+I_3 , \tag{5.18}$$

where

$$I_1=\int_{d(x, y)\leqq\delta} (\partial/\partial t')H(\lambda ; t', \tau ; x, y)[P(y, x)\xi(t, x)-\xi(\tau, y)] d\mu_g(y) ,$$

$$I_2=\int_{d(x, y)\geqq\delta} (\partial/\partial t')H(\lambda ; t', \tau ; x, y)[P(y, x)\xi(t, x)-\xi(\tau, y)] d\mu_g(y) ,$$

$$I_3=\int_M H(\lambda ; t', \tau ; x, y)\sharp_y P(y, x)\xi(t, x) d\mu_g(y) .$$

So, there exists a constant $C_6''>0$ such that

$$|I_1|_x \leqq C_6''(t-\tau)^{-(1-\gamma/2)}, \quad |I_2|_x \leqq C_6'' , \quad |I_3|_x \leqq C_6''(t-\tau)^{-1/2} \tag{5.19}$$

which implies if $t-\delta\leqq t'$, then

$$|(\partial/\partial t)\xi(\lambda ; t, \tau ; x)|_x \leqq C_6'''(t'-\tau)^{-(1-\gamma/2)} \tag{5.20}$$

with some constant $C_6'''>0$. On the other hand, if $s_1\leqq\tau\leqq t-s$, then $t'-\tau\geqq\delta>0$. Therefore, we see that $(\partial/\partial t')\xi(\lambda ; t', \tau ; x)$ is uniformly bounded in (t', τ, x), because of the form $(\partial/\partial t)H(\lambda ; t', \tau ; x, y)$ and (iii) of the properties of ξ. So, we have the estimate (5.16). Other estimates are obviously obtained. Now, by properties (iii) and (iv) of $\xi(\tau, x)$, we have for some constant C_7,

$$|D_x\xi(\lambda ; t, \tau ; x)|_x \leqq C_7(t-\tau)^{-(1-\gamma/2)}$$

$$|\Delta_x^D\xi(\lambda ; t, \tau ; x)|_x \leqq C_7(t-\tau)^{-(1-\gamma/2)}, \qquad s_1\leqq\tau\leqq t<t_1 .$$

Remarking $\int_s^t (t-\tau)^{-(1-\gamma/2)} d\tau < +\infty$, we can interchanging the operator $\mathcal{H}_x$ and the integral. So, we have (5.15). Similarly, we get (5.16).

As a direct consequence of Proposition 5.4, and Lemma 3.3, we have

COROLLARY 5.5. *Under the same notations and assumptions as in Proposition 5.4, we have*

$$\mathcal{H}_x \Xi(t, x) = \int_s^t d\tau \int_M \mathcal{H}_x H(\lambda; t, \tau; x, y) \xi(\tau, y) d\mu_g(y), \tag{5.21}$$

$$(\partial/\partial t)\Xi(t, x) = \xi(t, x) + \int_s^t d\tau \int_M (\partial/\partial t) H(\lambda; \tau, s; x, y) \xi(\tau, y) d\mu_g(y). \tag{5.22}$$

§ 6. Construction of the fundamental solution.

To prove the main theorem stated in the introduction, we shall construct the fundamental solution for the following system of parabolic equations:

$$(\partial/\partial t)\xi(t, x) = \lambda^{-1} \mathcal{H}_x \xi(t, x), \quad \xi(0, x) = \xi_0(x) \in C_0(E), \tag{6.1}$$

where $\mathcal{H}_x = (1/2)\Delta_x^D - (1/12) Scal_g(x)$.

Throughout this section, we assume the assumptions (A.0)-(A.2). We denote by L the differential operator $-(\partial/\partial t) + \lambda^{-1}\mathcal{H}_x$. Recall the kernel function $H(\lambda; t, s; x, y)$ of $H(\lambda; t, s)$, $0 \leqq s < t < T$ in (4.1). Set

$$\begin{aligned} J_0(\lambda; t, s; x, y) &= LH(\lambda; t, s; x, y) \\ &= -[(\partial/\partial t) - \lambda^{-1}\mathcal{H}_x] H(\lambda; t, s; x, y). \end{aligned} \tag{6.2}$$

LEMMA 6.1. *For any $0 \leqq s < t < T$ and $x, y \in M$, there exists positive constants M_0, k_6 and $M_1 = M_1(\lambda; T)$ such that*

$$\begin{aligned} &|H(\lambda; t, s; x, y)|_{(x,y)} \\ &\quad \leqq M_0 \lambda^{m/2} (t-s)^{-m/2} \exp[-\lambda(d^2(x, y)/(2(t-s))] \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} &|J_0(\lambda; t, s; x, y)|_{(x,y)} \\ &\quad \leqq M_0 (M_1/\varepsilon)(\exp(k_6 T/\varepsilon\lambda)) \lambda^{m/2-2} (t-s)^{-(m-1)/2} \\ &\qquad \times \exp[-\lambda(\varepsilon^* d^2(x, y)/(2(t-s))], \end{aligned} \tag{6.4}$$

where $\varepsilon^* = 1 - \varepsilon$.

Proof. (6.3) is easily obtained by the form (3.1) and Lemma 2.4. For (6.4), we have by (4.1)

$$(6.5)\qquad \begin{aligned}&|J_0(\lambda\,;t,\ s\,;\ x,\ y)|_{(x,y)}\\ &\leqq M_0\lambda^{m/2-1}k_5(t-s)^{-m/2}d(x,\ y)\\ &\qquad\qquad \times\exp[-\lambda(d^2(x,\ y)/(2(t-s))-k_5d(x,\ y))]\end{aligned}$$

where k_5 is a positive constant. Putting the function $F(r)$, $r\geqq 0$, by

$$F(r)=r[\exp(k_5r-\lambda\varepsilon r^2/(2(t-s))]\,,$$

we have

$$(6.6)\qquad \begin{aligned}F(r)&\leqq(2\lambda\varepsilon)^{-1}(t-s)^{1/2}[k_5(t-s)^{1/2}+((k_5)^2(t-s)+4\lambda\varepsilon)^{1/2}]\\ &\quad\times[\exp((4\lambda\varepsilon)^{-1}(k_5(t-s)+((k_5)^2(t-s)+4\lambda\varepsilon))^{1/2}(t-s)^{1/2}))]\\ &\leqq C_8'\lambda^{-1}\varepsilon^{-1}[\exp(k_6T/\varepsilon\lambda)](t-s)^{1/2},\end{aligned}$$

with some constants $C_8'=C_8'(\lambda\,;T)$ and k_6. Substituting (6.6) into (6.5), we get, for any $x,\ y\in M$,

$$(6.7)\qquad \begin{aligned}&|J_0(\lambda\,;t,\ s\,;\ x,\ y)|_{(x,y)}\\ &\leqq M_0C_8'\lambda^{m/2-2}\varepsilon^{-1}[\exp(k_6T/\varepsilon\lambda)](t-s)^{-(m-1)/2}\\ &\qquad\qquad \times\exp[-\lambda(\varepsilon^*d^2(x,\ y)/(2(t-s))]\end{aligned}$$

So, we get Lemma 6.1.

Now, we put

$$(6.8)\qquad J_1(\lambda\,;t,\ s\,;\ x,\ y)=\int_s^t d\sigma\int_M J_0(\lambda\,;t,\ \sigma\,;\ x,\ z)_z^{\#}J_0(\lambda\,;\ \sigma,\ s\,;z,\ y)d\mu_g(z)\,.$$

LEMMA 6.2. *For any* $0\leqq s<t<T$, *and* $x,\ y\in M$, *there exists positive constants* $M_2=M_2(\lambda\,;T)$ *such that for any* $0<\varepsilon<1/2$,

$$(6.9)\qquad \begin{aligned}&|J_1(\lambda\,;t,\ s\,;\ x,\ y)|_{(x,y)}\\ &\leqq M_0(M_2/\varepsilon)^2(\varepsilon^*)^{-m/2}[\exp(k_6T/\varepsilon\lambda)]\lambda^{-m/2-2}(t-s)^{-(m/2)+2}\\ &\qquad\qquad \times B(3/2:3/2)[\exp(-\lambda(\varepsilon^{**}d^2(x,\ y)/(2(t-s)))]\,,\end{aligned}$$

where $\varepsilon^{**}=1-2\varepsilon$, *and* $B(\ :\ )$ *is the Beta function.*

Proof. First, we put

$$(6.10)\qquad J_1(\lambda\,;t,\ s,\ \sigma\,;\ x,\ y)=\int_M J_0(\lambda\,;t,\ \sigma\,;\ x,\ z)_z^{\#}J_0(\lambda\,;\ \sigma,\ s\,;z,\ y)d\mu_g(z)\,.$$

By the comparison theorem, we have $d^2(z,\ y)\geqq|Z-Y|^2$, if we can write $y=Exp_xY$ and $z=Exp_xZ$. Thus, we get

$$|J_1(\lambda; t, s, \sigma; x, y)|_{(x,y)}$$
$$\leqq M_0^2 M_1^2[\exp(k_6 T/\varepsilon\lambda)]^2 \lambda^{m-4}\varepsilon^{-2}(t-\sigma)^{-(m-1)/2}(\sigma-s)^{-(m-1)/2}$$
$$\times\int_{T_xM}\exp -(\lambda(\varepsilon^*|Z|^2/(2(t-\sigma))$$
$$+\varepsilon^*|Z-Y|^2/(2(\sigma-s)))-k|Z|)dZ,$$

with some constant $k_5>0$. Since

$$|Z|^2/(2(t-\sigma))+|Z-Y|^2/(2(\sigma-s))$$
$$=(t-s)/(2(t-\sigma)(\sigma-s))|Z-((t-\sigma)/(t-s))Y|^2$$
$$+(1/(2(t-s)))|Y|^2,$$

we have

$$|J_1(\lambda; t, s, \sigma; x, y)|_{(x,y)} \tag{6.11}$$
$$\leqq M_0^2(M_1/\varepsilon)^2[\exp(2k_6T/\varepsilon\lambda)]\lambda^{m-4}(2/\lambda\varepsilon^*)^{m/2}(t-s)^{-m/2}$$
$$\times(t-\sigma)^{1/2}(\sigma-s)^{1/2}[\exp -((\lambda\varepsilon^* d^2(x, y)/(2(t-s))-k_5 d(x, y))]$$
$$\times\int_{T_xM}\exp(-(|Z'|^2-k_5(2(t-\sigma)(\sigma-s)/(\lambda\varepsilon^*(t-s)))^{1/2}|Z'|)dZ'$$

because of $0<(t-\sigma)/(t-s)<1$. By an easy computation, we get

$$|J_1(\lambda; t, s, \sigma; x, y)|_{(x,y)} \tag{6.12}$$
$$\leqq(\pi/2)^{1/2}(2^{m/2})M_0^2(M_1/\varepsilon)^2(\varepsilon^*)^{-m/2}\lambda^{(m/2)-4}[\exp(2k_6T/\varepsilon\lambda)]$$
$$\times(t-\sigma)^{1/2}(\sigma-s)^{1/2}(t-s)^{-m/2}\mathrm{vol}(S^{m-1})[\exp(t-s)k_5^2/8\lambda\varepsilon^*]$$
$$\times[\exp(-\lambda(\varepsilon^* d^2(x, y)/(2(t-s))-k_5 d(x, y)))].$$

Therefore, there exists positive constants M_2 and k_7 such that we have

$$|J_1(\lambda; t, s; x, y)|_{(x,y)} \tag{6.13}$$
$$\leqq M_0^2(M_1/\varepsilon)^2(\varepsilon^*)^{-m/2}\lambda^{(m/2)-4}[\exp(kT/\varepsilon\lambda)]^2(t-s)^{-m/2}$$
$$\times[\exp(-\lambda(\varepsilon^{**}d^2(x, y)/(2(t-s)))]\int_s^t(t-\sigma)^{1/2}(\sigma-s)^{1/2}d\sigma$$
$$\leqq M_0^2(M_2/\varepsilon)^2(\varepsilon^*)^{-m/2}\lambda^{(m/2)-4}[\exp(k_7T/\varepsilon\lambda)]^2(t-s)^{-(m-4)/2}$$
$$\times B(3/2; 3/2)[\exp(-\lambda(\varepsilon^{**}d^2(x, y)/(2(t-s)))],$$

where $\varepsilon^{**}=1-2\varepsilon$, which gives Lemma 6.2.

Successively, we define, for $n\geqq 1$,

$$J_n(\lambda; t, s; x, y)=\int_s^t d\sigma\int_M J_0(\lambda; t, \sigma; x, z)_z^\# J_{n-1}(\lambda; \sigma, s; z, y)d\mu_g(z). \tag{6.14}$$

By a similar computation as above, we get

LEMMA 6.3. *For any* $0 \leqq s < t < T$ *and* $x, y \in M$, *the following estimate holds: Given any* $0 < \varepsilon < 1/2$,

$$(6.15) \qquad |J_n(\lambda ; t, s ; x, y)|_{(x,y)} \leqq M_0^{n+1}(M_2/\varepsilon)^{n+1}[\exp(k_7 T/\varepsilon\lambda)]^{n+1}(\varepsilon^*)^{-m+n/2}(t-s)^{-(m-1-3n)/2} \times \lambda^{m/2-2(n+1)} \prod_{a=1}^{n} B(3/2 : 3(a+1)/2)[\exp(-\lambda(\varepsilon^{**}d^2(x, y)/(2(t-s)))] \quad (n \geqq 1),$$

where $\varepsilon^{**} = 1-2\varepsilon$.

Remark that

$$\prod_{a=1}^{n} B(3/2 : 3(a+1)/2) = \Gamma(3/2)^n/\Gamma(3(n+1)/2) \leqq k_8^n/n!.$$

for some positive constant k_8. Then, there exists positive constants $M_3 = M_3(\lambda ; T)$ and k_9 such that

$$(6.16) \qquad \sum_{n=0}^{\infty} |J_n(\lambda ; t, s ; x, y)|_{(x,y)} \leqq M_0 M_3 \lambda^{m/2-2}[\exp((M_3\varepsilon^{-1}\lambda^{-2}(\varepsilon^*)^{-m/2})\exp(k_9 T/\lambda\varepsilon))](t-s)^{-(m-1)/2} \times [\exp M_3(t-s)^{3/2}][\exp -\lambda(\varepsilon^{**}d^2(x, y)/(2(t-s)))].$$

Thus, on $\{(t-s) | 0 \leqq s < t < T\} \times M \times M$, we can define a function

$$(6.17) \qquad K(\lambda ; t, s ; x, y) = \sum_{n=0}^{\infty} J_n(\lambda ; t, s ; x, y)$$

and for any $C > 1$, on $\{(t, s) | 0 \leqq s < t < T, C^{-1} < t-s < C\} \times M \times M$, the infinite sum of (6.17) converges uniformly on each compact set, and we have

$$(6.18) \qquad |K(\lambda ; t, s ; x, y)|_{(x,y)} \leqq M_0 M_3 \lambda^{m/2-2}[\exp(M_3\varepsilon^{-1}\lambda^{-2}(\varepsilon^*)^{-m/2} \exp(k_9 T/\lambda\varepsilon))][\exp M_3(t-s)] \times (t-s)^{-(m-1)/2}[\exp -\lambda(\varepsilon^{**}d^2(x, y)/(2(t-s))].$$

Moreover, by a direct computation, we get

LEMMA 6.4. *Let* $J_n(\lambda ; t, s ; x, y)$ *be the function defined by* (6.14). *For any* $0 \leqq s < t < T$, *there exist constants* $M_4 = M_4(\lambda ; T) > 0$ *and* $k_{10} > 0$ *such that*

$$(6.19) \qquad \sum_{n=0}^{\infty} \int_M |J_n(\lambda ; t, s ; x, y)|_{(x,y)} d\mu_g(x) \leqq M_0 M_4 \lambda^{m/2-2}[\exp((M_4\varepsilon^{-1}\lambda^{-2}(\varepsilon^*)^{-m/2} \exp(k_{10}T/\lambda\varepsilon))][\exp(M_4(t-s))]$$

$$(6.20) \qquad \sum_{n=0}^{\infty} \int_M |J_n(\lambda ; t, s ; x, y)|_{(x,y)} d\mu_g(y) \leqq M_0 M_4 \lambda^{m/2-2}[\exp((M_4\varepsilon^{-1}\lambda^{-2}(\varepsilon^*)^{-m/2})\exp(k_{10}T/\lambda\varepsilon))][\exp(M_4(t-s))]$$

Therefore, we have

$$\int_M |K(\lambda;t,s;x,y)|_{(x,y)}d\mu_g(y) \leqq M_0M_4\lambda^{m/2-2}[\exp((M_4\varepsilon^{-1}\lambda^{-2}(\varepsilon^*)^{-m/2}\exp(k_{10}T/\lambda\varepsilon))] \times[\exp(M_4(t-s))](t-s)^{1/2} \tag{6.21}$$

$$\int_M |K(\lambda;t,s;x,y)|_{(x,y)}d\mu_g(x) \leqq M_0M_4\lambda^{m/2-2}[\exp((M_4\varepsilon^{-1}\lambda^{-2}(\varepsilon^*)^{-m/2}\exp(k_{10}T/\lambda\varepsilon))] \times[\exp(M_4(t-s))](t-s)^{1/2}. \tag{6.22}$$

Now, fix (s, y) and consider $\xi(t, z)=K(\lambda;t,s;x,z)$. Applying Corollary 5.5, we have

$$\mathcal{H}_x\int_s^t d\sigma\int_M H(\lambda;t,\sigma;x,z)^{\#}_z K(\lambda;\sigma,s;z,y)d\mu_g(z) = \int_s^t d\sigma\int_M \mathcal{H}_x H(\lambda;t,\sigma;x,y)^{\#}_z K(\lambda;\sigma,s;z,y)d\mu_g(z), \tag{6.23}$$

Thus, we get

$$(\partial/\partial t)\int_s^t d\sigma\int_M H(\lambda;t,\sigma;x,z)^{\#}_z K(\lambda;\sigma,s;z,y)d\mu_g(z) = K(\lambda;t,s;x,y)+\int_s^t d\sigma\int_M (\partial/\partial t)H(\lambda;t,\sigma;x,z)^{\#}_z K(\lambda;\sigma,s;z,y)d\mu_g(z). \tag{6.24}$$

So, we have

$$-[(\partial/\partial t)-\lambda^{-1}\mathcal{H}_x]\int_s^t d\sigma\int_M H(\lambda;t,\sigma;x,z)^{\#}_z K(\lambda;\sigma,s;z,y)d\mu_g(z)$$
$$=K(\lambda;t,s;x,y)+\int_s^t d\sigma\int_M J_0(\lambda;t,\sigma;x,z)^{\#}_z K(\lambda;\sigma,s;z,y)d\mu_g(z)$$
$$=-J_0(\lambda;t,s;x,y).$$

Therefore, we obtain the following:

PROPOSITION 6.5. *Under the same assumptions and notations as above, put*

$$\boldsymbol{H}(\lambda;t,s;x,y) = H(\lambda;t,s;x,y)+\int_s^t d\sigma\int_M H(\lambda;t,\sigma;x,z)^{\#}_z K(\lambda;\sigma,s;z,y)d\mu_g(z). \tag{6.25}$$

Then, the following properties hold:

(i) $\boldsymbol{H}(\lambda;t,s;x,y)$ *is continuous in* $\{(t,s)\,|\,0\leqq s<t<T\}\times M\times M$.

(ii) $\boldsymbol{H}(\lambda;t,s;x,y)$ *satisfies*

(6.26) $$(\partial/\partial t)\boldsymbol{H}(\lambda;t,s;x,y)=\lambda^{-1}\mathcal{H}_x\boldsymbol{H}(\lambda;t,s;x,y)$$

(iii) *There exists a positive constant* $M_5=M_5(\lambda;T)$ *such that*

(6.27) $$|\boldsymbol{H}(\lambda;t,s;x,y)|_{(x,y)}$$
$$\leqq M_0M_5\lambda^{m/2}[\exp((M_5\varepsilon^{-1}\lambda^{-2}(\varepsilon^*)^{-m/2}\exp(k_{10}T/\lambda\varepsilon))][\exp(M_5(t-s))]$$
$$\times[\exp-\lambda(\varepsilon^{**}d^2(x,y)/(2(t-s))]$$

and

$$\int_M|\boldsymbol{H}(\lambda;t,s;x,y)|_{(x,y)}d\mu_g(x)$$
$$\leqq M_0M_5\lambda^{m/2}[\exp((M_5\varepsilon^{-1}\lambda^{-2}(\varepsilon^*)^{-m/2}\exp(k_{10}T/\lambda\varepsilon))][\exp(M_5(t-s))]$$

$$\int_M|\boldsymbol{H}(\lambda;t,s;x,y)|_{(x,y)}d\mu_g(y)$$
$$\leqq M_0M_5\lambda^{m/2}[\exp((M_5\varepsilon^{-1}\lambda^{-2}(\varepsilon^*)^{-m/2}\exp(k_{10}T/\lambda\varepsilon))][\exp(M_5(t-s))].$$

(iv) *Therefore, defining* $\boldsymbol{H}(\lambda;t,s)\xi(x)=\int_M\boldsymbol{H}(\lambda;t,s;x,y)\xi(y)d\mu_g(y)$, *we have a bounded linear operator* $\boldsymbol{H}(\lambda;t,s)$ *on* $L^2(E)$ *and is* C^0 *semi-group with infinitesimal generator* $\lambda^{-1}\mathcal{H}$ *in* (6.26).

Proposition 6.5 shows that $\boldsymbol{H}(\lambda;t,s;x,y)$ defined by (6.25) determines a fundamental solution for the parabolic equation (6.1).

Remark. Cheng et al. [5] gives the upper estimate for the heat kernel of Laplace-Beltrami operator acting on functions only by assuming the boundedness of the curvature tensor of g.

By a similar argument as in Lemma 4.4, we have

LEMMA 6.6. *Let* $\xi(\tau,z)$ *be a bounded continuous mapping from* $[s,t]\times M$ *to* E *such that for any fixed* $\tau\in[s,t]$, $\xi(\tau,\cdot)\in C(E)$. *Then, the following equalities hold uniformly on any compact set on* M:

(6.28) $$\begin{cases}\lim\limits_{\tau\to s}\displaystyle\int_M\xi(\tau,z)^{\#}_z\boldsymbol{H}(\lambda;\tau,s;x,y)d\mu_g(y)=\xi(s,y),\\ \lim\limits_{\tau\to t}\displaystyle\int_M\boldsymbol{H}(\lambda;t,\tau;z,y)^{\#}_z\xi(\tau,z)d\mu_g(z)=\xi(t,y).\end{cases}$$

§7. Convergence of the product integral as the kernel function.

In this section, we shall prove Theorem 1.3, which gives the main theorem in the introduction, using the fundamental solution constructed in §6. Let $[0,t]$ be any closed interval such that $0<t<T$, and T be any fixed positive number. Let σ_N be a N-equal subdivision of $[0,t]$;

(7.1) $$\sigma_N:0=t_0<t_1<\cdots<t_{N-1}<t_N=t,\quad t_j=(j/N)t.$$

We define a operator $H(\lambda;\sigma_N|t)$ associated with the subdivision σ_N:

$$H(\lambda;\sigma_N|t)=H(\lambda;t,t_{N-1})H(\lambda;t_{N-1},t_{N-2})\cdots H(\lambda;t_1,0)\,, \tag{7.2}$$

and we denote by $H(\lambda;\sigma_N|t;x,y)$ the kernel function of the operator (7.2), i.e.

$$\begin{aligned} H(\lambda;\sigma_N|t;x,y)=&\int_M\cdots\int_M H(\lambda;t,t_{N-1};x,z_{N-1})\#_{z_{N-1}}\cdots \\ &\cdots\#_{z_1}H(\lambda;t_1,0;z_1,y)d\mu_g(z_{N-1})\cdots d\mu_g(z_1)\,, \end{aligned} \tag{7.3}$$

where $H(\lambda;t,s;x,y)$ is defined in (4.1).

To prove Theorem 1.3, we need several steps as below. First, put

$$R(\lambda;t,s)=H(\lambda;t,s)-\boldsymbol{H}(\lambda;t,s) \tag{7.4}$$

and denote by $R(\lambda;t,s;x,y)$ the kernel function of (7.4). Then, we get the following, which gives Corollary stated in §0:

PROPOSITION 7.1. *For any* $0<\varepsilon<1/2$, *there exists a positive constant* $\gamma_1=\gamma_1(\lambda;T,\varepsilon)$ *such that*

$$\begin{aligned} &|R(\lambda;t,s;x,y)|_{(x,y)} \\ &\leqq M_0\gamma_1(t-s)^{-(m-3)/2}[\exp -\lambda(\varepsilon^{**}d^2(x,y)/(2(t-s)))] \end{aligned} \tag{7.5}$$

where $\varepsilon^{**}=1-2\varepsilon$.

Proof. Combining (6.25) with (7.4), we have

$$\begin{aligned} &|R(\lambda;t,s;x,y)|_{(x,y)} \\ &\leqq M_0\gamma_1\int_s^t(t-\sigma)^{-m/2}(\sigma-s)^{-(m-1)/2} \\ &\quad\times\int_{T_xM}\exp -\lambda(\varepsilon^{**}(|Z|^2/(2(t-\sigma))-|Z-Y|^2/(2(\sigma-s)))dZd\sigma \\ &\leqq M_0\gamma_1(t-s)^{-(m-3)/2}[\exp -\lambda(\varepsilon^{**}d^2(x,y)/(2(t-s))]\,, \end{aligned} \tag{7.6}$$

by a similar computations as in §6. Thus, we get (7.5).

Now, we obtain

$$\begin{aligned} &H(\lambda;\sigma_N|t)-\boldsymbol{H}(\lambda;t) \\ &=H(\lambda;t,t_{N-1})\cdots H(\lambda;t_1,0)-\boldsymbol{H}(\lambda;t) \\ &=[\boldsymbol{H}(\lambda;t,t_{N-1})+R(\lambda;t,t_{N-1})]\cdots \\ &\qquad\cdots[\boldsymbol{H}(\lambda;t_1,0)+R(\lambda;t_1,0)]-\boldsymbol{H}(\lambda;t,0)\,. \end{aligned} \tag{7.7}$$

Using the evolutional property of $\boldsymbol{H}(\lambda;t,s)$, we shall write down the right hand side of (7.7). Let

(7.8) $$\mathcal{P}=\left\{\begin{array}{c}(\alpha_1, \cdots, \alpha_k ; \beta_1, \cdots, \beta_{k+1}) ; k=1, \cdots, N, \alpha_i>0, \beta_i \geqq 0, \\ \sum_{i=1}^{k}[\alpha_i+\beta_i]+\beta_{k+1}=N\end{array}\right\}.$$

Also, we denote by

(7.9) $$A_j=\alpha_1+\cdots+\alpha_j, \quad B_j=\beta_1+\cdots+\beta_j$$
$$j \geqq 1, \quad A_0=0, \quad B_0=0.$$

Thus, $A_k+B_k=N$. The right hand side of (7.7) is written by

(7.10) $$H(\lambda ; \sigma_N | t)-\boldsymbol{H}(\lambda ; t)$$
$$=\sum_{(\alpha_1, \cdots, \alpha_k ; \beta_1, \cdots, \beta_{k+1}) \in \mathcal{P}} \mathcal{T}(\alpha_1, \cdots, \alpha_k ; \beta_1, \cdots, \beta_{k+1})$$

where

(7.11) $$\mathcal{T}(\alpha_1, \cdots, \alpha_k ; \beta_1, \cdots, \beta_{k+1})$$
$$=\boldsymbol{H}(\lambda ; t, (A_k+B_k) t / N)$$
$$\times R(\lambda ;(A_k+B_k) t / N, (A_k-1+B_k) t / N) \cdots$$
$$\cdots R(\lambda ;(A_{k-1}+1+B_k) t / N, (A_{k-1}+B_k) t / N)$$
$$\times \boldsymbol{H}(\lambda ;(A_{k-1}+B_k) t / N, (A_{k-1}+B_{k-1}) t / N)$$
$$\times R(\lambda ;(A_1+B_1) t / N, (A_1-1+B_1) t / N) \cdots$$
$$\cdots R(\lambda ;(1+B_1) t / N, B_1 t / N)$$
$$\times \boldsymbol{H}(\lambda ; B_1 t / N, 0).$$

Now, we put

(7.12) $$R^{(j)}(\lambda ; A_j, B_j | t)=R(\lambda ;(A_j+B_j) t / N, (A_j-1+B_j) t / n) \cdots$$
$$\cdots R(\lambda ;(A_{j-1}+1+B_j) t / N, (A_{j-1}+B_j) t / N)$$

and denote by $R^{(j)}(\lambda ; A_j, B_j | t ; x, y)$ the kernel function of (7.12).

LEMMA 7.2. *Given any* $0<\varepsilon<1/3$, *there exists a positive constant* $\gamma_2=\gamma_2(\lambda ; T, \varepsilon)$ *and* $\gamma_3=\gamma_3(\lambda ; T, \varepsilon)$ *such that*

(7.13) $$|R^{(j)}(\lambda ; A_j, B_j, t ; x, y)|_{(x, y)}$$
$$\leqq M_0^{\alpha_j} \gamma_2^{\alpha_j}[\exp (\gamma_3(\alpha_j / N) t](t / N)^{3 \alpha_j / 2}(\alpha_j t / N)^{-m / 2}$$
$$\times[\exp -\lambda(\varepsilon^{***} d^2(x, y) /(2 t_j / N))]$$

where $\varepsilon^{***}=1-3 \varepsilon$.

Proof. Generally, take $t_1, \cdots, t_a \in[0, t)$, $o \leqq t_1<\cdots<t_a<t$, and put

(7.14) $$R(\lambda ; t_a, \cdots, t_1)=R(\lambda ; t_a, t_{a-1}) \cdots R(\lambda ; t_2, t_1).$$

We denote by $R(\lambda; t_a, \cdots, t_1; x, y)$ the kernel function of (7.14). To prove (7.13), it is sufficient to get the following estimate for (7.14):

$$(7.15)\qquad \begin{aligned} &|R(\lambda; t_a, \cdots, t_1)|_{(x,y)} \\ &\leqq M_0^{a-1}\gamma_2^{a-1}[\exp(\gamma_3(t_a-t_1))]\prod_{k=1}^{a-1}(t_{k+1}-t_k)^{3/2}(t_a-t_1)^{-m/2} \\ &\qquad \times[\exp -\lambda(\varepsilon^{***}d^2(x, y)/(2(t_a-t_1)))]. \end{aligned}$$

We shall show (7.15) by induction. Remark that (7.15) holds for $a=1$ easily by Lemma 7.1. Assume that (7.15) holds for $a-1\geqq 1$. Then, by a similar computation as in Lemma 7.1, we have

$$(7.16)\qquad \begin{aligned} &|R(\lambda; t_a, \cdots, t_1)|_{(x,y)} \\ &\leqq M_0^{a-1}\gamma_2^{a-2}[\exp(\gamma_3(t_{a-1}-t_1))\prod_{k=1}^{a-2}(t_{k+1}-t_k)^{3/2}(t_a-t_{a-1})^{-(m-3)/2} \\ &\quad \times(t_a-t_1)^{-m/2}\int_{T_xM}\exp -(\lambda\varepsilon^{***}(|Z|^2/(2(t_a-t_{a-1}) \\ &\qquad\qquad -|Z-Y|^2/(2(t_a-t_1)))dZ \end{aligned}$$

because of $\exp -(\lambda\varepsilon|Z|^2/(2(t_a-t_{a-1}))-k|Z|)\leqq\exp((t_a-t_{a-1})k^2/2\lambda\varepsilon$. By (7.16), we get

$$(7.17)\qquad \begin{aligned} &|R(\lambda; t_a, \cdots, t_1)|_{(x,y)} \\ &\leqq M_1^a\gamma_2^{a-2}\gamma_1[\exp\gamma_3(t_a-t_1))]\prod_{k=1}^{a-1}(t_{k+1}-t_k)^{3/2}(t_a-t_1)^{-m/2} \\ &\quad \times[\exp -(\lambda\varepsilon^{***}(d^2(x, y)/(2(t_a-t_1))))] \\ &\quad \times\int_{T_xM}\exp -(\lambda\varepsilon^{***}|Z'|^2/2)dZ'. \end{aligned}$$

Therefore, we have (7.15).

Define a operator $S^{(j)}(\lambda; t)$ by

$$(7.18)\qquad \begin{aligned} S^{(j)}(\lambda; t)=&\boldsymbol{H}(\lambda; (A_j+B_{j+1})t/N, (A_j+B_j)t/N)R^{(j)}(\lambda; A_j, B_jt)\cdots \\ &\cdots\boldsymbol{H}(\lambda; (A_1+B_1)t/N, (A_1+B_1)t/N)R^{(1)}(\lambda; A_1, B_1t)\boldsymbol{H}(\lambda; B_1t/N, 0) \end{aligned}$$

and we denote by $S^{(j)}(\lambda; t; x, y)$ the kernel function of $S^{(j)}(\lambda; t)$. Using (7.13) and doing the similar computations as above, we have

$$(7.19)\qquad \begin{aligned} &|S^{(j)}(\lambda; t; x, y)|_{(x,y)} \\ &\leqq M_0^{2A_j}\gamma_2^{2A_j}[\exp(\gamma_3(t/N)B_j)](t/N)^{3A_j/2}((A_j+B_j)t/N)^{-m/2} \\ &\quad \times[\exp -(\lambda\varepsilon^{(4)}(d^2(x, y)/(2(A_j+B_j)t/N)))] \end{aligned}$$

where $\varepsilon^{(4)}=1-4\varepsilon$.

Proof of Theorem 1.3. Combining (7.11) and (7.19), we get

$$(7.20)\quad \sum_{(\alpha_1,\cdots,\alpha_k;\beta_1,\cdots,\beta_{k+1})\in\mathcal{P}} |\mathcal{T}(\alpha_1, \cdots, \alpha_k ; \beta_1, \cdots, \beta_{k+1} ; x, y)|_{(x,y)}$$
$$\leqq M_0^{2Ak+1}\gamma_2^{2Ak+1}[\exp(\gamma_3 t)](t/N)^{3Ak/2}t^{-m/2}[\exp -(\lambda\varepsilon^{(4)} d^2(x, y)/2t))]$$
$$\leqq M_0\gamma_2 t^{-m/2}[\exp(\gamma_3 t)]((1+M_0\gamma_2(t/N)^{3/2})^N-1)[\exp -(\lambda\varepsilon^{(4)} d^2(x, y)/2t)]$$
$$\leqq M_0\gamma_2[\exp(\gamma_3 t)]t^{-m/2}(t/N)^{3/2}[\exp(M_0\gamma_3(t/N)^{1/2})][\exp -(\lambda\varepsilon^{(4)} d^2(x, y)/2t)]$$

where $\mathcal{T}(\cdots;\cdots;x, y)$ is the kernel function of (7.11) and $\varepsilon^{(4)}=1-4\varepsilon$. So, Theorem 1.3 is obtained.

Remark. The above computation can be slightly moved for general subdivision of $[0, t]$ (Cf. [11]).

Appendix. Growth of the higher order derivatives of $\rho(x, y)$ and $P(x, y)$.

In this appendix, we shall show the growth estimates of $\rho(x, y)$ and $P(x, y)$, defined by § 2, under the assumptions (A.0)-(A.2). First, we give the estimate for $\rho(x, y)$. Namely, we have

PROPOSITION A.1. *Assume that* (A.0)-(A.1) *holds. Then, there exists a positive constant k_4 such that*

$$(1)\qquad |\nabla_y^\alpha \rho(x, y)|_y \leqq k_4(\exp k_4 r), \qquad r=d(x, y), \quad 0\leqq|\alpha|\leqq 3,$$

for any $x, y\in M$.

To show the above proposition, we prepare some lemmas. Remark that the exponential mapping is a diffeomorphism from T_xM onto M by the assumptions. Thus, we can introduce the normal coordinate around x (Cf. [13] for the precise notation). By the identification $T_xM\cong S_xM\times \boldsymbol{R}^+$, we shall use the normal polar coordinate (r, ω), where $\omega=(\omega^2, \cdots, \omega^m)$ in a local coordinate of $S_xM=\{\omega\in T_xM ; |\omega|_x=1\}$ and $r\in \boldsymbol{R}^+$. Choosing an orthonormal vectors $e_2(\omega), \cdots, e_m(\omega)$, at a point (r, ω), which are perpendicular to radial axis, we may assume that $\{e_2(\omega), \cdots, e_m(\omega)\}$ depends smoothly on ω locally. We put, for $a=2, \cdots, m$,

$$(2)\qquad K_a(\tau, \omega_1)=Exp_x\tau(\omega+(\varepsilon_1/r)e_a(\omega)),$$

for sufficiently small ε_1. Since (2) is a geodesic variation, $(\partial/\partial\varepsilon_1)K_a$ is a Jacobi field along the curve $K_a(\tau, \omega_1)$ for each fixed ε_1. Therefore, we can apply the comparison theorem and we have for some constant $k_{41}>0$,

$$(3)\qquad \begin{cases} |(\partial/\partial\varepsilon_1)K_a(r, 0)|_y\leqq k_{41}(\exp k_{41}r), \\ |(\delta/\delta\tau)(\partial/\partial\varepsilon_1)K_a(r, 0)|_y\leqq k_{41}(\exp k_{41}r), \end{cases}$$

where $r=d(x, y)$ and $Exp_x r\omega=y$.

Let us use the idices $A, B, C, \cdots=1, 2, \cdots, m$ and $a, b, c, \cdots=2, 3, \cdots, m$. Denote by g_{AB} the component of the Riemannian metric g with respect to the coordinate (r, ω), i.e.

$$(4) \qquad \begin{cases} g_{11}(r,\omega)=g((dExp_x)_{r\omega}\omega,\ (dExp_x)_{r\omega}\omega)=1\,, \\ g_{1a}(r,\omega)=g((dExp_x)_{r\omega}\omega,\ (dExp_x)_r e_a(\omega))=0\,, \end{cases}$$

$$(5) \qquad g_{ab}(r,\omega)=g((dExp_x)_{r\omega}e_a(\omega),\ (dExp_x)_{r\omega}e_b(\omega))\,.$$

Differentiating (5) directly and noticing that $(\partial/\partial\varepsilon_1)K_{a|\varepsilon_1=0}=(dExp_x)_{r\omega}e_a(\omega)$, we have for any $2\leqq a,\ b\leqq m$, with some constant $k_{42}>0$,

$$(6) \qquad \begin{cases} |g_{ab}(r,\omega)|\leqq k_{42}(\exp k_{42}r)\,, \\ |\partial_r g_{ab}(r,\omega)|\leqq k_{42}(\exp k_{42}r)\,. \end{cases}$$

LEMMA A.2. *Under the same assumptions as in Proposition* A.1, *there exists a positive constant* k_{43} *such that for any* $2\leqq a,\ b\leqq m$,

$$(7) \qquad |\partial_c g_{ab}(r,\omega)|\leqq k_{43}(\exp k_{43}r)\,.$$

Proof. We take a smooth curve $\omega(\varepsilon_2)$ in S_xM for sufficiently small ε_2 such that $(\partial/\partial\varepsilon_2)\omega(0)=e_c$, $c=2,\cdots,m$. Consider

$$(8) \qquad K_a(\tau,\varepsilon_1,\varepsilon_2)=Exp_x\tau(\omega(\varepsilon_2)+(\varepsilon_1/r)e_a(\omega(\varepsilon_2)))\,.$$

Then, $K_a(\tau,\varepsilon_1,\varepsilon_2)$ is also geodesic variation in two parameters ε_1, ε_2 and has the following initial conditions

$$(9) \qquad (\partial/\partial\varepsilon_1)K_a(0,\varepsilon_1,\varepsilon_2)=0\,,\quad (\partial/\partial\varepsilon_2)K_a(0,\varepsilon_1,\varepsilon_2)=0\,,$$

and

$$(10) \qquad \begin{cases} (\delta/\delta\tau)(\partial/\partial\varepsilon_1)K_a(0,\varepsilon_1,\varepsilon_2)=(1/r)e_a(\omega(\varepsilon_2))\,, \\ (\delta/\delta\tau)(\partial/\partial\varepsilon_2)K_a(0,\varepsilon_1,\varepsilon_2)=\omega'(\varepsilon_2)+(\varepsilon_1/r)(d/d\varepsilon_2)e_a(\omega(\varepsilon_2))\,. \end{cases}$$

By differentiating the Jacobi equation with respect to ε_1 and ε_2, and putting $\varepsilon_1=\varepsilon_2=0$, we get

$$(11) \qquad \begin{aligned} &(\delta^2/\delta\varepsilon_i\delta\varepsilon_j)(\delta^2/\delta\tau^2)K_a(\tau,0,0)+R(\dot\gamma(\tau),(\delta^2/\delta\varepsilon_i\delta\varepsilon_j)K_a(\tau,0,0))\dot\gamma(\tau) \\ &\quad =F_{a,i,j}(\tau)\,, \end{aligned}$$

$i,j=1,2$, $\gamma(\tau)=Exp_x\tau\omega$, where $F_{a,i,j}(\tau)$ is the function of R, ∇R, $(\partial/\partial\varepsilon_i)K_a$, $(\delta/\delta\tau)(\partial/\partial\varepsilon_i)K_a$ and $(\ /\ )(\ /\ _j)K_a$. Also, we have the following setimate by (2.3) and (2.5),

$$(12) \qquad |F_{a,i,j}(r)|_{\gamma(r)}\leqq k_{44}(\exp k_{44}r)\,,\qquad \gamma(r)=Exp_x r\omega\,,$$

with some constant $k_{44}>0$. Therefore, we get by variation of constant

$$(13) \qquad |(\delta^2/\delta\varepsilon_i\delta\varepsilon_j)K_a(r,0,0)|_{\gamma(r)}\leqq k_{45}(\exp k_{45}r)\,,$$

$\gamma(r)=Exp_x r\omega$, with some positive constant $k_{45}>0$. Then, in accordance of

$\omega=\omega^a e_a$, $\omega=(\omega^2, \cdots, \omega^m)$ as the coordinate of ω, using (13), we have

$$(14) \qquad \partial_c g_{ab}(r, \omega)=(d/d\varepsilon_2)g_{ab}(r, \omega(\varepsilon_2))|_{\varepsilon_2=0}$$
$$=g((\partial^2/\partial\varepsilon_2\partial\varepsilon_1)K_a(r, 0, 0), (\partial/\partial\varepsilon_1)K_b(r, 0, 0))_{\gamma(r)}$$
$$+g((\partial/\partial\varepsilon_1)K_a(r, 0, 0), (\partial^2/\partial\varepsilon_2\partial\varepsilon_1)K_b(r, 0, 0))_{\gamma(r)},$$

we get (7).

By Lemma A.2 and the definitions of the Christoffel symbols and $\rho(x, y)$, we have

LEMMA A.3. *Under the same assumptions as in Proposition* A.1, *there exists a positive constant* k_{46} *such that the following estimates hold*:

$$(15) \qquad |g^{AB}(r, \omega)| \leqq k_{46}(\exp k_{46}r),$$

$$(16) \qquad |\Gamma^A_{BC}(r, \omega)| \leqq k_{46}(\exp k_{46}r), \qquad r=d(x, y)$$

where $g^{AB}(r, \omega)$ *and* $\Gamma^A_{BC}(r, \omega)$ *are the inverse matrix of* $g=(g_{AB}(r, \omega))$ *and the Christoffel symbol of* g *with respect to the coordinate* (r, ω) *respectively. Moreover, we have*

$$(17) \qquad |\nabla_y \rho(x, y)|_y \leqq k_{46}(\exp k_{46}r), \qquad r=k(x, y).$$

Now, let $\omega^2, \cdots, \omega^m$ be the coordinates on part of S_xM. We denote by $D^{p,v}$ the differential operator, $v=(v_2, \cdots, v_m)$,

$$D^{p,v}=(\partial/\partial r)^p(\partial/\partial\omega^2)^{v_2}\cdots(\partial/\partial\omega^m)^{v_m}.$$

Differentiating the Jacobi equation successively and using the variation of constant, we get the following, which gives Proposition A.1, because of the definition of $\rho(x, y)$ (Cf. [3]).

LEMMA A.4. *Under the same assumptions as in Proposition* A.1, *there exists a positive constant* k_{47} *such that*

$$(18) \qquad |D^{p,v}J_a(r, \omega)|_{\gamma(r)} \leqq k_{47}(\exp k_{47}r), \qquad a=2, \cdots, m,$$

$\gamma(r)=Exp_x r\omega$, *where* $J_a(\tau, \omega)=(\partial/\partial\varepsilon_1)K_a(\tau, 0)$.

Remark. (i) Bérard [1] has a similar estimate for $\rho(x, y)$ when the case that M is a universal coverting space of a compact manifold.
(ii) Assuming the boundedness of higher order derivatives of the curvature tensor, we get the more higher order growth estimate for $\rho(x, y)$.

Next, we give the higher order estimate for $P(x, y)$. Namely, we get

PROPOSITION A.5. *Under the assumptions* (A.0)-(A.2), *there exists a positive constant* k_{48} *such that for any* $0 \leqq |\alpha| \leqq 3$,

$$|(D_x)^\alpha P(x, y)|_{(x,y)} \leqq k_{48}(\exp k_{48}r),$$

$$|(D_y)^\alpha P(x, y)|_{(x,y)} \leqq k_{48}(\exp k_{48}r), \qquad r=d(x, y).$$

Remark. If we obtain the above proposition, we have Proposition 2.6, because the adjoint operator D^* can be written by using D.

Before proving Proposition A.5 generally, we first observe the following:

LEMMA A.6. *Under the same assumptions as in Proposition* A.5, *there exists a positive constant* k_{49} *such that*

$$\left\{\begin{array}{l} |D_xP(x, y)|_{(x,y)} \leqq k_{49}(\exp k_{49}r), \\ |D_yP(x, y)|_{(x,y)} \leqq k_{49}(\exp k_{49}r), \qquad r=d(x, y). \end{array}\right. \tag{20}$$

Proof. Let $\{e_1(y), \cdots, e_m(y)\}$ be an orthonormal basis of T_yM and put $\xi_i(x, y)=P(x, y)e_i(y)$. Take $\{f_1(x), \cdots, f_m(x)\}$ as an orthonormal basis of T_xM also. Let $\eta_j(\varepsilon_1)$ be a smooth curve such that $\eta_j(0)=x$ and $(d/d\varepsilon_1)\eta_j(0)=f_j(x)$, $j=1, \cdots, m$. Then, we get

$$D_{f_j(x)}\xi(x, y)=(\delta/\delta\varepsilon_1)\xi_j(\eta_j(\varepsilon_1), y)_1=0.$$

Consider the variation $K_j(\tau, \varepsilon_1)=Exp_y\tau(\omega+(\varepsilon_1/r)f_j(y))$, where $Exp_yr\omega=x$ and $f_j(y)$ is the parallel transport along the geodesic from x to y, i.e. $f_j(y)=P(y, x)f_j(x)$. Also, we define $\xi_j(\tau, \varepsilon_1)\in E_{K_j(\tau,\varepsilon_1)}$, $j=1, \cdots, m$ by

$$(\delta/\delta\tau)\xi_j(\tau, \varepsilon_1)=0, \quad \xi_j(0, \varepsilon_1)=e_j(y),$$

for each field ε_1. Differentiating covariantly (2.15) with respect to ε_1, we get

$$(\delta/\delta\tau)(\delta/\delta\varepsilon_1)\xi_j(\tau, \varepsilon_1)+\Omega((\delta/\delta\tau)K_j, (\delta/\delta\varepsilon_1)\xi_j)=0, \tag{21}$$

where Ω denotes the curvature tensor of D. Since $(\delta/\delta\varepsilon_1)K_{j|\varepsilon_1=0}$ is a Jacobi field along $\gamma(\tau)=Exp_xr\omega$, we get by Lemma 2.4

$$\begin{aligned} &(d/d\tau)|(\delta/\delta\varepsilon_1)\xi_j(\tau, 0)| \\ &\leqq 2|\Omega|_{\gamma(\tau)}|(\delta/\delta\tau)K_j(\tau, 0)|_{\gamma(\tau)}|(\delta/\delta\varepsilon_1)K_j(\tau, 0)|_{\gamma(\tau)}|\xi_j|_{\gamma(\tau)} \\ &\leqq 2k'_{49}(\exp k'_{49}r) \end{aligned} \tag{22}$$

with some constant $k'_{49}>0$. Thus, we have

$$\begin{aligned} |D_xP(x, y)|^2_{(x,y)} &= \sum_{i,j=1}^{m} |D_{f_j(x)}\xi_i(x, y)|^2_x \\ &\leqq 2k_{49}(\exp k_{49}r), \end{aligned} \tag{23}$$

which proves the first inequality of (20). The second one is obvious by using $P(x, y)P(y, x)=Id$.

Proof of Proposition A.5 is easily obtained by differentiating (2.15) covariantly and doing the similar computations as in the proof of Lemma A.6. Thus, we get the desired results.

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
KEIO UNIVERSITY
YOKOHAMA, 223, JAPAN