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NEARLY KÄHLER MANIFOLDS WITH POSITIVE HOLOMORPHIC SECTIONAL CURVATURE

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§1. Introduction.

An almost Hermitian manifold $(M, J, \langle , \rangle)$ is called a nearly Kähler manifold provided that $(\nabla_X J)Y+(\nabla_Y J)X=0$ for all $X, Y \in \mathfrak{X}(M)$ ($\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M). From the definition, it follows immediately that a Kähler manifold is necessarily a nearly Kähler manifold. In the present paper, we shall study the structure of nearly Kähler manifolds with positive holomorphic sectional curvature. In §2, we recall some elementary formulas in a nearly Kähler manifold. In §3, we establish an integral formula on the unit sphere bundle over a compact Einstein nearly Kähler manifold. In §4, we discuss the pinching problem on the holomorphic sectional curvature of a compact non-Kähler, nearly Kähler manifold and show some results related to the ones obtained by Tanno [18], Takamatsu and the second named author [17].

In [7], Gray studied the structure of positively curved compact nearly Kähler manifolds and proposed the following conjecture:

Conjecture: Let $M=(M, J, \langle , \rangle)$ be a compact nearly Kähler manifold with positive sectional curvature. If the scalar curvature of M is constant, then M is isometric to a complex projective space with a Kähler metric of constant holomorphic sectional curvature or a 6-dimensional sphere with a Riemannian metric of constant sectional curvature.

For Kähler manifolds, this conjecture is positive (cf. [5], [10], etc.). However, for non-Kähler, nearly Kähler manifolds, this conjecture is negative. Namely, we shall give a counter example to this conjecture in the last section.

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§2. Preliminaries.

In this section, we prepare some elementary formulas in a nearly Kähler manifold. Let $M=(M, J, \langle , \rangle)$ be an $n(=2m)$-dimensional connected nearly Kähler manifold. We denote by ∇ and R the Riemannian connection and the curvature tensor of M, respectively. We assume that the curvature tensor R is defined by

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(2.1) $$R(X, Y)Z=\nabla_{[X,Y]}Z-[\nabla_X, \nabla_Y]Z, \qquad X, Y, Z\in\mathfrak{X}(M).$$

We denote by R_1 and R_1^* the Ricci tensor and the Ricci *-tensor of M, respectively. The tensor field R_1 and R_1^* are defined respectively by

(2.2) $$R_1(x, y)=\text{Trace of } (z\mapsto R(x, z)y),$$

and

(2.3) $$R_1^*(x, y)=(1/2)\text{Trace of } (z\mapsto R(Jy, x)Jz),$$

for $x, y, z\in M_p$ (the tangent space of M at p) (cf. [9], [19]). Then it is known that the tensor fields R_1 and R_1^* satisfy the following equalities:

(2.4) $$R_1(X, Y)=R_1(Y, X), \quad R_1(JX, JY)=R_1(X, Y),$$

(2.5) $$R_1^*(X, Y)=R_1^*(Y, X), \quad R_1^*(JX, JY)=R_1^*(X, Y),$$

for $X, Y\in\mathfrak{X}(M)$. The first Chern form γ of M is given by

(2.6) $$8\pi\gamma(X, Y)=5R_1^*(JX, Y)-R_1(JX, Y),$$

for all $X, Y\in\mathfrak{X}(M)$ ([9], p. 238).

We denote by S the scalar curvature of M. The sectional curvature, the holomorphic sectional curvature and the holomorphic bisectional curvature are defined respectively by

(2.7) $$K(x, y)=\frac{\langle R(x, y)x, y\rangle}{\|x\|^2\|y\|^2},$$

for $x, y\in M_p$ $(p\in M)$ with $x\neq 0$, $y\neq 0$, $\langle x, y\rangle=0$,

(2.8) $$H(x)=K(x, Jx),$$

for $x\in M_p$ $(p\in M)$ with $x\neq 0$, and

(2.9) $$B(x, y)=\frac{\langle R(x, Jx)y, Jy\rangle}{\|x\|^2\|y\|^2},$$

for $x, y\in M_p$ $(p\in M)$ with $x\neq 0$, $y\neq 0$.

A nearly Kähler manifold M is said to be of holomorphically δ-pinched $(0\leqq\delta\leqq 1)$ if there exists a positive constant l such that

(2.10) $$\delta l\leqq H(x)\leqq l,$$

for all non-zero $x\in M_p$, for all $p\in M$. Since we are dealing with nearly Kähler manifolds, the size $\|(\nabla_x J)y\|^2$ will be important in the pinching estimates. A nearly Kähler manifold M is said to satisfy the condition $T(\rho, \sigma)$ if

(2.11) $$\rho H(x)\leqq\|(\nabla_x J)y\|^2\leqq\sigma H(x),$$

for $x, y \in M_p$ with $\|x\|=\|y\|=1$, $\langle x, y\rangle=\langle x, Jy\rangle=0$ for all $p\in M$ ([7]).

In the present paper, we shall adopt the following notational convention. For an orthonormal basis $\{e_i\}=\{e_\alpha, e_{m+\alpha}=Je_\alpha\}$ $(1\leqq\alpha, \beta, \cdots\leqq m;\ 1\leqq a, b, \cdots i, j, k, \cdots\leqq n=2m)$, of M_p $(p\in M)$, we put

$$e_{\bar{i}}=Je_i \quad (\text{and hence } e_{\bar{\alpha}}=e_{m+\alpha},\ e_{\overline{m+\alpha}}=-e_\alpha), \tag{2.12}$$

$$R_{hijk}=\langle R(e_h, e_i)e_j, e_k\rangle, \quad R_{\bar{h}ijk}=\langle R(e_{\bar{h}}, e_i)e_j, e_k\rangle, \tag{2.13}$$
$$\cdots, R_{\bar{h}\bar{i}\bar{j}\bar{k}}=\langle R(e_{\bar{h}}, e_{\bar{i}})e_{\bar{j}}, e_{\bar{k}}\rangle,$$

$$\nabla_l R_{hijk}=\langle(\nabla_{e_l}R)(e_h, e_i)e_j, e_k\rangle, \quad \nabla_{\bar{l}} R_{hijk}=\langle(\nabla_{e_{\bar{l}}}R)(e_h, e_i)e_j, e_k\rangle, \tag{2.14}$$
$$\cdots, \nabla_{\bar{l}} R_{\bar{h}\bar{i}\bar{j}\bar{k}}=\langle(\nabla_{e_{\bar{l}}}R)(e_{\bar{h}}, e_{\bar{i}})e_{\bar{j}}, e_{\bar{k}}\rangle, \quad \text{etc.},$$

and

$$R_{ij}=R_1(e_i, e_j), \quad R^*_{ij}=R^*_1(e_i, e_j). \tag{2.15}$$

The following equalities in M are well-known ([7], [9], etc.):

$$\langle R(w, x)y, z\rangle-\langle R(w, x)Jy, Jz\rangle=\langle(\nabla_w J)x, (\nabla_y J)z\rangle, \tag{2.16}$$

$$\langle R(w, x)y, z\rangle=\langle R(Jw, Jx)Jy, Jz\rangle, \tag{2.17}$$

$$\langle(\nabla^2_{e_ie_j}J)x, y\rangle=\frac{1}{2}(\langle R(e_i, Je_j)x, y\rangle-\langle R(Jy, e_i)e_j, x\rangle \tag{2.18}$$
$$+\langle R(Jx, e_i)e_j, y\rangle),$$

$$\|\nabla R_1-\nabla R^*_1\|^2=(1/8)\,\text{Trace of } \{(R^1-(R^*)^1)\circ(R^1-5(R^*)^1)\circ(R^1-(R^*)^1)\}, \tag{2.19}$$

where $\langle R^1x, y\rangle=R_1(x, y)$, $\langle(R^*)^1x, y\rangle=R^*_1(x, y)$, $w, x, y, z\in M_p$ $(p\in M)$. By (2.2), (2.3) and (2.18), we have

$$\sum_{i=1}^n\langle(\nabla^2_{e_ie_i}J)x, y\rangle=R^*_1(Jx, y)-R_1(Jx, y), \tag{2.20}$$

for $x, y\in M_p$ $(p\in M)$. By (2.2), (2.3) and (2.16), we have

$$\sum_{i=1}^n\langle(\nabla_{e_i}J)x, (\nabla_{e_i}J)y\rangle=R_1(x, y)-R^*_1(x, y), \tag{2.21}$$

for $x, y\in M_p$. By (2.3), (2.4), (2.5), (2.16) and (2.21), we have

$$\sum_{a=1}^n R_{a\bar{a}i\bar{j}}=2R^*_{ij}, \tag{2.22}$$
$$\sum_{a=1}^n R_{a\bar{i}j\bar{a}}=R^*_{ij}.$$

We note that $\langle(\nabla_x J)y, z\rangle(x, y, z\in M_p)$ satisfies the followings:

$$\langle(\nabla_x J)y, z\rangle = -\langle(\nabla_y J)x, z\rangle = -\langle(\nabla_x J)z, y\rangle, \tag{2.23}$$

and

$$\langle(\nabla_{Jx} J)Jy, z\rangle = -\langle(\nabla_x J)y, z\rangle .$$

By (2.7), (2.8), (2.9) and (2.16), we have

$$\begin{aligned} K(x, y) = (1/8)\{&3H(x+Jy)+3H(x-Jy)-H(x+y)-H(x-y) \\ &-H(x)-H(y)\} + (3/4)\|(\nabla_x J)y\|^2, \end{aligned} \tag{2.24}$$

$$B(x, y) = K(x, y) + K(x, Jy) - 2\|(\nabla_x J)y\|^2, \tag{2.25}$$

for $x, y \in M_p$ $(p \in M)$ with $\|x\| = \|y\| = 1$, $\langle x, y\rangle = \langle x, Jy\rangle = 0$.

§3. An integral formula on the unit sphere bundle.

The following fact is well-known and useful for our arguments ([2]):

PROPOSITION 3.1. *Let $\boldsymbol{R}^n$ be an n-dimensional Euclidean space and f a homogeneous polynomial of degree r $(\geqq 1)$ defined on $\boldsymbol{R}^n$. Then we have*

$$\int_{S^{n-1}(1)} (Df)\omega_2 = r(n+r-2)\int_{S^{n-1}(1)} (f|_{S^{n-1}(1)})\omega_2 ,$$

where D denotes the Laplace operator of $\boldsymbol{R}^n$ and ω_2 denotes the volume element of an $(n-1)$-dimensional unit sphere $S^{n-1}(1)$ with the canonical Riemannian metric.

Let $M = (M, \langle\,,\rangle)$ be an n-dimensional connected Riemannian manifold. We denote by $T(M)$ and $S(M)$ the tangent bundle and the unit sphere bundle over M, respectively:

$$T(M) = \{(p, x) \mid p \in M, x \in M_p\},$$

$$S(M) = \{(p, x) \in T(M) \mid \|x\| = 1\}.$$

For each point $p \in M$, we put

$$S_p = \{x \in M_p \mid \|x\| = 1\}.$$

Then S_p is isometric to $S^{n-1}(1)$. We now recall the Sasaki metric $\langle\,,\rangle^s$ on $T(M)$ (cf. [12]). We denote by X^h (resp. X^v) the horizontal lift (resp. the vertical lift) of $X \in \mathfrak{X}(M)$. Then the Sasaki metric $\langle\,,\rangle^s$ on $T(M)$ is defined by

$$\langle X^h, Y^h\rangle^s = \langle X, Y\rangle, \quad \langle X^v, Y^v\rangle^s = \langle X, Y\rangle, \quad \langle X^h, Y^v\rangle^s = 0, \tag{3.1}$$

for $X, Y \in \mathfrak{X}(M)$. From (3.1), we get easily

$$(\overset{s}{\nabla}_{X^h} Y^h)_{(p,x)} = (\nabla_X Y)^h + \frac{1}{2}(R(X, Y)x)^v, \tag{3.2}$$

where $\overset{s}{\nabla}$ denotes the Riemannian connection on $T(M)$ with respect to the Sasaki metric $\langle\, ,\rangle^s$. From (3.2), we see that any horizontal lift of a geodesic in M is a geodesic in $T(M)=(T(M), \langle\, ,\rangle^s)$. We denote by using the same notation $\langle\, ,\rangle^s$ the induced metric on $S(M)$ which is induced from the Sasaki metric $\langle\, ,\rangle^s$ on $T(M)$. Let ω (resp. ω_1) be the volume element on $S(M)$ (resp. M) with respect to the metric $\langle\, ,\rangle^s$ (resp. $\langle\, ,\rangle$). Then we have easily

$$\omega(p, x)=\omega_1(p)\wedge\omega_2(x)\,, \qquad (p, x)\in S(M)\,. \tag{3.3}$$

If M is compact and orientable, by (3.3), for any smooth function f on $S(M)$, we have

$$\int_{S(M)} f\omega=\int_M\left\{\int_{S_p} f(p, x)\omega_2(x)\right\}\omega_1(p)\,. \tag{3.4}$$

Let (p, x) be any point of $S(M)$. We take an orthonormal basis $\{e_i\}=\{e_1, \cdots, e_n\}$ of M_p such that $x=e_1$. Then $\{e_1{}^h, \cdots, e_n{}^h, e_2{}^v, \cdots, e_n{}^v\}$ is an orthonormal basis of the tangent space $S(M)_{(p,x)}$. For each $y\in M_p$, the tangent space $(M_p)_y$ (i.e., the vertical subspace of $T(M)_{(p,y)}$) is identified with M_p by means of parallel translation. Under this identification, $e_i{}^v$ corresponds to e_i $(1\leqq i\leqq n)$. We denote by $(u_1, \cdots, u_n, v_2, \cdots, v_n)$ the normal coordinate system on a neighborhood of (p, x) in $S(M)$ with respect to the orthonormal basis $\{e_1{}^h, \cdots, e_n{}^h, e_2{}^v, \cdots, e_n{}^v\}$. In [10], Gray has introduced a second order linear differential operator L by

$$L_{(p,x)}=\left\{\sum_{i=1}^n \frac{\partial^2}{\partial u_i^2}+\frac{1}{2}\sum_{i,j\geqq 2} h_{ij}\frac{\partial^2}{\partial v_i\partial v_j}\right\}_{(p,x)}, \tag{3.5}$$

where $h_{ij}(p, x)=\langle R(e_i, x)e_j, x\rangle$. We denote by Δ^h the horizontal Laplacian of $S(M)$. Then in terms of the normal coordinate system $(u_1, \cdots, u_n, v_2, \cdots, v_n)$, Δ^h is given by

$$\Delta^h_{(p,x)}=\left\{\sum_{i=1}^n \frac{\partial^2}{\partial u_i^2}\right\}_{(p,x)}. \tag{3.6}$$

For a smooth function f on $S(M)$, we denote by $\mathrm{grad}^h f$ (resp. $\mathrm{grad}^v f$) the horizontal (resp. the vertical) component of $\mathrm{grad}\, f$.

Now, let $M=(M, J, \langle\, ,\rangle)$ be an n $(=2m)$-dimensional nearly Kähler manifold. We may regard holomorphic sectional curvature $H=H(x)$ as a smooth function on $S(M)$. Then we have

$$(\mathrm{grad}^h H)_{(p,x)}=\sum_{i=1}^n \{\langle(\nabla_{e_i}R)(x, Jx)x, Jx\rangle+2\langle R(x, Jx)x, (\nabla_{e_i}J)x\rangle\}\, e_i{}^h, \tag{3.7}$$

$$\begin{aligned}(\mathrm{grad}^v H)_{(p,x)}&=(\mathrm{grad}\, H)_{(p,x)}-(\mathrm{grad}^h H)_{(p,x)}-\langle(\mathrm{grad}\, H)_{(p,x)}, x^v\rangle^s x^v\\ &=4\sum_{i=2}^n \langle R(x, Jx)x, Je_i\rangle e_i{}^v.\end{aligned} \tag{3.8}$$

By (3.8), we see that

$$\langle(\mathrm{grad}^v H)_{(p,x)},\ x^v\rangle^s=\langle(\mathrm{grad}^v H)_{(p,x)},\ (Jx)^v\rangle^s=0\,.$$

From the result due to Tanno [18] and (3.8), we may note the following

PROPOSITION 3.2. *Let $M=(M, J, \langle\ ,\ \rangle)$ be a nearly Kähler manifold. Then M is a space of constant holomorphic sectional curvature if and only if* $\mathrm{grad}^v H=0$ *on* $S(M)$.

We assume that $M=(M, J, \langle\ ,\ \rangle)$ is a connected compact Einstein nearly Kähler manifold. First, we estimate the value $L(H)(p, x)$ at any point $(p, x)\in S(M)$. By (3.6) and (3.7), we get

$$\begin{aligned}
(3.9)\qquad \sum_{i=1}^{n}\frac{\partial^2 H}{\partial u_i^2}(p,\ x)&=(\Delta^h H)(p,\ x)\\
&=\sum_{i=1}^{n}\langle\overset{s}{\nabla}_{e_i}{}^h(\mathrm{grad}^h H),\ e_i{}^h\rangle\\
&=\sum_{i=1}^{n}\{\langle(\nabla^2_{e_ie_i}R)(x,\ Jx)x,\ Jx\rangle+\langle(\nabla_{e_i}R)(x,\ (\nabla_{e_i}J)x)x,\ Jx\rangle\\
&\qquad+\langle(\nabla_{e_i}R)(x,\ Jx)x,\ (\nabla_{e_i}J)x\rangle+2\langle(\nabla_{e_i}R)(x,\ Jx)x,\ (\nabla_{e_i}J)x\rangle\\
&\qquad+2\langle R(x,\ (\nabla_{e_i}J)x)x,\ (\nabla_{e_i}J)x\rangle+2\langle R(x,\ Jx)x,\ (\nabla^2_{e_ie_i}J)x\rangle\}\\
&=\sum_{i=1}^{n}\{\langle(\nabla^2_{e_ie_i}R)(x,\ Jx)x,\ Jx\rangle+4\langle(\nabla_{e_i}R)(x,\ Jx)x,\ (\nabla_{e_i}J)x\rangle\\
&\qquad+2\langle R(x,\ (\nabla_{e_i}J)x)x,\ (\nabla_{e_i}J)x\rangle+2\langle R(x,\ Jx)x,\ (\nabla^2_{e_ie_i}J)x\rangle\}\,.
\end{aligned}$$

Taking account of the first Bianchi, the second Bianchi and the Ricci identities, and (2.16), (2.17), (2.20), we get

$$\begin{aligned}
(3.10)\qquad &\frac{1}{2}\sum_{i=1}^{n}\langle(\nabla^2_{e_ie_i}R)(x,\ Jx)x,\ Jx\rangle\\
&=\frac{1}{2}\sum_{i=1}^{n}\{\langle(\nabla^2_{e_ix}R)(e_i,\ Jx)x,\ Jx\rangle-\langle(\nabla^2_{e_iJx}R)(e_i,\ x)x,\ Jx\rangle\}\\
&=\frac{1}{2}\Big\{\sum_{i=1}^{n}\langle(\nabla^2_{xe_i}R)(e_i,\ Jx)x,\ Jx\rangle\\
&\qquad+\sum_{i,j=1}^{n}(R_{ixij}R_{j\bar{x}x\bar{x}}+R_{ix\bar{x}j}R_{ijx\bar{x}}+R_{ixxj}R_{i\bar{x}j\bar{x}}+R_{ix\bar{x}j}R_{i\bar{x}xj})\\
&\qquad-\sum_{i=1}^{n}\langle(\nabla^2_{Jxe_i}R)(e_i,\ x)x,\ Jx\rangle\\
&\qquad-\sum_{i,j=1}^{n}(R_{i\bar{x}ij}R_{jxx\bar{x}}+R_{i\bar{x}xj}R_{ijx\bar{x}}+R_{i\bar{x}xj}R_{ixj\bar{x}}+R_{i\bar{x}\bar{x}j}R_{ixxj})\Big\}\\
&=\sum_{i,j=1}^{n}\{R_{ixjx}\delta_{ij}H(x)-R_{ixjx}R_{i\bar{x}j\bar{x}}-R_{ix\bar{x}j}(R_{ix\bar{x}j}+2R_{i\bar{x}jx})\}
\end{aligned}$$

$$=\sum_{i,j=1}^{n} R_{ixjx}(\delta_{ij}H(x)-R_{i\bar{x}j\bar{x}})$$

$$+\sum_{i,j=1}^{n} R_{ixj\bar{x}}(-R_{ixx\bar{j}}+\langle(\nabla_{e_i}J)x, (\nabla_{Jx}J)e_j\rangle+2R_{i\bar{x}\bar{j}\bar{x}} +2\langle(\nabla_{e_i}J)Jx, (\nabla_{e_j}J)x\rangle)$$

$$=\sum_{i,j=1}^{n} \{R_{ixjx}(\delta_{ij}H(x)-R_{i\bar{x}j\bar{x}}) +(-R_{ixjx}+\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle)(R_{ixjx}+2R_{i\bar{x}j\bar{x}} -3\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle)\}$$

$$=\sum_{i,j=1}^{n} R_{ixjx}\{\delta_{ij}H(x)-R_{ixjx}-3R_{i\bar{x}j\bar{x}}+3\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle\}$$

$$+3\sum_{i,j=1}^{n} R_{ix\bar{j}\bar{x}}\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle,$$

where we put $R_{ixjx}=R_{i1j1}$, $R_{i\bar{x}j\bar{x}}=R_{i1j1}$, $\cdots$, etc. Thus, by (3.9) and (3.10), we have

$$\sum_{i=1}^{n} \frac{\partial^2 H}{\partial u_i^2}(p, x) =2\Big\{\sum_{i,j=1}^{n} R_{ixjx}(\delta_{ij}H(x)-R_{ixjx}-3R_{i\bar{x}j\bar{x}}+3\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle) +3\sum_{i,j=1}^{n} R_{ix\bar{j}\bar{x}}\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle\Big\}$$
$$+4\sum_{i=1}^{n}\langle(\nabla_{e_i}R)(x, Jx)x, (\nabla_{e_i}J)x\rangle+2\sum_{i=1}^{n}\langle R(x, (\nabla_{e_i}J)x)x, (\nabla_{e_i}J)x\rangle +2\sum_{i=1}^{n}\langle R(x, Jx)x, (\nabla^2_{e_ie_i}J)x\rangle. \tag{3.11}$$

Similarly, we have

$$\frac{\partial^2 H}{\partial v_i \partial v_j}(p, x)=-4\{\delta_{ij}H(x)-R_{ixjx}-3R_{i\bar{x}j\bar{x}}+3\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle\}. \tag{3.12}$$

We now define smooth functions f_λ ($\lambda=1, 2, 3, 4$) on $S(M)$ by

$$f_1(p, x)=\sum_{i,j=1}^{n} R_{ix\bar{j}\bar{x}}\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle, \tag{3.13}$$

$$f_2(p, x)=\sum_{i=1}^{n}\langle(\nabla_{e_i}R)(x, Jx)x, (\nabla_{e_i}J)x\rangle,$$

$$f_3(p, x)=\sum_{i=1}^{n}\langle R(x, (\nabla_{e_i}J)x)x, (\nabla_{e_i}J)x\rangle,$$

$$f_4(p,\ x)=\sum_{i=1}^{n}\langle R(x,\ Jx)x,\ (\nabla^2_{e_ie_i}J)x\rangle$$

$$=-\langle R(x,\ Jx)x,\ (R^1-(R^*)^1)Jx\rangle .$$

From (3.5), (3.11), (3.12) and (3.13), we have

$$L(H)(p,\ x)=6f_1(p,\ x)+4f_2(p,\ x)+2f_3(p,\ x)+2f_4(p,\ x), \tag{3.14}$$

for all $(p,\ x)\in S(M)$. Since M is an Einstein space, it follows that the operator L is self-adjoint (cf. [10]). Thus, we have the following equality ([10], p. 42):

$$0=\int_{S(M)} L(H^2)\omega \tag{3.15}$$

$$=\int_{S(M)}\{2HL(H)+2\|\mathrm{grad}^h H\|^2+\langle R(x,\ \mathrm{grad}^v H)x,\ \mathrm{grad}^v H\rangle\}\,\omega .$$

We shall evaluate the integral $\int_{S(M)}\|\mathrm{grad}^h H\|^2\omega$. We define smooth functions g_μ ($\mu=1,\ 2,\ 3$) on $S(M)$ by

$$g_1(p,\ x)=\sum_{i=1}^{n}\langle(\nabla_{e_i}R)(x,\ Jx)x,\ Jx\rangle^2, \tag{3.16}$$

$$g_2(p,\ x)=\sum_{i=1}^{n}\langle(\nabla_{e_i}R)(x,\ Jx)x,\ Jx\rangle\langle R(x,\ Jx)x,\ (\nabla_{e_i}J)x\rangle ,$$

$$g_3(p,\ x)=\sum_{i=1}^{n}\langle R(x,\ Jx)x,\ (\nabla_{e_i}J)x\rangle^2,$$

for $(p,\ x)\in S(M)$. Then, by (3.7) and (3.16), we get

$$\int_{S(M)}\|\mathrm{grad}^h H\|^2\omega=\int_{S(M)}g_1\omega+4\int_{S(M)}(g_2+g_3)\omega . \tag{3.17}$$

Taking account of (3.4), (3.13), (3.16), Proposition 3.1 and Green's theorem, we have

$$\int_{S(M)}g_2\omega=-2\int_{S(M)}g_3\omega-\int_{S(M)}H(f_2+f_3+f_4)\omega . \tag{3.18}$$

From the results due to Gray [8] and the second named author [13], we may note that M is a Riemannian locally 3-symmetric space if and only if g_1 is identically zero.

By (3.14), (3.15), (3.17) and (3.18), we have finally

$$\int_{S(M)}[2\{g_1-4g_3+H(6f_1-2f_3-2f_4)\}+\langle R(x,\ \mathrm{grad}^v H)x,\ \mathrm{grad}^v H\rangle]\omega=0 . \tag{3.19}$$

The integral formula (3.19) together with (3.13) and (3.16) plays an important role in the arguments of the next section.

In the rest of this section, we assume that $M=(M,\ J,\ \langle\ ,\ \rangle)$ is a connected non-Kähler, Einstein nearly Kähler manifold with vanishing first Chern form (i.e., $R_1=5R_1^*$). By making use of (2.22), (3.8) and Proposition 3.1, we have the followings:

$$\int_{S_p} H\omega_2=\frac{8S}{5n(n+2)}V_2, \tag{3.20}$$

$$\int_{S_p} H^2\omega_2=\frac{1}{4(n+2)}\int_{S_p}\|\mathrm{grad}^v H\|^2\omega_2+\frac{64S^2}{25n^2(n+2)^2}V_2, \tag{3.21}$$

where $V_2=\mathrm{Vol}(S^{n-1}(1))$. By (3.8) and (3.21), we have

$$\begin{aligned}\int_{S_p}\sum_{k=1}^{n}(R_{x\bar{x}xk})^2\omega_2&=\frac{1}{16}\int_{S_p}\|\mathrm{grad}^v H\|^2\omega_2+\int_{S_p}H^2\omega_2\\&=\frac{n+6}{16(n+2)}\int_{S_p}\|\mathrm{grad}^v H\|^2\omega_2+\frac{64S^2}{25n^2(n+2)^2}V_2.\end{aligned} \tag{3.22}$$

If M is holomorphically δ-pinched, by (2.10) and (3.20), we get

$$\delta l\leqq\frac{8S}{5n(n+2)}\leqq l. \tag{3.23}$$

§ 4. Some results.

S. Tanno [18] has proved the following

PROPOSITION 4.1. *If a 6-dimensional nearly Kähler manifold $M=(M, J, \langle\,,\rangle)$ is of constant holomorphic sectional curvature H, then either M is Kählerian, or M is of constant sectional curvature $H>0$.*

First, in connection with the above result, we shall show some results. Let $M=(M, J, \langle\,,\rangle)$ be a 6-dimensional connected non-Kähler, nearly Kähler manifold. Then it is known that M is an Einstein space with positive scalar curvature and vanishing first Chern form (i.e., $R_1=5R_1^*$), and furthermore the following equalities hold ([11]):

$$\begin{aligned}\langle(\nabla_{e_h}J)e_i, (\nabla_{e_j}J)e_k\rangle=-\frac{S}{30}\{&\langle e_i, e_j\rangle\langle e_h, e_k\rangle-\langle e_h, e_j\rangle\langle e_i, e_k\rangle\\&-\langle Je_i, e_j\rangle\langle Je_h, e_k\rangle+\langle Je_h, e_j\rangle\langle Je_i, e_k\rangle\},\end{aligned} \tag{4.1}$$

$$\begin{aligned}\langle(\nabla^2_{e_k e_j}J)e_i, e_h\rangle=-\frac{S}{30}\{&\langle e_k, e_j\rangle\langle Je_i, e_h\rangle+\langle e_k, e_i\rangle\langle Je_h, e_j\rangle\\&+\langle e_k, e_h\rangle\langle Je_j, e_i\rangle\},\end{aligned} \tag{4.2}$$

where $\{e_i\}=\{e_\alpha, e_{3+\alpha}=Je_\alpha\}$ $(\alpha=1, 2, 3)$ is an orthonormal basis of M_p $(p\in M)$.

We now evaluate the values $f_\lambda(p, x)$ $(\lambda=1, 3, 4)$. By (2.22), (3.13) and (4.1), we get

$$f_1(p,\ x)=\frac{S}{30}\left(\frac{S}{30}-H(x)\right), \tag{4.3}$$

$$f_3(p,\ x)=-\frac{S}{30}\left(H(x)-\frac{S}{6}\right). \tag{4.4}$$

Since M is an Einstein space with $R_1=5R_1^*$, by (3.13), we get

$$f_4(p,\ x)=-\frac{2S}{15}H(x). \tag{4.5}$$

By (3.16), (3.22) and (4.1), we get

$$\int_{S_p} g_3\omega_2=\frac{S}{480}\int_{S_p}\|\mathrm{grad}^v H\|^2\omega_2. \tag{4.6}$$

THEOREM 4.2. *Let $M=(M, J, \langle\,,\rangle)$ be a 6-dimensional connected complete non-Kähler, nearly Kähler manifold satisfying the condition*

$$K(x,\ y)>\frac{S}{120},$$

for $x, y\in M_p$ with $\|x\|=\|y\|=1$, $\langle x, y\rangle=\langle x, Jy\rangle=0$, for all $p\in M$. Then M is isometric to a 6-dimensional sphere of constant curvature $S/30$.

Proof. Since M is an Einstein space with positive scalar curvature, M is compact by Myer's theorem. By (3.19)~(3.21), (4.3)~(4.6), we have

$$\int_{S(M)}\left\{2g_1+\langle R(x,\ \mathrm{grad}^v H)x,\ \mathrm{grad}^v H\rangle-\frac{S}{120}\|\mathrm{grad}^v H\|^2\right\}\omega=0. \tag{4.7}$$

From the hypothesis, (4.7) and Proposition 4.1, the theorem follows immediately. Q.E.D.

Furthermore, we have the following

THEOREM 4.3. *Let $M=(M, J, \langle\,,\rangle)$ be a 6-dimensional connected complete non-Kähler, nearly Kähler manifold. If M is holomorphically δ ($>2/5$)-pinched, then M is isometric to a 6-dimensional sphere of constant curvature $S/30$.*

Proof. By the hypothesis and (2.10), (2.24), (3.23) and (4.1), we have

$$\begin{aligned}K(x,\ y)&\geqq\frac{1}{4}(3\delta-2)l+\frac{S}{40}\\&>-\frac{1}{5}l+\frac{S}{40}\\&>-\frac{S}{60}+\frac{S}{40}=\frac{S}{120},\end{aligned}$$

for $x, y\in M_p$ with $\|x\|=\|y\|=1$, $\langle x, y\rangle=\langle x, Jy\rangle=0$, for all $p\in M$. Thus the theorem follows immediately from Theorem 4.2. Q.E.D.

Next, we shall deal with general cases where the dimension of M is not necessarily equal to 6. In connection with the results obtained by Bishop and Goldberg ([3], [4], [5]), we have the following

THEOREM 4.4. *Let $M=(M, J, \langle , \rangle)$ be an n $(=2m)$-dimensional connected compact non-Kähler, nearly Kähler manifold with constant scalar curvature. If M satisfies the condition*

$$K(x, y)+K(x, Jy)+B(x, y)>0\,, \tag{4.8}$$

for $x, y\in M_p$ with $x\neq 0$, $y\neq 0$, $\langle x, y\rangle=\langle x, Jy\rangle=0$, for all $p\in M$, then the Ricci tensor R_1 of M is parallel and the first Chern form of M vanishes.

Proof. Since M is compact and the scalar curvature S of M is constant, by the result due to Tachibana [16], the first Chern form γ is a harmonic 2-form.

For each point $p\in M$, we may choose an orthonormal basis $\{e_i\}=\{e_\alpha, e_{\bar\alpha}\}$ which diagonalizes the symmetric linear endomorphism $5(R^*)^1-R^1$ of M_p. By the choice of $\{e_i\}$, we get

$$\gamma(e_i, e_j)=0 \qquad \text{for} \quad e_j\neq \pm e_{\bar i}\,. \tag{4.9}$$

For the 2-form γ, we put

$$F(\gamma)=\sum_{i,j,k} R_{ij}\gamma_{ik}\gamma_{jk}-\frac{1}{2}\sum_{h,i,j,k} R_{hijk}\gamma_{hi}\gamma_{jk}\,, \tag{4.10}$$

where $\gamma_{ij}=\gamma(e_i, e_j)$. By (4.9), (4.10) reduces to

$$F(\gamma)=2\sum_{\alpha,\beta}\{(R_{\alpha\beta\alpha\beta}+R_{\alpha\bar\beta\alpha\bar\beta})(\gamma_{\alpha\bar\alpha})^2-R_{\alpha\bar\alpha\beta\bar\beta}\gamma_{\alpha\bar\alpha}\gamma_{\beta\bar\beta}\}\,. \tag{4.11}$$

By (2.25) and (4.11), we get

$$F(\gamma)=2\sum_{\alpha<\beta}\{R_{\alpha\bar\alpha\beta\bar\beta}(\gamma_{\alpha\bar\alpha}-\gamma_{\beta\bar\beta})^2+2\|(\nabla_{e_\alpha}J)e_\beta\|^2(\gamma_{\alpha\bar\alpha}^2+\gamma_{\beta\bar\beta}^2)\}\,, \tag{4.12}$$

or

$$F(\gamma)=2\sum_{\alpha<\beta}\{(R_{\alpha\beta\alpha\beta}+R_{\alpha\bar\beta\alpha\bar\beta})(\gamma_{\alpha\bar\alpha}-\gamma_{\beta\bar\beta})^2+4\|(\nabla_{e_\alpha}J)e_\beta\|^2\gamma_{\alpha\bar\alpha}\gamma_{\beta\bar\beta}\}\,.$$

By (4.12), we have finally

$$F(\gamma)=\sum_{\alpha<\beta}\{(R_{\alpha\bar\alpha\beta\bar\beta}+R_{\alpha\beta\alpha\beta}+R_{\alpha\bar\beta\alpha\bar\beta})(\gamma_{\alpha\bar\alpha}-\gamma_{\beta\bar\beta})^2 +2\|(\nabla_{e_\alpha}J)e_\beta\|^2(\gamma_{\alpha\bar\alpha}+\gamma_{\beta\bar\beta})^2\}\,. \tag{4.13}$$

Since γ is a harmonic 2-form and $F(\gamma)\geqq 0$, according to Yano and Bochner [22], it follows that $F(\gamma)=0$ and γ is parallel. Thus, by (4.12) and (4.13), we get

$$\gamma_{\alpha\bar\alpha}-\gamma_{\beta\bar\beta}=0\,, \quad \text{and} \quad \|(\nabla_{e_\alpha}J)e_\beta\|^2(\gamma_{\alpha\bar\alpha}+\gamma_{\beta\bar\beta})^2=0\,, \tag{4.14}$$

for $1 \leqq \alpha < \beta \leqq m$.

Since M is non-Kählerian, it follows that

$$(4.15) \qquad (\nabla_{e_\alpha} J) e_\beta \neq 0 \qquad \text{for some} \quad \alpha < \beta .$$

Thus, by (4.14) and (4.15), we have

$$(4.16) \qquad \gamma = 0 \qquad (\text{i.e. } R_1 = 5R_1^*) .$$

Therefore, by (2.19) and (4.16), we have

$$\nabla(R_1 - R_1^*) = 0 ,$$

and hence

$$\nabla R_1 = 0 .$$

Q. E. D.

Furthermore, we have the following

THEOREM 4.5. *Let $M=(M, J, \langle\, ,\rangle)$ be an n $(=2m)$-dimensional connected compact non-Kähler, nearly Kähler manifold with constant scalar curvature. If M satisfies the condition $T(\rho, \sigma)$ $(\rho > 0)$, and is holomorphically δ $(>2/(\rho+3))$-pinched, then M is an Einstein space and the first Chern form of M vanishes.*

Proof. By the hypothesis and (2.10), (2.11), (2.24), we get

$$(4.17) \qquad K(x, y) \geqq (1/4)(3\delta - 2 + 3\rho\delta) l$$

$$> \frac{\rho}{\rho+3} l \quad (>0) ,$$

for $x, y \in M_p$ with $\|x\| = \|y\| = 1$, $\langle x, y\rangle = \langle x, Jy\rangle = 0$, for all $p \in M$. Thus, by (2.11), (2.25) and (4.17), we get

$$\begin{aligned} & K(x, y) + K(x, Jy) + B(x, y) \\ & \quad = 2\{K(x, y) + K(x, Jy) - \|(\nabla_x J) y\|^2\} \\ & \quad \geqq (3\delta - 2) l + \|(\nabla_x J) y\|^2 \\ & \quad \geqq \{(3+\rho)\delta - 2\} l > 0 , \end{aligned}$$

and hence M satisfies the condition (4.8) in Theorem 4.4. Thus, from Theorem 4.4, it follows that

$$\nabla R_1 = 0 \quad \text{and} \quad R_1 = 5R_1^* .$$

Thus, taking account of (2.4), (2.5) and (4.14), we may easily see that M is an Einstein space. Q. E. D.

In [17], Takamatsu and the second named author have proved the following

PROPOSITION 4.6. *There does not exist any dimensional, except 6-dimensional, non-Kähler, nearly Kähler manifold of constant holomorphic sectional curvature.*

From Propositions 4.1 and 4.6, it follows immediately that a non-Kähler, nearly Kähler manifold of constant holomorphic sectional curvature is a 6-dimensional space of positive constant curvature, and satisfies the condition $T(1, 1)$. In the rest of this section, we shall prove a result (Theorem 4.10) related to Proposition 4.6. We assume that $M=(M, J, \langle , \rangle)$ is an n $(=2m)$-dimensional connected non-Kähler, Einstein nearly Kähler manifold with vanishing first Chern form, and furthermore satisfies the condition $T(\rho, \sigma)$ with $5\rho>4\sigma$ and is holomorphically $\delta(>2/(\rho+3))$-pinched. First, we estimate the values of the functions f_λ $(\lambda=1, 3, 4)$ on $S(M)$.

LEMMA 4.7. *For each point $(p, x)\in S(M)$, we have*

$$f_1(p, x)\geqq\frac{l}{8}\{(5\rho-4\sigma)(n+2)\delta-8\rho\}H(x).$$

Proof. Let $\{e_i\}=\{e_\alpha, e_{\bar\alpha}\}$ $(x=e_1)$ be an orthonormal basis of M_p which diagonalizes the matrix $(\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle)$ $(1\leqq i, j\leqq n)$. Then, by the hypothesis for M and (2.10), (2.11), (2.16), (3.13), (3.23) and (4.17), we get

$$\begin{aligned}
f_1(p, x)&=\sum_{i,j} R_{ixj\bar x}\langle(\nabla_{e_i}J)x, (\nabla_{e_j}J)x\rangle\\
&=\sum_i R_{ixix}\|(\nabla_{e_i}J)x\|^2-\sum_i \|(\nabla_{e_i}J)x\|^4\\
&\geqq\frac{S}{5n}(5\rho-4\sigma)H(x)-\rho H(x)^2\\
&\geqq\frac{n+2}{8}(5\rho-4\sigma)\delta lH(x)-\rho lH(x)\\
&=\frac{l}{8}\{(5\rho-4\sigma)(n+2)\delta-8\rho\}H(x).
\end{aligned}$$

Q.E.D.

LEMMA 4.8. *For each point $(p, x)\in S(M)$, we have*

$$f_3(p, x)\leqq\delta l\Big(\frac{S}{n}-H(x)\Big).$$

Proof. Let $\{e_i\}=\{e_\alpha, e_{\bar\alpha}\}$ $(x=e_1)$ be an orthonormal basis of M_p as in the proof of Lemma 4.7. Then, by (2.10), (2.11), (3.13) and (4.17), we get

$$\begin{aligned}
f_3(p, x)&=\sum_i \langle R(x, (\nabla_{e_i}J)x)x, (\nabla_{e_i}J)x\rangle\\
&\leqq\sigma H(x)\Big(\frac{S}{n}-H(x)\Big)\\
&\leqq\sigma l\Big(\frac{S}{n}-H(x)\Big).
\end{aligned}$$

Q.E.D.

LEMMA 4.9. *For each point $(p, x)\in S(M)$, we have*

$$f_4(p, x)\leqq -\frac{n+2}{2}\delta l H(x).$$

Proof. By (2.10), (2.20), (3.13) and (3.23), we get

$$f_4(p, x)=-\frac{4S}{5n}H(x)$$
$$\leqq -\frac{n+2}{2}\delta l H(x). \qquad \text{Q.E.D.}$$

Next, we estimate the value $\int_{S_p} g_3\omega_2$. By (2.10), (2.11), (3.16), (3.20), (3.21) and (3.22), we have

$$\int_{S_p} g_3\omega_2 \leqq \sigma l\left[\int_{S_p}\sum_k (R_{x\bar{x}xk})^2\omega_2-\int_{S_p}H^2\omega_2\right]=\frac{\sigma l}{16}\int_{S_p}\|\mathrm{grad}^v H\|^2\omega_2. \tag{4.18}$$

We are now in a position to prove the following

THEOREM 4.10. *Let $M=(M, J, \langle\,,\rangle)$ be an n $(\geqq 6)$-dimensional connected compact non-Kähler, nearly Kähler manifold with constant scalar curvature. If M satisfies the condition $T(\rho, \sigma)$ with $5\rho>4\sigma$, $3\rho\geqq 4\sigma-1$, and is holomorphically δ-pinched $(\delta>2/(\rho+3)$ and $\delta\geqq(4\sigma+3\rho)/(15\rho-12\sigma+4))$, then M is isometric to a 6-dimensional sphere of constant curvature.*

Proof. First of all, we note

$$1-\frac{4\sigma+3\rho}{15\rho-12\sigma+4}=\frac{4(3\rho-4\sigma+1)}{15\rho-12\sigma+4}\geqq 0,$$

and

$$\frac{4\sigma+3\rho}{15\rho-12\sigma+4}-\frac{5(n+2)\sigma+24\rho-8\sigma}{(n+2)(15\rho-12\sigma+4)} \tag{4.19}$$
$$=\frac{(n-6)(3\rho-\sigma)}{(n+2)(15\rho-12\sigma+4)}$$
$$\geqq\frac{(n-6)(3\sigma-2\rho)}{(n+2)(15\rho-12\sigma+4)}\geqq 0.$$

Next, from the hypothesis for M and Theorem 4.5, it follows that M is an Einstein space with vanishing first Chern form. Furthermore, by (3.19), (4.18), (4.19) and Lemmas 4.7~4.9, we have

$$0\geqq 2\int_{S(M)} g_1\omega \tag{4.20}$$
$$+\frac{l}{2}\left\{\frac{(n+2)(15\rho-12\sigma+4)\delta+8\sigma-24\rho-4(n+2)\sigma}{4(n+2)}+\frac{2\rho}{\rho+3}\right\}\int_{S(M)}\|\mathrm{grad}^v H\|^2\omega$$

$$+\frac{32S^2 l}{25n^2(n+2)^2}\{(15\rho-12\sigma+4)(n+2)\delta+8\sigma-24\rho-5(n+2)\sigma\}V_2\mathrm{Vol}(M)$$

$$\geqq 2\int_{S(M)} g_1\omega+\frac{l}{2}\left(\frac{\sigma}{4}+\frac{2\rho}{\rho+3}\right)\int_{S(M)}\|\mathrm{grad}^v H\|^2\omega\geqq 0\,.$$

Thus, from (4.20) and Proposition 3.2, it follows that M is a space of constant holomorphic sectional curvature. Therefore, the theorem follows immediately from Propositions 4.1 and 4.6. Q.E.D.

§ 5. An example.

We shall recall some elementary facts about Riemannian 3-symmetric spaces (cf. [8], [21]). Let $(G/K, J, \langle\,,\rangle)$ be a compact Riemannian 3-symmetric space such that the Riemannian metric $\langle\,,\rangle$ is determined by a biinvariant Riemannian metric on G and J is the canonical almost complex structure. Then it is known that $(G/K, J, \langle\,,\rangle)$ is a nearly Kähler manifold ([8]). We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Then we have the following direct sum decomposition ([8]):

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{m}\,,\quad \mathrm{Ad}(K)\mathfrak{m}=\mathfrak{m}\,, \tag{5.1}$$

where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. We may identify the subspace $\mathfrak{m}$ with the tangent space $(G/K)_{eK}$ of G/K at the origin $eK\in G/K$. Under this identification, we have the following formulas ([8], [20]):

$$(\nabla_x J)y=-J[x,\,y]_{\mathfrak{m}}\,,\qquad x,\,y\in\mathfrak{m}\,, \tag{5.2}$$

$$K(x,\,y)=\frac{1}{4}\|[x,\,y]_{\mathfrak{m}}\|^2+\|[x,\,y]_{\mathfrak{k}}\|^2, \tag{5.3}$$
$$x,\,y\in\mathfrak{m}\quad\text{with}\quad \|x\|=\|y\|=1\,,\ \langle x,\,y\rangle=0\,.$$

In particular, we consider the 6-dimensional compact Riemannian 3-symmetric space $(Sp(2)/(U(1)\times Sp(1)), J, \langle\,,\rangle)$ in which the Riemannian metric $\langle\,,\rangle$ is induced from the inner product

$$(x,\,y)=-\text{Real part of }(\mathrm{Trace}\,xy)\,,\qquad x,\,y\in\mathfrak{sp}(2)\,.$$

We put $G=Sp(2)$ and $K=U(1)\times Sp(1)$. Let $\boldsymbol{H}$ be the algebra of quaternions, i.e.,

$$\boldsymbol{H}=\{q=a_0+a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+a_3\boldsymbol{e}_3\mid a_0,\,a_1,\,a_2,\,a_3\in\boldsymbol{R}\,,\quad \boldsymbol{e}_i^2=-1\ (1\leqq i\leqq 3)\,,$$
$$\boldsymbol{e}_1\boldsymbol{e}_2=-\boldsymbol{e}_2\boldsymbol{e}_1=\boldsymbol{e}_3\,,\quad \boldsymbol{e}_2\boldsymbol{e}_3=-\boldsymbol{e}_3\boldsymbol{e}_2=\boldsymbol{e}_1\,,\quad \boldsymbol{e}_3\boldsymbol{e}_1=-\boldsymbol{e}_1\boldsymbol{e}_3=\boldsymbol{e}_2\}\,.$$

Then it is well known that the Lie algebra $\mathfrak{sp}(2)$ of $Sp(2)$ is given by

$$\mathfrak{sp}(2)=\{x\in\mathfrak{gl}(2,\,\boldsymbol{H})\mid {}^t x=-\bar{x}\}\,.$$

We put

(5.4)
$$x_1=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \qquad y_1=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & \boldsymbol{e}_1\\ \boldsymbol{e}_1 & 0\end{bmatrix},$$
$$x_2=\begin{bmatrix}-\boldsymbol{e}_2 & 0\\ 0 & 0\end{bmatrix}, \qquad y_2=\begin{bmatrix}\boldsymbol{e}_3 & 0\\ 0 & 0\end{bmatrix},$$
$$x_3=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & \boldsymbol{e}_2\\ \boldsymbol{e}_2 & 0\end{bmatrix}, \qquad y_3=\frac{1}{\sqrt{2}}\begin{bmatrix}0 & \boldsymbol{e}_3\\ \boldsymbol{e}_3 & 0\end{bmatrix},$$
$$s_1=\begin{bmatrix}0 & 0\\ 0 & \boldsymbol{e}_2\end{bmatrix}, \qquad s_2=\begin{bmatrix}0 & 0\\ 0 & \boldsymbol{e}_3\end{bmatrix},$$
$$t_1=\begin{bmatrix}\boldsymbol{e}_1 & 0\\ 0 & 0\end{bmatrix}, \qquad t_2=\begin{bmatrix}0 & 0\\ 0 & \boldsymbol{e}_1\end{bmatrix}.$$

Then we see that $\{x_i, y_i\ (1\leqq i\leqq 3),\ s_1, s_2, t_1, t_2\}$ is an orthonormal basis of $\mathfrak{g}=\mathfrak{sp}(2)$ and the Lie algebra $\mathfrak{k}$ of K (resp. the subspace $\mathfrak{m}$ of $\mathfrak{g}$ in the decomposition (5.1)) is linearly spanned by $\{s_1, s_2, t_1, t_2\}$ (resp. $\{x_i, y_i\ (1\leqq i\leqq 3)\}$) over $\boldsymbol{R}$ (cf. [15]).

The canonical almost complex structure J is given by

(5.5)
$$Jx_i=y_i\,,\quad Jy_i=-x_i \qquad (1\leqq i\leqq 3)\,.$$

By (5.2), we get

(5.6)
$$\|(\nabla_x J)y\|^2=1\,,$$

for $x, y\in\mathfrak{m}$ with $\|x\|=\|y\|=1$, $\langle x, y\rangle=\langle x, Jy\rangle=0$. By (5.6), we see that $(Sp(2)/(U(1)\times Sp(1)), J, \langle\,,\rangle)$ is a non-Kähler, nearly Kähler manifold.

By (5.3), (5.4) and (5.5), by direct computation, we get

(5.7)
$$\begin{aligned} H(x)&=\|[x, Jx]_{\mathfrak{k}}\|^2\\ &=2\Big\{5\Big(a_1^2+b_1^2+a_3^2+b_3^2-\frac{3}{5}\Big)^2+\frac{1}{5}\Big\}, \end{aligned}$$

for any unit vector $x=a_1x_1+b_1y_1+a_2x_2+b_2y_2+a_3x_3+b_3y_3\in\mathfrak{m}$. By (5.7), we have easily

(5.8)
$$\frac{2}{5}\leqq H(x)\leqq 4\,.$$

Thus, by (5.6) and (5.8), we see that $(Sp(2)/(U(1)\times Sp(1)), J, \langle\,,\rangle)$ is holomorphically 1/10-pinched and satisfies the condition $T(1/4, 5/2)$.

Let x be any unit vector in $\mathfrak{m}$ and y any unit vector in $\mathfrak{m}$ which is orthogonal to x. Then we may put

(5.9)
$$y=aJx+bz\,,$$

where z is a unit vector in $\mathfrak{m}$ with $\langle x, z\rangle=0$, $\langle Jx, z\rangle=0$, and $a, b\in\boldsymbol{R}$ with $a^2+b^2=1$. By (5.3), taking account of (5.2), (5.6) and (5.9), we have

(5.10) $$K(x, y)=\frac{1}{4}\|b[x, z]_{\mathfrak{m}}\|^2+\|a[x, Jx]_{\mathfrak{k}}+b[x, z]_{\mathfrak{k}}\|^2$$
$$=\frac{b^2}{4}+\|a[x, Jx]_{\mathfrak{k}}+b[x, z]_{\mathfrak{k}}\|^2.$$

Therefore, by (5.7), (5.8) and (5.10), we may easily see that $(Sp(2)/(U(1)\times Sp(1)), J, \langle , \rangle)$ has strictly positive sectional curvature.

We remark that $Sp(2)/(U(1)\times Sp(1))$ is diffeomorphic to a complex projective space of complex dimension 3 ([15]). We also note that K. Furukawa has obtained the estimation (5.8) in unpublished work.

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