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GROWTH OF COMPOSITE ENTIRE FUNCTIONS

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Introduction. If f and g are transcendental entire functions then Clunie [1] proved that $\lim_{r\to\infty}\frac{T(r, f(g))}{T(r, f)}=\infty$. An obvious question arises is, what can be said about the ratio

$$\frac{\log T(r, f(g))}{T(r, f)} \tag{1}$$

when $r\to\infty$? In general by considering $g(z)=e^{e^z}$, and $f(z)=e^z$, we see that the ratio (1) also tends to infinity. However if we put some restriction on the orders of f and g then we can show that the above ratio is bounded above by a finite quantity. Thus the purpose of this paper will be to prove some results dealing with the ratios that are of the form (1). We start with

THEOREM 1. *Let $f(z)$ and $g(z)$ be entire functions of finite order such that $g(0)=0$ and $\rho_g<\lambda_f\leqq\rho_f$ where ρ, λ denote respectively the order and the lower order for the corresponding functions. Then*

$$\limsup_{r\to\infty}\frac{\log T(r, f(g))}{T(r, f)}\leqq\rho_f .$$

Note. (i) From the hypothesis it is clear that f must necessarily be transcendental.

(ii) The theorem does not hold true when $\rho_g=\rho_f$, for let $f(z)=e^z$ and $g(z)=e^z-1$, then $\rho_g=\rho_f=1$ and $T(r, f(g))\sim\frac{e^r}{(2\pi^3 r)^{1/2}}$ see [2, 7], so that

$$\limsup_{r\to\infty}\frac{\log T(r, f(g))}{T(r, f)}=\pi .$$

(iii) In case $\rho_g>\rho_f$ we shall show that the limit superior will tend to infinity. Thus we shall prove

THEOREM 2. *Let $f(z)$ and $g(z)$ be entire functions of finite order with $\rho_g>\rho_f$. Then*

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$$\limsup_{r\to\infty}\frac{\log T(r, f(g))}{T(r, f)}=\infty .$$

For the proof of Theorem 1 we shall need the following lemma of K. Niino and N. Suita [3].

LEMMA. *Let $f(z)$ and $g(z)$ be entire functions. Let $g(0)=0$. Then $T(r, f(g)) \leqq T(M(r, g), f)$ for all $r>0$.*

Proof of theorem 1. By definition of order and lower order we have

$$T(r, f)<r^{\rho_f+\varepsilon} \qquad \text{for all} \quad r\geqq r_0$$

$$T(r, f)>r^{\lambda_f-\varepsilon} \qquad \text{for all} \quad r\geqq r_0$$

(r_0 need not be the same at every stage).

Now by the lemma

$$\begin{aligned}\log T(r, f(g)) &\leqq \log T(M(r, g), f)\\ &<(\rho_f+\varepsilon)\log(M(r, g)) && \text{for all} \quad r\geqq r_0\\ &<(\rho_f+\varepsilon)r^{\rho_g+\varepsilon} && \text{for all} \quad r\geqq r_0\\ &<(\rho_f+\varepsilon)r^{\lambda_f-\varepsilon} && \text{by choosing}\end{aligned}$$

$\varepsilon>0$ so small that $\rho_g+\varepsilon<\lambda_f-\varepsilon$.

On the other hand, $T(r, f)>r^{\lambda_f-\varepsilon}$ for all $r\geqq r_0$. Thus for large r,

$$\frac{\log T(r, f(g))}{T(r, f)}<(\rho_f+\varepsilon) .$$

The theorem now follows since $\varepsilon>0$ is arbitrary.

Proof of theorem 2. We prove this theorem on the same lines as K. Niino and C. C. Yang [4].

$$\begin{aligned}T(r, f(g)) &\geqq \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)+o(1), f\right) && \text{see [4].}\\ &\geqq \frac{1}{3}\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right)+o(1)\right\}^{\lambda_f-\varepsilon} && \text{for all} \quad r\geqq r_0\\ &\geqq \frac{1}{3}\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right)\right\}^{\lambda_f-\varepsilon} && \text{for all} \quad r\geqq r_0\\ &\geqq \frac{1}{3}\left(\frac{1}{9}\right)^{\lambda_f-\varepsilon}\{\exp(r/4)^{\rho_g-\varepsilon}\}^{\lambda_f-\varepsilon} && \text{for a sequence}\end{aligned}$$

$r=r_n\to\infty$. Thus for a sequence $\{r_n\}$

$$\log T(r_n, f(g))\geqq\log A+(\lambda_f-\varepsilon)(r_n/4)^{\rho_g-\varepsilon} \tag{2}$$

where $A=\frac{1}{3}\left(\frac{1}{9}\right)^{\lambda_f-\varepsilon}$.

On the other hand for all $r\geqq r_0$, $T(r, f)<r^{\rho_f+\varepsilon}$. Thus for a sequence $\{r_n\}$ (where each $r_n\geqq r_0$) we have

$$\frac{\log T(r_n, f(g))}{T(r_n, f)}>\frac{\log A}{r_n^{\rho_f+\varepsilon}}+\frac{(\lambda_f-\varepsilon)}{r_n^{\rho_f+\varepsilon}}\left(\frac{r_n}{4}\right)^{\rho_g-\varepsilon}.$$

And so, $\limsup_{r\to\infty}\frac{\log T(r, f(g))}{T(r, f)}=\infty$, since we can choose $\varepsilon>0$ such that $\rho_g-\varepsilon>\rho_f+\varepsilon$. This proves theorem 2.

An immediate consequence of theorem 1 is the following corollary

COROLLARY 1. *Let f and g be entire functions satisfying the conditions of theorem* 1. *Further let* $\liminf_{r\to\infty}\frac{\log T(r, f(g))}{T(r, f)}\geqq\rho_f$. *Then the hyper order of $f(g)$ is ρ_f. (Hyper order of a function f is defined to be* $\limsup_{r\to\infty}\frac{\log\log T(r, f)}{\log r}$).

The proof follows easily since the hypothesis and the theorem 1 imply that $\log T(r, f(g))\sim\rho_f T(r, f)$.

We now give an application of theorem 2.

COROLLARY 2. *Let f and g be transcendental entire functions of finite order. Further let $\rho_g>\rho_f$ then $f(g)$ is of infinite order.*

Proof.

$$\limsup_{r\to\infty}\frac{\log T(r, f(g))}{\log r}=\limsup_{r\to\infty}\left\{\frac{\log T(r, f(g))}{T(r, f)}\cdot\frac{T(r, f)}{\log r}\right\}$$

$$\geqq\limsup_{r\to\infty}\frac{\log T(r, f(g))}{T(r, f)}\liminf_{r\to\infty}\frac{T(r, f)}{\log r}.$$

But for a transcendental entire function f, it is well known that $\lim_{r\to\infty}\frac{T(r, f)}{\log r}=\infty$. The result now follows using theorem 2.

In [theorem 2, 1], Clunie has proved that if f and g are transcendental entire functions then $\lim_{r\to\infty}\frac{T(r, f(g))}{T(r, g)}=\infty$. So the obvious question is what can be said about $\lim_{r\to\infty}\frac{\log T(r, f(g))}{T(r, g)}$? This we have been unable to solve. However if we consider the ratio $\frac{\log\log T(r, f(g))}{\log T(r, g)}$ or $\frac{\log T(r, f(g))}{\log T(r, g)}$ we have obtained the following two theorems.

THEOREM 3. *Let f and g be transcendental entire functions of finite order. Let $g(0)=0$ and let $\lambda_g>0$. Then*

$$\limsup_{r\to\infty}\frac{\log\log T(r, f(g))}{\log T(r, g)}\leqq\frac{\rho_g}{\lambda_g}.$$

Proof. As in theorem 1,

$$\log T(r, f(g)) < (\rho_f + \varepsilon) r^{\rho_g + \varepsilon} \quad \text{for all} \quad r \geqq r_0 .$$

Thus for all $r \geqq r_0$ we have

$$\log\log T(r, f(g)) < \log(\rho_f + \varepsilon) + (\rho_g + \varepsilon) \log r .$$

On the other hand,

$$\log T(r, g) > (\lambda_g - \varepsilon) \log r \quad \text{for all} \quad r \geqq r_0 .$$

Thus

$$\limsup_{r \to \infty} \frac{\log\log T(r, f(g))}{\log T(r, g)} \leqq \frac{\rho_g}{\lambda_g} .$$

THEOREM 4. *Let f and g be transcendental entire functions of finite order with $\rho_g > 0$, then* $\limsup_{r \to \infty} \dfrac{\log T(r, f(g))}{\log T(r, g)} = \infty$.

Proof. From (2), for a sequence $\{r_n\}$ we have,

$$\log T(r, f(g)) \geqq \log A + (\lambda_f - \varepsilon) \left(\frac{r_n}{4} \right)^{\rho_g - \varepsilon}$$

where $A = \dfrac{1}{3} \left(\dfrac{1}{9} \right)^{\lambda_f - \varepsilon}$. Also

$$\log T(r, g) < (\rho_g + \varepsilon) \log r \quad \text{for all} \quad r \geqq r_0 .$$

Thus

$$\frac{\log T(r_n, f(g))}{\log T(r_n, g)} \geqq \frac{\log A}{(\rho_g + \varepsilon) \log r_n} + \frac{\lambda_{f - \varepsilon}}{4^{\rho_g - \varepsilon}} \cdot \frac{(r_n)^{\rho_g - \varepsilon}}{(\rho_g + \varepsilon) \log r_n}$$

which tends to infinity as $r_n \to \infty$, since $\rho_g > 0$. This yields the desired result.

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