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COMPLEX SUBMANIFOLDS OF CERTAIN NON-KAEHLER MANIFOLDS

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§ 0. Introduction.

Complex submanifolds of Kaehlerian manifolds have been studied extensively by many differential geometers (see, for example, the bibliography of Ogiue's paper [5]), but complex submanifolds of non-Kaehlerian Hermitian manifold have not been explored to any great extent.

On the other hand, E. Calabi and B. Eckmann [2] proved that the product of two odd-dimensional spheres which admits a Hermitian structure is called Calabi—Eckmann manifold. A Calabi—Eckmann manifold has two structures, namely the product Riemannian structure and the complex structure that is mentioned above. Thus submanifolds of Calabi—Eckmann manifold have two-sided property. One is that they are submanifolds of a product manifold and another is that they are submanifolds of a complex manifold.

In § 1, we study first of all, submanifolds of Riemannian product manifolds using the same method by G. D. Ludden and M. Okumura [3].

In § 2, we study properties of complex submanifolds of a Riemannian product of two Sasakian manifolds and prove that any compact complex submanifold of certain non-Kaehlerian, Hermitian manifold, which is a generalization of Calabi—Eckmann manifold is minimal.

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§ 1. Submanifold of Riemannian product manifolds.

Let $\overline{M}_1$, $\overline{M}_2$ be respectively differentiable manifolds of dimensions n and m, and we consider the product manifold $\overline{M}_1 \times \overline{M}_2$. We denote by $\overline{P}_i$ ($i=1, 2$) the projection mappings of the tangent space of $\overline{M}_1 \times \overline{M}_2$ to that of $\overline{M}_i$ ($i=1, 2$), where the tangent space to $\overline{M}_1$ (resp. $\overline{M}_2$) is identified with that of $\overline{M}_1 \times$(point) (resp. (point)$\times \overline{M}_2$). Then we have

$$\overline{P}_1+\overline{P}_2=I\,, \quad \overline{P}_1{}^2=\overline{P}_1\,, \quad \overline{P}_2{}^2=\overline{P}_2\,, \quad \overline{P}_1\overline{P}_2=\overline{P}_2\overline{P}_1=0\,,$$

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where I denotes the identity transformation of the tangent space of $\bar{M}_1 \times \bar{M}_2$. We put $\bar{P}=\bar{P}_1-\bar{P}_2$. Then it follows that

$$\bar{P}^2=I, \quad \operatorname{tr} \bar{P}=m-n,$$

where $\operatorname{tr} \bar{P}$ denotes the trace $\bar{P}$. We call $\bar{P}$ an *almost product structure* on $\bar{M}_1 \times \bar{M}_2$ (cf. [3]).

If the manifolds $\bar{M}_1$, $\bar{M}_2$ are both Riemannian manifolds, we define a Riemannian metric of $\bar{M}_1 \times \bar{M}_2$ by

$$\bar{g}(X, Y)=\bar{g}_1(\bar{P}_1 X, \bar{P}_1 Y)+\bar{g}_2(\bar{P}_2 X, \bar{P}_2 Y),$$

where $\bar{g}_1$ and $\bar{g}_2$ are respectively the Riemannian metrics of $\bar{M}_1$ and $\bar{M}_2$. Then it follows that

$$\bar{g}(\bar{P}X, Y)=\bar{g}(X, \bar{P}Y), \quad \bar{\nabla}_X \bar{P}=0,$$

where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to the Riemannian connection of $\bar{g}$.

Let M^n be a submanifold of codimension p in $\bar{M}_1^m \times \bar{M}_2^{n+p-m}$ and suppose $\imath: M \to \bar{M}_1 \times \bar{M}_2$ the immersion. For a tangent vector field X to M and orthonormal normal vectors N_α $(\alpha=1, \cdots, p)$ to M, the transforms $\bar{P}i_*X$ and $\bar{P}N_\alpha$ $(\alpha=1, \cdots, p)$ by $\bar{P}$ can be written as follows;

$$\bar{P}i_*X=i_*PX+\textstyle\sum_{\alpha=1}^p u^\alpha(X)N_\alpha, \tag{1.1}$$

$$\bar{P}N_\alpha=i_*U_\alpha+\textstyle\sum_{\beta=1}^p \lambda_{\alpha\beta}N_\beta \qquad (\beta=1, \cdots, p), \tag{1.2}$$

where P defines a symmetric linear transformation of the tangent bundle $T(M)$ of M, while u^α, U_α and $\lambda_{\alpha\beta}$ define 1-forms, vector fields and functions on a neighborhood of a point of M respectively. Moreover, we easily see that $g(U_\alpha, X)=u^\alpha(X)$, where g is the induced Riemannian metric on M.

We denote by ∇ the operator of covariant differentiation with respect to the Riemannian connection of g. Then Gauss and Weingarten equations are given by

$$\begin{aligned}
\bar{\nabla}_{i_*X}\imath_*Y &= i_*\nabla_X Y+\sigma(X, Y) \\
&= i_*\nabla_X Y+\textstyle\sum_{\alpha=1}^p g(A_{N_\alpha}X, Y)N_\alpha, \\
\bar{\nabla}_{i_*X}N_\alpha &= -\imath_*A_{N_\alpha}X+\nabla_X^\perp N_\alpha, \\
&= -\imath_*A_{N_\alpha}X+\textstyle\sum_{\beta=1}^p s_{\alpha\beta}(X)N_\beta,
\end{aligned}$$

where σ and A are respectively the second fundamental form and the corresponding second fundamental tensor, while $\nabla^\perp$, $s_{\alpha\beta}$ the normal connection and the third fundamental tensor respectively. They satisfy

$$\sigma(X, Y)=\sigma(Y, X), \quad s_{\alpha\beta}(X)=-s_{\beta\alpha}(X).$$

§2. Complex submanifold of certain non-Kaehler manifolds.

Let $\bar{M}=\bar{M}^{2n+1}(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a Sasakian manifold of dimension $2n+1$. The structure tensors $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ satisfy (cf. [1])

$$\bar{\phi}=-I+\bar{\eta}\otimes\bar{\xi}\,,\quad \bar{\phi}\bar{\xi}=0\,,\quad \bar{\eta}(\bar{\phi}X)=0\,,\quad \bar{\eta}(\bar{\xi})=1\,,$$

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y)=\bar{g}(X, Y)-\bar{\eta}(X)\bar{\eta}(Y)\,,\quad \bar{\eta}(X)=\bar{g}(X, \bar{\xi})\,,$$

$$\bar{\nabla}_X\bar{\xi}=\bar{\phi}X\,,\quad (\bar{\nabla}_X\bar{\phi})Y=\bar{\eta}(Y)X-\bar{g}(X, Y)\bar{\xi}\,.$$

Let $\bar{M}_1^{2m+1}\times\bar{M}_2^{2n+2p-2m-1}$ be the Riemannian product manifold of Sasakian manifolds, and let $\Sigma_1=(\bar{\phi}'_1, \bar{\eta}'_1, \bar{\xi}'_1, \bar{g}'_1)$ (resp. $\Sigma_2=(\bar{\phi}'_2, \bar{\eta}'_2, \bar{\xi}'_2, \bar{g}'_2)$) be the Sasakian structure on $\bar{M}_1$ (resp. $\bar{M}_2$). Then actions of $\bar{\phi}_i$ $(i=1, 2)$ on $\bar{M}_1\times\bar{M}_2$ are defined by $\bar{\phi}_i=\bar{\phi}'_i\bar{P}_i$ where $\bar{P}_1$ (resp. $\bar{P}_2$) is the projection of tangent space of $\bar{M}_1\times\bar{M}_2$ to that of $\bar{M}_1$ (resp. $\bar{M}_2$), where the tangent space to $\bar{M}_1$ (resp. $\bar{M}_2$) is identified with that of $\bar{M}_1\times$(point) (resp. (point)$\times\bar{M}_2$). Similarly, $\bar{\xi}_i$ and $\bar{\eta}_i$ are defined by $\bar{P}_i\bar{\xi}_i=\bar{\xi}'_i$, $\bar{P}_i\bar{\xi}_j=0$ $(i, j=1, 2, i\neq j)$ and $\bar{\eta}_i(X)=\bar{\eta}'_i(\bar{P}_iX)$ on $\bar{M}_1\times\bar{M}_2$.

For any tangent vector X of $\bar{M}_1\times\bar{M}_2$, we define

$$JX=\bar{\phi}_1X-\bar{\eta}_2(X)\bar{\xi}_1+\bar{\phi}_2X+\bar{\eta}_1(X)\bar{\xi}_2\,. \tag{2.1}$$

Then J defines a complex structure on $\bar{M}_1\times\bar{M}_2$ (cf. [4]). Moreover it is easily checked that Riemannian product metric $\bar{g}$ on $\bar{M}_1\times\bar{M}_2$ is a non-Kaehler Hermitian metric on the complex manifold.

Let M^{2n} be a complex submanifold of codimension $2p$ of $\bar{M}_1\times\bar{M}_2$ with the Hermitian structure which is defined as above, and $i: M\to\bar{M}_1\times\bar{M}_2$ be the immersion.

EXAMPLE. Let $P_n^i(\boldsymbol{C})$ $(i=1, 2)$ be complex projective spaces with homogeneous coordinates $(z_0^i, \cdots, z_n^i)$ and constant holomorphic sectional curvature 4. Let M be a complex hypersurface of $P_n^1(\boldsymbol{C})\times P_n^2(\boldsymbol{C})$ defined by $\sum_{j=0}^n z_j^1 z_j^2=0$ and (M, T^2) be the torus bundle over M such that the following diagram is commutative;

$$\begin{array}{ccc}(M, T^2) & \longrightarrow & S^{2n+1}\times S^{2n+1}\\ \downarrow & & \downarrow\\ M & \longrightarrow & P_n(\boldsymbol{C})\times P_n(\boldsymbol{C})\,.\end{array}$$

Then (M, T^2) is the complex hypersurfaces of $S^{2n+1}\times S^{2n+1}$. Moreover, M is diffeomorphic to $U(n+1)/U(n-1)\times T^2$ (Kaehler C-space) and (M, T^2) is diffeomorphic to $U(n+1)/U(n-1)$ (Complex Stiefel manifold).

We take orthonormal normal vectors $N_1, \cdots, N_{2p}$ to M in such a way that $N_{2q}=JN_{2q-1}$ $(q=1, \cdots, p)$. Suppose that the vector fields $\bar{\xi}_i$ $(i=1, 2)$ are not always tangent to M. Then there exists such a point $x\in M$ that the normal parts of $\bar{\xi}_i$ $(i=1, 2)$ do not vanish, because $\bar{\xi}_2=J\bar{\xi}_1$. At this point, we can choose the unit normal frame to M in such a way that, N_i $(i=1, 2)$ are the normal directions of $\bar{\xi}_i$ ($i=1, 2$) and extend them to local fields. Hence $\bar{\xi}_i$ can be written as a sum of the tangential components and the normal components in the following way,

$$\bar{\xi}_j = i_*\xi_j + rN_j \qquad (j=1, 2). \tag{2.2}$$

Then ξ_j and r define vector fields and a function on M respectively. Let X be a tangent vector field on M. Then we immediately get

$$\bar{\eta}_j(i_*X) = \bar{g}(\bar{\xi}_j, i_*X) = g(\xi_j, X),$$

$$\bar{\eta}_j(N_j) = \bar{g}(\bar{\xi}_j, N_j) = r \qquad (j=1, 2, \quad 0 \leqq r \leqq 1), \tag{2.3}$$

$$\bar{\eta}_j(N_k) = 0 \qquad (j=1, 2, \ k=1, \cdots, 2p, \ j \neq k), \tag{2.4}$$

where g is the induced Riemannian metric on M. The transforms $\bar{\phi}_j i_* X$ and $\bar{\phi}_j N_\alpha$ ($j=1, 2$, $\alpha=1, 2, \cdots, 2p$) of X and N_α by $\bar{\phi}_j$ can be written as

$$\begin{aligned} \bar{\phi}_1 i_* X &= i_*\phi_1 X + \textstyle\sum_{\alpha=1}^{2p} v^\alpha(X) N_\alpha \\ \bar{\phi}_2 i_* X &= i_*\phi_2 X + \textstyle\sum_{\alpha=1}^{2p} w^\alpha(X) N_\alpha \end{aligned} \tag{2.5}$$

$$\begin{aligned} \bar{\phi}_1 N_\alpha &= -i_* V_\alpha + \textstyle\sum_{\beta=1}^{2p} \mu_{\alpha\beta} N_\beta, \\ \bar{\phi}_2 N_\alpha &= -i_* W_\alpha + \textstyle\sum_{\beta=1}^{2p} \nu_{\alpha\beta} N_\beta, \end{aligned} \tag{2.6}$$

where ϕ_j's define skew-symmetric linear transformations of the tangent bundle of M, while v^α, w^α, V_α, W_α, $\mu_{\alpha\beta}$ and $\nu_{\alpha\beta}$ define 1-forms, vector fields and functions on a neighborhood of a point of M respectively. We easily see that $\mu_{\alpha\beta}$ and $\nu_{\alpha\beta}$ are skew-symmetric with respect to α and β and that

$$g(V_\alpha, X) = v^\alpha(X), \quad g(W_\alpha, X) = w^\alpha(X) \qquad (\alpha=1, \cdots, 2p).$$

Since $JN_1 = N_2$ and $JN_2 = -N_1$ hold, using (2.1), (2.3) and (2.6), we have

$$V_1 + W_1 - r\xi_2 = 0, \quad V_2 + W_2 + r\xi_1 = 0, \tag{2.7}$$

$$\mu_{1,2} + \nu_{1,2} + r^2 = 1, \tag{2.8}$$

$$\mu_{1,\beta} + \nu_{1,\beta} = \mu_{2,\beta} + \nu_{2,\beta} = 0 \qquad (\beta=3, 4, \cdots, 2p). \tag{2.9}$$

Similarly, from $JN_{2q-1} = N_{2q}$ and $JN_{2q} = -N_{2q-1}$ ($q=2, 3, \cdots, p$), we have

$$V_{2\alpha-1} + W_{2\alpha-1} = V_{2\alpha} + W_{2\alpha} = 0, \tag{2.10}$$

$$\begin{aligned} &\mu_{2\alpha-1,2\alpha} + \nu_{2\alpha-1,2\alpha} = 1, \\ &\mu_{2\alpha-1,\gamma} + \nu_{2\alpha-1,\gamma} = \mu_{2\beta,\gamma} + \nu_{2\beta,\gamma} = 0 \qquad (\gamma \neq 2\alpha, 2\beta-1). \end{aligned} \tag{2.11}$$

Let $\bar{P}$ be the almost product structure defined in § 1. Since $\bar{P}_1\bar{\xi}_1 = \bar{\xi}_1'$ and $\bar{P}_2\bar{\xi}_2 = \bar{\xi}_2'$ holds, $\bar{P}\bar{\xi}_1 = \bar{\xi}_1$ and $\bar{P}\bar{\xi}_2 = -\bar{\xi}_2$. From this, using (1.1), (1.2) and (2.2), we get

$$P\xi_1=\xi_1-rU_1\,,\quad P\xi_2=-\xi_2-rU_2\,,$$

$$u^1(\xi_1)=r(1-\lambda_{1,1})\,,\quad u^2(\xi_2)=-r(1+\lambda_{2,2})\,, \tag{2.12}$$

$$u^\alpha(\xi_1)=-r\lambda_{1,\alpha}\,,\quad u^\beta(\xi_2)=-r\lambda_{2,\beta}\qquad(\alpha\neq 1,\ \beta\neq 2)\,. \tag{2.13}$$

From the definition of $\bar{P}$, we have

$$\bar{P}\bar{\phi}_1=\bar{\phi}_1=\bar{\phi}_1\bar{P}\,,\quad \bar{P}\bar{\phi}_2=-\bar{\phi}_2=\bar{\phi}_2\bar{P}\,. \tag{2.14}$$

Applying $\bar{P}$ to $JN_1=N_2$ and $JN_2=-N_1$, using (2.1), we have

$$U_1=V_2-W_2+r\xi_1\,,\quad U_2=-V_1+W_1-r\xi_2\,,$$

$$\lambda_{1,1}=-\mu_{2,1}+\nu_{2,1}+r^2,\quad \lambda_{2,2}=\mu_{1,2}-\nu_{1,2}-r^2,$$

$$\lambda_{1,\alpha}=-\mu_{2,\alpha}+\nu_{2,\alpha}\,,\quad \lambda_{2,\beta}=\mu_{1,\beta}-\nu_{1,\beta}\,,\qquad(\alpha\neq 1,\ \beta\neq 2)\,,$$

because of (2.14). From these equations and (2.7)$\sim$(2.9), we obtain

$$U_2=-2V_1\,,\quad U_1=-2W_2\,,$$

$$\lambda_{2,2}=2\mu_{1,2}-1\,,\quad \lambda_{1,1}=2\nu_{2,1}+1\,, \tag{2.15}$$

$$\begin{aligned}&\lambda_{2,1}=0\,,\quad \lambda_{2,\alpha}=2\mu_{1,\alpha}\qquad(\alpha\neq 2)\,,\\ &\lambda_{1,2}=0\,,\quad \lambda_{1,\alpha}=2\nu_{2,\alpha}\qquad(\alpha\neq 1)\,.\end{aligned} \tag{2.16}$$

In the same way, we have, for N_k $(k=3, 4, \cdots, 2p)$

$$U_{2q}=-V_{2q-1}+W_{2q-1}\,,\quad U_{2q-1}=V_{2q}-W_{2q}\qquad(q=2, 3, \cdots. p)\,,$$

$$\lambda_{2q,\beta}=\mu_{2q-1,\beta}-\nu_{2q-1,\beta}\,,\quad \lambda_{2q-1,\beta}=-\mu_{2q,\beta}+\nu_{2q,\beta}\qquad(\beta=1, 2, \cdots, 2p)\,.$$

These equations, (2.10) and (2.11) imply that

$$U_{2q}=-2V_{2q-1}\,,\quad U_{2q-1}=-2W_{2q}\,,$$

$$\lambda_{2q,2q}=2\mu_{2q-1,2q}-1\,,\quad \lambda_{2q-1,2q-1}=2\nu_{2q,2q-1}+1\,,$$

$$\lambda_{2q,\beta}=2\mu_{2q-1,\beta}\,,\quad \lambda_{2q-1,\gamma}=2\nu_{2q,\gamma}\,,$$

$$\lambda_{2q,2q-1}=0\qquad(\beta\neq 2q,\ \gamma\neq 2q-1)\,.$$

On the other hand, since $\bar{\phi}_j\xi_k=0$ $(j=1, 2,\ k=1, 2)$ hold, using (2.2), (2.5) and (2.6), we have

$$\phi_1\xi_j=rV_j\,,\quad \phi_2\xi_j=rW_j\,,$$

$$v^\alpha(\xi_j)=r\mu_{\alpha,j}\,,\quad w^\alpha(\xi_j)=r\nu_{\alpha,j}\qquad(\alpha=1, 2, \cdots, 2p)\,. \tag{2.17}$$

Since $\bar{M}_1$ and $\bar{M}_2$ are Sasakian manifolds, one obtains that

$$\bar{\nabla}_X\xi_1=\bar{\phi}_1X\,,\quad \bar{\nabla}_X\xi_2=\bar{\phi}_2X\,.$$

Then, using Gauss, Weingarten equations and (2.5), we have

$$\nabla_X\xi_1=\phi_1X+rA_1X\,,\quad \nabla_X\xi_2=\phi_2X+rA_2X\,,$$

(2.18) $$\operatorname{grad} r=V_1-A_1\xi_1=W_2-A_2\xi_2\,,$$

(2.19) $$V_\alpha=A_\alpha\xi_1+rS_{1,\alpha}\qquad(\alpha\neq1)\,,$$
$$W_\alpha=A_\alpha\xi_2+rS_{2,\alpha}\qquad(\alpha\neq2)\,,$$

where $S_{\alpha\beta}$ are dual vectors of 1-forms $s_{\alpha\beta}(X)$. Hence

(2.20) $$\operatorname{div}\xi_1=r\operatorname{trace}A_1\,,\quad \operatorname{div}\xi_2=r\operatorname{trace}A_2\,.$$

Here, for simplicity, we have written A_α instead of A_{N_α} for a frame $N_1, N_2, \cdots, N_{2p}$ for N_xM. In the same way, for any $X\in T(M)$, we have

$$(\bar{\nabla}_X\bar{\phi}_j)N_\alpha=\bar{\eta}_j(N_\alpha)\bar{P}_jX-\bar{g}(\bar{P}_jX,\,N_\alpha)\bar{\xi}_j\qquad(j=1,\,2,\ \alpha=1,\,2,\,\cdots,\,2p)\,.$$

Since $\bar{P}_1=(I+P)/2$ and $\bar{P}_2=(I-P)/2$, making use of Gauss, Weingarten equations and (2.2)$\sim$(2.6), we get

$$\nabla_XV_1=-(r/2)(X+PX)+(1/2)u^1(X)\xi_1+\phi_1A_1X+\textstyle\sum_{\alpha=1}^{2p}(s_{1,\alpha}(X)V_\alpha-\mu_{1,\alpha}A_\alpha X)\,,$$

$$\nabla_XV_\alpha=(1/2)u^\alpha(X)\xi_1+\phi_1A_\alpha X+\textstyle\sum_{\alpha=1}^{2p}(s_{\alpha\beta}(X)V_\beta-\mu_{\alpha\beta}A_\beta X)\qquad(\alpha\neq1)\,.$$

$$\nabla_XW_2=-(r/2)(X-PX)-(1/2)u^2(X)\xi_2+\phi_2A_2X+\textstyle\sum_{\alpha=1}^{2p}(s_{2,\alpha}(X)W_\alpha-\nu_{2,\alpha}A_\alpha X)\,,$$

$$\nabla_XW_\alpha=-(1/2)u^\alpha(X)\xi_2+\phi_2A_\alpha X+\textstyle\sum_{\alpha=1}^{2p}(s_{\alpha\beta}(X)W_\beta-\nu_{\alpha\beta}A_\beta X)\qquad(\alpha\neq2)\,.$$

From above, using (2.12) and (2.13), we obtain

(2.21) $$\operatorname{div}V_1=-rn-(r/2)\operatorname{trace}P+(r/2)(1-\lambda_{1\,1})$$
$$+\textstyle\sum_{\alpha=1}^{2p}(s_{1,\alpha}(V_\alpha)-\mu_{1,\alpha}\operatorname{trace}A_\alpha)\,,$$

$$\operatorname{div}V_\alpha=-(r/2)\,\lambda_{1,\alpha}+\textstyle\sum_{\beta=1}^{2p}(s_{\alpha\beta}(V_\beta)-\mu_{\alpha\beta}\operatorname{trace}A_\beta)\qquad(\alpha\neq1)$$

$$\operatorname{div}W_2=-rn+(r/2)\operatorname{trace}P+(r/2)(1+\lambda_{2,2})$$
$$+\textstyle\sum_{\alpha=1}^{2p}(s_{2,\alpha}(W_\alpha)-\nu_{2,\alpha}\operatorname{trace}A_\alpha)\,,$$

(2.22) $$\operatorname{div}W_\alpha=(r/2)\lambda_{2,\alpha}+\textstyle\sum_{\beta=1}^{2p}(s_{\alpha\beta}(W_\beta)-\nu_{\alpha\beta}\operatorname{trace}A_\beta)\qquad(\alpha\neq2)\,.$$

Now we prove the following,

THEOREM. *Let $\bar{M}_1\times\bar{M}_2$ be the Riemannian product of Sasakian manifolds, and M be its compact complex submanifold with respect to the complex structure that is defined by* (2.1). *Then*

(1) $\bar{\xi}_i$ $(i=1, 2)$ *are tangent to* M,
(2) M *is a minimal submanifold of* $\overline{M}_1\times\overline{M}_2$.

Proof. First of all, we calculate $\operatorname{div} r\xi_i$ and $\operatorname{div} r\xi_2$. Making use of (2.17)~(2.20), we get

$$\begin{aligned}\operatorname{div} r\xi_2 &= (\xi_2 r)+r^2 \operatorname{trace} A_2\\ &= g(\operatorname{grad} r,\ \xi_2)+r^2 \operatorname{trace} A_2\\ &= g(V_1-A_1\xi_1,\ \xi_2)+r^2 \operatorname{trace} A_2\\ &= r\mu_{1,2}-g(A_1\xi_2,\ \xi_1)+r^2 \operatorname{trace} A_2\\ &= r\mu_{1,2}-g(W_1-rS_{2,1},\ \xi_1)+r^2 \operatorname{trace} A_2\\ &= r\mu_{1,2}+s_{2,1}(r\xi_1)+r^2 \operatorname{trace} A_2\,.\end{aligned}$$

In the same way, we have

$$\operatorname{div} r\xi_1=r\nu_{2,1}+s_{1,2}(r\xi_2)+r^2 \operatorname{trace} A_1\,.$$

Hence, making use of (2.7), (2.10), (2.15), (2.16), (2.21) and (2.22), we have

$$\begin{aligned}0&=\operatorname{div} V_1+\operatorname{div} W_1-\operatorname{div} r\xi_2\\ &=-rn-(r/2)\operatorname{trace} P+(r/2)(1-\lambda_{1,1})\\ &\quad+\textstyle\sum_{\alpha=1}^{2p}(s_{1,\alpha}(V_\alpha)-\mu_{1,\alpha}\operatorname{trace} A_\alpha)\\ &\quad+(r/2)\lambda_{2,1}+\textstyle\sum_{\alpha=1}^{2p}(s_{1,\alpha}(W_\alpha)-\nu_{1,\alpha}\operatorname{trace} A_\alpha)\\ &\quad-r\mu_{1,2}-s_{2,1}(r\xi_1)-r^2\operatorname{trace} A_2\\ &=-rn-(r/2)\operatorname{trace} P-(r/2)(\lambda_{1,1}+\lambda_{2,2})-\operatorname{trace} A_2\,.\end{aligned}$$

Consequently, we get

$$\operatorname{trace} A_2=-r(n+(\operatorname{trace} P)/2+(\lambda_{1,1}+\lambda_{2,2})/2)\,. \tag{2.23}$$

Similarly, we have

$$\begin{aligned}0&=\operatorname{div} V_2+\operatorname{div} W_2+\operatorname{div} r\xi_1\\ &=-(r/2)\lambda_{1,2}+\textstyle\sum_{\alpha=1}^{2p}(s_{2,\alpha}(V_\alpha)-\mu_{2,\alpha}\operatorname{trace} A_\alpha)\\ &\quad-rn+(r/2)\operatorname{trace} P+r(1+\lambda_{2,2})/2\\ &\quad+\textstyle\sum_{\alpha=1}^{2p}(s_{2,\alpha}(W_\alpha)-\nu_{2,\alpha}\operatorname{trace} A_\alpha)\\ &\quad+r\nu_{2,1}+s_{1,2}(r\xi_2)+r^2\operatorname{trace} A_1\\ &=\operatorname{trace} A_1-rn+(r/2)\operatorname{trace} P+r(\lambda_{1,1}+\lambda_{2,2})/2\,.\end{aligned}$$

Hence we obtain

$$\text{trace } A_1 = r(n-(1/2)\,\text{trace } P-(\lambda_{1,1}+\lambda_{2,2})/2)\,. \tag{2.24}$$

By quite the same computation, using (2.10) and (2.11), we have

$$\begin{aligned} 0 &= \operatorname{div} V_{2q-1} + \operatorname{div} W_{2q-1} \\ &= -(r/2)\,\lambda_{1,2q-1} + \textstyle\sum_{\alpha=1}^{2p}(s_{2q-1,\alpha}(V_\alpha) - \mu_{2q-1,\alpha}\,\text{trace } A_\alpha) \\ &\quad + (r/2)\,\lambda_{2,2q-1} + \textstyle\sum_{\alpha=1}^{2p}(s_{2q-1,\alpha}(W_\alpha) - \nu_{2q-1,\alpha}\,\text{trace } A_\alpha) \\ &= (r/2)(\lambda_{2,2q-1} - \lambda_{1,2q-1}) - \text{trace } A_{2q} - r(s_{2q-1,2}(\xi_1) - s_{2q-1,1}(\xi_2))\,, \end{aligned}$$

and

$$\begin{aligned} 0 &= \operatorname{div} V_{2q} + \operatorname{div} W_{2q} \\ &= (r/2)(\lambda_{2,2q} - \lambda_{1,2q}) + \text{trace } A_{2q-1} - r(s_{2q,2}(\xi_1) - s_{2q,1}(\xi_2))\,. \end{aligned}$$

Consequently, we get

$$\text{trace } A_{2q-1} = r((\lambda_{1,2q} - \lambda_{2,2q})/2 - s_{2q,1}(\xi_2) + s_{2q,2}(\xi_1))\,, \tag{2.25}$$

$$\text{trace } A_{2q} = r((\lambda_{2,2q-1} - \lambda_{1,2q-1})/2 - s_{2q-1,2}(\xi_1) + s_{2q-1,1}(\xi_2)) \tag{2.26}$$

$$(q=2,\ 3,\ \cdots,\ 2p)\,.$$

Then from (2.20), (2.23) and (2.24), we obtain

$$\operatorname{div} \xi_1 - \operatorname{div} \xi_2 = r(\text{trace } A_1 - \text{trace } A_2) = 2r^2 n\,.$$

Since M is compact, by Green's theorem, we have

$$0 = \int_M (\operatorname{div} \xi_1 - \operatorname{div} \xi_2) {*1} = 2n \int_M r^2 {*1}\,,$$

where $*1$ is a volume element on M. From this, $r^2=0$ on M, that is $\xi_j \in T_x M$ $(j=1, 2)$. Hence by (2.23)~(2.26), trace $A_\alpha=0$ $(\alpha=1, 2, \cdots, 2p)$. Consequently, M is a minimal submanifold.

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