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AN INTRINSIC FIBRE METRIC ON THE n-TH SYMMETRIC TENSOR POWER OF THE TANGENT BUNDLE

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0. Introduction. Let $H(M)$ be the Hilbert space consisting of all square-integrable holomorphic m-forms on an m-dimensional complex manifold M. The Bergman form K is defined as a specific holomorphic $2m$-form on the product manifold $M\times\overline{M}$, where $\overline{M}$ is the conjugate complex manifold of M. Let $z=(z^1, \cdots, z^m)$ be a coordinate system with defining domain U_z, and k_z be the Bergman function relative to z, i.e. $K(p, \bar{p})=k_z(p)(dz^1\wedge\cdots\wedge dz^m)_p\wedge(d\bar{z}^1\wedge\cdots\wedge d\bar{z}^m)_{\bar{p}}$, $p\in U_z$. In general, $k_z\geqq 0$. In Kobayashi [4], the following conditions are considered:

(A.1) For every $p\in M$, there exists $\alpha\in H(M)$ such that $\alpha(p)\neq 0$.

(A.2) For every non-zero tangent vector X at $p\in M$, there exists $\alpha\in H(M)$ such that $\alpha(p)=0$ and $X.\alpha(p)\neq 0$.

Suppose (A.1) holds. Then $k_z>0$ for every z, and the Bergman pseudo-metric g, with components $g_{a\bar{b}}=\partial_a\overline{\partial_b}.\log k_z$, is defined. Furthermore, the following is known ([4]):

(K_1) g is a metric if and only if (A.2) holds.

If M satisfies (A.1) and (A.2), and if $R_{a\bar{b}c\bar{d}}$ are the components of the hermitian curvature tensor of the Bergman metric, then the following are known ([4]):

(K_2) Set $\hat{R}_{ac\overline{bd}}=R_{a\bar{b}c\bar{d}}+g_{a\bar{b}}g_{c\bar{d}}+g_{a\bar{d}}g_{c\bar{b}}$. Then $\sum \hat{R}_{ac\overline{bd}}v^a v^c \bar{v}^b \bar{v}^d\geqq 0$ for every $(v^1, \cdots, v^m)\in \boldsymbol{C}^m$.

(K_3) $\hat{R}_{ac\overline{bd}}=k^{-1}(k_{ac\overline{bd}}-k^{-1}k_{ac}k_{\overline{bd}})-k^{-2}\sum g^{\bar{t}s}(k_{ac\bar{t}}-k^{-1}k_{ac}k_{\bar{t}})(k_{s\overline{bd}}-k^{-1}k_{\overline{bd}}k_s)$, where $k=k_z$, $k_{ac}=\partial_a\partial_c.k$, etc., and $(g^{\bar{t}s})=(g_{a\bar{b}})^{-1}$.

In the preceding joint paper [2] with Burbea, conditions (C_n) are defined so that (C_0) (resp. (C_1)) coincides with (A.1) (resp. (A.2)). Furthermore, under assumption (C_0), non-negative functions $\mu_{0,n}$, which are biholomorphic invariants, on the tangent bundle are introduced.

In the present paper, we first note (Proposition 1.2) that the functions $\mu_{0,n}$ on the tangent bundle are, in general, upper semi-continuous, and show (Theorem 2.1) that when M satisfies condition (C_0) there exists a unique fibre pseudo-metric $g^{(n)}$ on the n-th symmetric tensor power $S^nT(M)$ of the tangent bundle

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$T(M)$ for $n\in \boldsymbol{N}$ such that

$$(n\,!)^{-2}\mu_{0,n}(X)=g^{(n)}(X^{n}, \bar{X}^{n}), \qquad X\in T(M);$$

in particular, the pseudo-metric $g^{(1)}$ coincides with the Bergman one stated before. In addition, if M satisfies also $(C_1), \cdots, (C_{n-1})$, then $g^{(n)}$ is differentiable (Theorem 2.5), and assertion (K_1) is generalized as follows (Theorem 2.6): $g^{(n)}$ is a metric if and only if (C_n) holds. Finally, we consider the curvature of the hermitian connection of the hermitian vector bundle $(S^{n}T(M), g^{(n)})$ in the sense of Kobayashi and Nomizu [6]. In view of Fuks [3], the component $g^{(2)}_{ab\bar{c}\bar{d}}$ coincides with $\hat{R}_{ab\bar{c}\bar{d}}/4$ given in (K_2), and (K_2) gives a relationship between the curvature of $g^{(1)}$ and the metric $g^{(2)}$. We generalize this relationship to the one between the curvature of $g^{(n)}$ and the metric $g^{(n+1)}$ (Theorem 3.1). The proof of Theorem 3.1 is done by observing formula (K_3) and by the use of a recurrence formula (Proposition 3.5) for the components of $g^{(n)}$.

1. Preliminaries. Throughout this paper, we are concerned with a fixed paracompact connected complex manifold M of dimension m. The term "coordinate z" stands for a local holomorphic coordinate system $z=(z^{1}, \cdots, z^{m})$ of M with defining domain U_z. For simplicity, we set $\partial_a^z=\partial/\partial z^a$ $(a=1, \cdots, m)$, and $dz=dz^1\wedge\cdots\wedge dz^m$. For a multi-index $A=(a_1, \cdots, a_n)\in \mathrm{MI}(n)=\{1, \cdots, m\}^n$, set $\partial_A^z=\partial_{a_1}^z\cdots\partial_{a_n}^z$. In particular, $\mathrm{MI}(0)=\{\phi\}$, and ∂_ϕ^z means the identity operator acting on functions on U_z. For a constant vector $v=(v^1, \cdots, v^m)$ in $\boldsymbol{C}^m$, set $\partial_v^z=\sum_{a=1}^m v^a\partial_a^z$. The powers $(\partial_v^z)^n$ $(n=0, 1, \cdots)$ are naturally defined. We denote by $\bar{M}$ the conjugate complex manifold of M, and denote by $\rho: M\ni p\mapsto \bar{p}\in \bar{M}$ the conjugation. For a coordinate z with defining domain U_z, we denote by $\bar{z}$ the conjugate coordinate of z with defining domain $\overline{U_z}$, i.e. $\bar{z}(\bar{p})=\overline{z(p)}$ for $p\in U_z$.

We denote by $H(M)$ the separable Hilbert space consisting of all holomorphic m-forms α on M which satisfy $\|\alpha\|^2=(\sqrt{-1}^{m^2}/2^m)\int_M \alpha\wedge\bar{\alpha}<+\infty$, and denote by $(\,,)$ the hermitian inner product on $H(M)$ corresponding to the norm $\|\cdot\|$. There exists a unique $(2m, 0)$-form K, called the *Bergman form*, on the product manifold $M\times\bar{M}$ such that $K(\cdot, \bar{p})/d\bar{z}_{\bar{p}}\in H(M)$ and $\alpha(p)/dz_p=(\alpha, K(\cdot, \bar{p})/d\bar{z}_{\bar{p}})$ for every $p\in M$ and $\alpha\in H(M)$, where z is a coordinate around p (cf., e.g., [2; Corollary 2.6]). Thus, $(1_M, \rho)^*K$ is an (m, m)-form on M. For every coordinate z, we call the function $k_z=(1_M, \rho)^*K/dz\wedge\overline{dz}$ on U_z the *Bergman function* of M relative to z. That is

$$K(p, \bar{p})=k_z(p)dz_p\wedge d\bar{z}_{\bar{p}}, \qquad p\in U_z.$$

The Bergman functions are non-negative (cf., e.g., [2; Proposition 2.7]). It holds (cf., e.g., [2; Proposition 2.5]) that for every multi-index A, the m-form $K_A^z(p)=\partial_A^{\bar{z}}.K(\cdot, \bar{p})/d\bar{z}_{\bar{p}}$ belongs to $H(M)$, and that for every $\alpha\in H(M)$,

$$\partial_A^z.\alpha(p)=(\alpha, K_A^z(p))dz_p. \tag{1.1}$$

In particular, if A and B are multi-indices, then

$$(1.2)\qquad (K_A^z(p),\ K_B^z(p))=\partial_B^z\overline{\partial_A^z}\,.\,k_z(p)\,.$$

Let $n\in\boldsymbol{Z}_+$ be a non-negative integer. For every $p\in M$, set

$$H_n(p)=\{K_A^z(p)\,;\ A\in\bigcup_{j=0}^{n-1}MI(j)\}^{\perp}\subset H(M)\,,$$

where z is a coordinate around p. The subspace $H_n(p)$ does not depend on the choice of z. Let $X\in T_p(M)$ be a tangent vector at p. For a coordinate z around p, represent X as $(\partial_v^z)_p$ for some $v\in\boldsymbol{C}^m$. Then $(\partial_{\bar v}^{\bar z})^n$ is a differential operator on $U_{\bar z}=\overline{U_z}$, and $K_{vn}^z(p)=(\partial_{\bar v}^{\bar z})^n.K(\cdot,\ \bar p)/d\bar z_{\bar p}$ belongs to $H(M)$. Set

$$\mu_n(X)=\max\{|(K_{vn}^z(p),\ \alpha)|^2\,;\ \alpha\in H_n(p),\ \|\alpha\|=1\}(dz\wedge\overline{dz})_p\,.$$

Then the $(m,\ m)$-form $\mu_n(X)$ does not depend on the representation of $X=(\partial_v^z)_p$ in terms of z ([2; Proposition 3.7]).

We recall a lemma on a pre-Hilbert space H over $\boldsymbol{C}$. We denote by $G(x_1,\ \cdots x_n)$ the Gramian of a system $(x_1,\ \cdots,\ x_n)$ in H (especially $G(\phi)=1$).

LEMMA 1.1 ([2; Lemma 3.9]). *Let* $(x_1,\ \cdots,\ x_n)$ $(n\in\boldsymbol{Z}_+)$ *be a linearly independent system in* H, *and let* $x_{n+1}\in H$. *Then the maximum of the set* $\{|(y,\ x_{n+1})|^2\,;\ y\in\{x_1,\ \cdots,\ x_n\}^{\perp},\ \|y\|=1\}$ *coincides with* $G(x_1,\ \cdots,\ x_{n+1})/G(x_1,\ \cdots,\ x_n)$.

Set $\mathrm{MII}(n)=\{(a_1,\ \cdots,\ a_n)\in\mathrm{MI}(n)\,;\ a_1\leqq a_2\leqq\cdots\leqq a_n\}$. We denote by $\varphi_n=\binom{m+n}{n}$ the cardinality of the set $\bigcup_{j=0}^{n}\mathrm{MII}(j)$, and fix a numbering Φ of $\bigcup_{j=0}^{\infty}\mathrm{MII}(j)$ such that $\mathrm{MII}(n)=\{\Phi(\varphi_{n-1}+1),\ \cdots,\ \Phi(\varphi_n)\}$. For a sequence $(j_1,\ \cdots,\ j_u,\ s,\ t)$ of positive integers, set

$$(1.3)\qquad \begin{cases} \mathcal{L}_z(j_1,\ \cdots,\ j_u)=[\partial_{\Phi(i)}^z\overline{\partial_{\Phi(l)}^z}\,.\,k_z]_{l=j_1,\cdots,j_u}^{i=j_1,\cdots,j_u} \\ L_z(j_1,\ \cdots,\ j_u)=\det\mathcal{L}_z(j_1,\ \cdots,\ j_u)\qquad (L_z(\phi)=1) \\ L_z(j_1,\ \cdots,\ j_u\,;\ s,\ t)=\det[\partial_{\Phi(i)}^z\overline{\partial_{\Phi(l)}^z}\,.\,k_z]_{l=j_1,\cdots,j_u,t}^{i=j_1,\cdots,j_u,s}\,. \end{cases}$$

By (1.2), $\mathcal{L}_z(j_1,\ \cdots,\ j_u)(p)$ is the transpose of the Gram matrix of the system $(K_{\Phi(j_1)}^z(p),\ \cdots,\ K_{\Phi(j_u)}^z(p))$, and $L_z(j_1,\ \cdots,\ j_u)(p)$ is its Gramian.

Now, let $f_{n,z}$ be the function on $U_z\times\boldsymbol{C}^m$ defined by

$$\mu_n((\partial_v^z)_p)=f_{n,z}(p,\ v)(dz\wedge\overline{dz})_p\,,\qquad (p,\ v)\in U_z\times\boldsymbol{C}^m.$$

If $\{K_{\Phi(j_1)}^z(p),\ \cdots,\ K_{\Phi(j_u)}^z(p)\}$ is a maximal linearly independent subset of $\{K_A^z(p)\,;\ A\in\bigcup_{j=1}^{n-1}\mathrm{MII}(j)\}$, then Lemma 1.1, together with (1.2), implies that

$$(1.4)\qquad \begin{aligned} f_{n,z}(p,\ v)=&L_z(j_1,\ \cdots,\ j_u)(p)^{-1}\\ &\times\textstyle\sum_{\varphi_{n-1}<s,t\leq\varphi_n}C_{\Phi(s)}C_{\Phi(t)}v^{\Phi(s)}\bar v^{\Phi(t)}L_z(j_1,\ \cdots,\ j_u\,;\ s,\ t)(p)\,. \end{aligned}$$

Here $C_A = n!/n_1! \cdots n_m!$ and $v^A = v^{a_1} \cdots v^{a_n}$ ($A = (a_1, \cdots, a_n) \in \mathrm{MII}(n)$, $v = (v^1, \cdots, v^m) \in \boldsymbol{C}^m$), where n_ν is the cardinality of the set $\{j \in \{1, \cdots, n\}\,;\, a_j = \nu\}$ ($\nu = 1, \cdots, m$).

PROPOSITION 1.2. *The function $f_{n,z}$ is upper semi-continuous on $U_z \times \boldsymbol{C}^m$.*

Proof. The proof is reduced to the following lemma.

LEMMA 1.3. *Let f be the function on the power H^{n+1} of a pre-Hilbert space H over $\boldsymbol{C}$ given by*

$$f(x_1, \cdots, x_{n+1}) = \max\{|(y, x_{n+1})|^2\,;\, y \in \{x_1, \cdots, x_n\}^\perp, \|y\| = 1\}.$$

Then f is upper semi-continuous on H^{n+1}.

Proof. Let $x^0 = (x_1^0, \cdots, x_{n+1}^0) \in H^{n+1}$ be fixed, and let $\{x_{\sigma(1)}^0, \cdots, x_{\sigma(u)}^0\}$ be a maximal linearly independent subset of $\{x_1^0, \cdots, x_n^0\}$. Then $G(x_{\sigma(1)}, \cdots, x_{\sigma(u)})$ is positive in a neighborhood of x^0. So, by Lemma 1.1 we have

$$\begin{aligned}\limsup_{x \to x^0} f(x) &\leqq \limsup_{x \to x^0} \max\{|(y, x_{n+1})|^2\,;\, y \in \{x_{\sigma(1)}, \cdots, x_{\sigma(u)}\}^\perp, \|y\| = 1\} \\ &= \limsup_{x \to x^0} G(x_{\sigma(1)}, \cdots, x_{\sigma(u)}, x_{n+1})/G(x_{\sigma(1)}, \cdots, x_{\sigma(u)}) \\ &= f(x^0),\end{aligned}$$

as desired.

2. An intrinsic fibre pseudo-metric on the holomorphic vector bundle $S^nT(M)$. For $n \in \boldsymbol{Z}_+$ and $p \in M$, we consider the following condition:

$(C_n)_p$ For every non-zero vector $(\xi^A)_{A \in \mathrm{MII}(n)}$ of dimension $\binom{m+n-1}{n}$, there exists $\alpha \in H_n(p)$ such that $\sum_A \xi^A \partial_A^z . \alpha(p) \neq 0$.

Condition (C_n) stands for that $(C_n)_p$ hold for all $p \in M$. From (1.1), we reduce the following ([2; Lemma 3.4]):

(2.1) $\left\{\begin{array}{l}\text{Conditions } (C_j)_p\ (j = 0, \cdots, n) \text{ hold if and only if the} \\ \text{set } \{K_A^z(p)\,;\, A \in \bigcup_{j=0}^n \mathrm{MII}(j)\} \text{ is linearly independent,} \\ \text{or } \mathcal{L}_z(1, \cdots, \varphi_n)(p) \text{ is positive definite.}\end{array}\right.$

Now, suppose M satisfies condition (C_0). Then (1.4) implies that $\mu_0(X) = k_z(p)(dz \wedge \overline{dz})_p$ for every $X \in T_p(M)$, and that $k_z > 0$ on U_z. So, $[0, +\infty)$-valued functions $\mu_{0,n} = \mu_n/\mu_0$ ($n \in \boldsymbol{N}$) on the holomorphic tangent bundle $T(M)$ are well defined. Every function $\mu_{0,n}$ is upper semi-continuous on $T(M)$ (by Proposition 1.2) and satisfies the following: $\mu_{0,n}(\xi X) = |\xi|^{2n} \mu_{0,n}(X)$ for $X \in T(M)$ and $\xi \in \boldsymbol{C}$; therefore $(\mu_{0,n})^{1/2n}$ is an upper semi-continuous Finsler pseudo-metric on M. Moreover, $\mu_{0,n}$ are biholomorphic invariants, i.e. $\mu_{0,n}(X) = \mu_{0,n}(f_* X)$, $X \in T(M)$ for every biholomorphic mapping f from M onto another complex manifold ([2;

Proposition 3.2]).

We denote by $S^nT_p(M)$ (resp. $S^nT(M)$) the n-th symmetric tensor power of $T_p(M)$ (resp. $T(M)$). $S^nT(M)$ is a holomorphic vector bundle over M, and $\{\partial_A^z; A\in \mathrm{MII}(n)\}$ forms its local frame on U_z.

We shall show the following assertion.

THEOREM 2.1. *If a complex manifold M satisfies condition (C_0), then for every $n\in \boldsymbol{N}$ and $p\in M$ there exists a unique hermitian pseudo-inner-product $g^{(n)}(\cdot, \bar{\cdot})$ on the space $S^nT_p(M)$ such that*

$$(2.2)\qquad (n\,!)^{-2}\mu_{0,n}(X)=g^{(n)}(X^n, \overline{X^n}), \qquad X\in T_p(M),$$

where $X^1=X$, $X^j=X\cdot X^{j-1}$ (the symmetric tensor product). Furthermore, the fibre pseudo-metric $g^{(n)}$ on $S^nT(M)$ is biholomorphic invariant, i.e. $g^{(n)}(Y, \overline{Y})= g^{(n)}(f_*Y, \overline{f_*Y})$ for $Y\in S^nT(M)$ and for any biholomorphic mapping f from M onto another complex manifold.

Remark 2.2. The constant $(n\,!)^{-2}$ in the formula (2.2) is chosen so that when M is the unit disk $\{\xi\in \boldsymbol{C}; |\xi|<1\}$ in $\boldsymbol{C}$ the inner product $g^{(n)}(\cdot, \bar{\cdot})$ on $S^nT_0(M)$ at the origin $0\in M$ has the simplest form, $g^{(n)}(X^n, \overline{X^n})=n+1$ for $X=(\partial/\partial\xi)_0\in T_0(M)$ (cf. [1]).

Proof of Theorem 2.1 (Existence). Let $\{K^z_{\Phi(j_1)}(p), \cdots, K^z_{\Phi(j_u)}(p)\}$ be a maximal linearly independent subset of $\{K^z_A(p); A\in\bigcup_{j=1}^{n-1}\mathrm{MII}(j)\}$. By (1.4) we have

$$\mu_{0,n}((\partial_v^z)_p)=L_z(j_1, \cdots, j_u)(p)^{-1}k_z(p)^{-1}$$
$$\times\textstyle\sum_{\varphi_{n-1}<s,t\leq\varphi_n}C_{\Phi(s)}C_{\Phi(t)}v^{\Phi(s)}\bar{v}^{\Phi(t)}L_z(j_1, \cdots, j_u; s, t)(p).$$

So, the function $g^{(n)}(\cdot, \bar{\cdot})$ defined by sesqui-bilinearity and by the requirement

$$(2.3)\qquad g^{(n)}((\partial^z_{\Phi(s)})_p, (\overline{\partial^z_{\Phi(t)}})_p)$$
$$=(n\,!)^{-2}L_z(j_1, \cdots, j_u)(p)^{-1}k_z(p)^{-1}L_z(j_1, \cdots, j_u; s, t)(p)$$

has the desired property. Thus, the existence is proved.

To complete the proof, we prepare two lemmas.

LEMMA 2.3. *Let $R=\sum_{n=0}^{\infty}R_n$ be a commutative, associative, graded algebra over $\boldsymbol{C}$. For every $n\in\boldsymbol{N}$, there exists a linear form $F_n(t_0, t_1, \cdots, t_{3n-1})$ on $\boldsymbol{C}^{3n}$ such that*

$$(x^n, y^n)_n=F_n(f(1), f(\rho), \cdots, f(\rho^{3n-1}))$$

for $x, y\in R_1$ and for any sesqui-bilinear form $(\ ,\ )_n$ on R_n, where $\rho=e^{2\pi\sqrt{-1}/3n}$ and $f(\xi)=f_{x,y}(\xi)=((x+\xi y)^n, (x+\xi y)^n)_n$, $\xi\in\boldsymbol{C}$.

Proof. Since

$$f(\rho^l)=\sum_{i,j=0}^n \binom{n}{i}\binom{n}{j}(x^{n-i}y^i,\ x^{n-j}y^j)_n \rho^{l(i-j)},$$

and since

$$\sum_{l=0}^{n-1}\rho^{3jl}=\begin{cases} n, & n \mid j \\ 0, & n \nmid j \end{cases}$$

for every $j\in \boldsymbol{Z}$, it follows that

$$\begin{cases} \sum_{l=0}^{n-1} f(\rho^{3l})=n(\bar{\eta}+\eta+\zeta) \\ \sum_{l=0}^{n-1} f(\rho^{3l+1})=n(\bar{\eta}\rho^n+\eta\rho^{-n}+\zeta) \\ \sum_{l=0}^{n-1} f(\rho^{3l+2})=n(\bar{\eta}\rho^{-n}+\eta\rho^n+\zeta), \end{cases}$$

where $\eta=(x^n, y^n)_n$, $\zeta=\sum_{j=0}^n \binom{n}{j}^2 (x^{n-j}y^j,\ x^{n-j}y^j)_n$. So, if $F^{(i)}(t_0, \cdots, t_{3n-1})=\sum_{l=0}^{n-1} t_{3l+i}$ $(i=0, 1, 2)$, and $\omega=\rho^n=e^{2\pi\sqrt{-1}/3}$, then the form $F_n=(F^{(0)}+\omega F^{(1)}+\omega^2 F^{(2)})/3n$ has the desired property.

Given $n, j\in \boldsymbol{N}$ with $j\leqq n$, denote by P_j^n the linear operator from $\boldsymbol{C}[t_1, \cdots, t_j]$ into $\boldsymbol{C}[t_1, \cdots, t_n]$, given by $P_j^n(f(t_1, \cdots, t_j))=\sum_{\sigma\in\Sigma(j,n)} f(t_{\sigma(1)}, \cdots, t_{\sigma(j)})$, $f(t_1, \cdots, t_j)\in \boldsymbol{C}[t_1, \cdots, t_j]$, where $\Sigma(j, n)$ means the family of all strictly increasing mappings from $\{1, \cdots, j\}$ into $\{1, \cdots, n\}$.

LEMMA 2.4. *For every $n\in \boldsymbol{N}$ it holds that*

$$n!\, t_1 \cdots t_n=\sum_{j=0}^{n-1}(-1)^j P_{n-j}^n((t_1+\cdots+t_{n-j})^n).$$

Proof. Let $f(t_1, \cdots, t_n)$ be the right hand side of the desired formula, and set

$$f_j(t_1, \cdots, t_n)=\sum_{\sigma\in\Sigma(n-j,n)}(t_{\sigma(1)}+\cdots+t_{\sigma(n-j)})^n$$

for $j=0, 1, \cdots, n-1$; thus $f=\sum_{j=0}^{n-1}(-1)^j f_j$. For every j,

$$f_j(0, t_2, \cdots, t_n)=g_j(t_2, \cdots, t_n)+h_j(t_2, \cdots, t_n),$$

where

$$\begin{cases} g_j(t_2, \cdots, t_n)=\sum_{\sigma\in\Sigma(n-j,n),\,\sigma(1)=1}(t_{\sigma(2)}+\cdots+t_{\sigma(n-j)})^n \\ h_j(t_2, \cdots, t_n)=\sum_{\sigma\in\Sigma(n-j,n),\,\sigma(1)\geqq 2}(t_{\sigma(1)}+\cdots+t_{\sigma(n-j)})^n. \end{cases}$$

It is easily seen that $g_{n-1}=0$, $h_0=0$, and $g_j=h_{j+1}$ $(j=0, 1, \cdots, n-2)$. From these we get $f(0, t_2, \cdots, t_n)=0$; therefore, the symmetry of f implies $f(t_1, \cdots, t_{j-1}, 0, t_{j+1}, \cdots, t_n)=0$ for any j. It follows from the remainder theorem that $f(t_1, \cdots, t_n)=c\, t_1 \cdots t_n$ for some constant c. Among expansions of f_j into monomials the term $t_1 \cdots t_n$ appears only in $f_0=(t_1+\cdots+t_n)^n$, for which the coefficient of $t_1 \cdots t_n$ is $n!$. So, the above constant must be $n!$, as desired.

Proof of Theorem 2.1 (Uniqueness). Lemmas 2.3 and 2.4 imply that every $g^{(n)}((\partial_A^z)_p, (\overline{\partial_B^z})_p)$ $(A, B\in \mathrm{MI}(n))$ can be written as a linear combination of terms

$g^{(n)}(X^n, \bar{X}^n)$ $(X \in T_p(M))$. From this we obtain the uniqueness of $g^{(n)}$. The invariant property of $g^{(n)}$ follows from the uniqueness and the invariant property of $\mu_{0,n}$ stated before. The proof is now complete.

THEOREM 2.5. *Suppose M satisfies conditions $(C_0), \cdots, (C_{n-1})$ with $n \geqq 1$. Then $g^{(n)}$ is a differential pseudo-metric, and its components $g^{(n)}_{z,A\bar{B}} = g^{(n)}(\partial^z_A, \overline{\partial^z_B})$ $(A, B \in \mathrm{MI}(n))$ relative to a coordinate z satisfy*

$$g^{(n)}_{z,\Phi(s)\overline{\Phi(t)}} = L_z(1, \cdots, \varphi_{n-1}; s, t)/\{(n!)^2 k_z L_z(1, \cdots, \varphi_{n-1})\}$$

on U_z for $s, t \in \{\varphi_{n-1}+1, \cdots, \varphi_n\}$. In particular, $g^{(1)}_{z,a\bar{b}} = \partial^z_a \overline{\partial^z_b} \cdot \log k_z$, i.e. $g^{(1)}$ is the usual Bergman pseudo-metric on M ([4; pp. 271-272]).

Proof. By (2.1) the hypothesis implies that the system $\{K^z_{\Phi(1)}(p), \cdots, K^z_{\Phi(\varphi_{n-1})}(p)\}$ itself is linearly independent for every $p \in U_z$. So, all the assertions follow from (2.3).

THEOREM 2.6. *Suppose M satisfies conditions $(C_0), \cdots, (C_{n-1})$ with $n \geqq 1$. Then the pseudo-inner-product $g^{(n)}(\cdot, \bar{\cdot})$ on $S^n T_p(M)$ is an inner product if and only if condition $(C_n)_p$ holds. In particular, the fibre pseudo-metric $g^{(n)}$ is a metric if and only if condition (C_n) holds.*

Proof. Let z be a coordinate around p. It follows from Theorem 2.5 that $g^{(n)}(\cdot, \bar{\cdot})$ is an inner product if and only if the following holds:

(2.4) $\left\{ \begin{array}{l} \text{The matrix } [L_z(1, \cdots, \varphi_{n-1}; s, t)]^{s=\varphi_{n-1}+1, \cdots, \varphi_n}_{t=\varphi_{n-1}+1, \cdots, \varphi_n}(p) \text{ is} \\ \text{positive definite.} \end{array} \right.$

If $j \in \boldsymbol{Z}$ with $j > \varphi_{n-1}$, applying Sylvester's theorem to the (j, j)-matrix $\mathcal{L}_z(1, \cdots, \varphi_{n-1}, \cdots, j)$ and its minnor determinants $L_z(1, \cdots, \varphi_{n-1}; s, t)$ $(\varphi_{n-1} < s, t \leqq j)$, we have

$$\det[L_z(1, \cdots, \varphi_{n-1}; s, t)]^{s=\varphi_{n-1}+1, \cdots, j}_{t=\varphi_{n-1}+1, \cdots, j}$$
$$= L_z(1, \cdots, j) L_z(1, \cdots, \varphi_{n-1})^{j-\varphi_{n-1}-1}.$$

Thereby, employing (2.1), one can see that the following four statements are mutually equivalent:

(i) Condition $(C_n)_p$ holds.
(ii) $L_z(1, \cdots, j)(p) > 0$ for any $j \in \boldsymbol{Z}$ with $\varphi_{n-1} < j \leqq \varphi_n$.
(iii) $\det[L_z(1, \cdots, \varphi_{n-1}; s, t)]^{s=\varphi_{n-1}+1, \cdots, j}_{t=\varphi_{n-1}+1, \cdots, j}(p) > 0$ for any $j \in \boldsymbol{Z}$ with $\varphi_{n-1} < j \leqq \varphi_n$.
(iv) Condition (2.4) holds.

This completes the proof of Theorem 2.6.

3. Connection of the hermitian vector bundle $(S^n T(M), g^{(n)})$. If M satisfies conditions $(C_0), \cdots, (C_n)$ for some $n \in \boldsymbol{N}$, then, as we have seen in Theorems 2.5 and 2.6, $g^{(n)}$ is a usual hermitian fibre metric on the holomorphic vector bundle $S^nT(M)$. We shall investigate the curvature of the hermitian connection of the hermitian vector bundle $(S^nT(M), g^{(n)})$ in the sense of Kobayashi and Nomizu [6; pp. 178-185] (also cf. [5; pp. 37-39]). Let z be a coordinate in $U_z \subset M$. We denote by $(g_z^{(n)\bar{B}A})_{A,B\in \mathrm{MII}(n)}$ the inverse matrix of $(g_{z,A\bar{B}}^{(n)})_{A,B\in \mathrm{MII}(n)}$ in the sense that

$$(3.1) \qquad \sum_{B\in \mathrm{MII}(n)} g_{z,A\bar{B}}^{(n)} g_z^{(n)\bar{B}C} = \delta_A^C, \qquad A, C\in \mathrm{MII}(n).$$

Let $R^{(n)}$ be the *curvature of the hermitian connection* of $(S^nT(M), g^{(n)})$, and let $R_{z,A\bar{B}|c\bar{d}}^{(n)} = g^{(n)}(R^{(n)}(\partial_c^z, \overline{\partial_d^z})\overline{\partial_B^z}, \partial_A^z)$ for $A, B\in \mathrm{MI}(n)$ and $c, d\in \{1, \cdots, m\} = \mathrm{MI}(1)$. It is known ([5, 6, 7]) that

$$(3.2) \qquad R_{z,A\bar{B}|c\bar{d}}^{(n)} = \partial_c^z \overline{\partial_d^z} \cdot g_{z,A\bar{B}}^{(n)} - \sum_{P,Q\in \mathrm{MII}(n)} g_z^{(n)\bar{Q}P} (\partial_c^z \cdot g_{z,A\bar{Q}}^{(n)})(\overline{\partial_d^z} \cdot g_{z,P\bar{B}}^{(n)}).$$

We shall show the following.

THEOREM 3.1. *Suppose M satisfies conditions $(C_0), \cdots, (C_n)$ with $n\in \boldsymbol{N}$. Then*

$$R_{z,A\bar{B}|c\bar{d}}^{(n)} = (n+1)^2 g_{z,Ac\overline{Bd}}^{(n+1)} - g_{z,c\bar{d}}^{(1)} g_{z,A\bar{B}}^{(n)}$$
$$- n^2 \sum_{P,Q\in \mathrm{MII}(n-1)} g_z^{(n-1)\bar{Q}P} g_{z,Pc\bar{B}}^{(n)} g_{z,A\overline{Qd}}^{(n)}$$

on U_z for $A, B\in \mathrm{MI}(n)$ and $c, d\in \mathrm{MI}(1)$, where $g_z^{(0)\phi\bar{\phi}} = 1$.

Taking $n=1$ in the above theorem we obtain the following result of Fuks [3; p. 525].

COROLLARY 3.2. *Suppose M satisfies conditions (C_0) and (C_1). Let $HSC(X)$ be the holomorphic sectional curvature of the Bergman metric $g^{(1)}$ on M in the direction $X\in T_p(M)-\{0\}$, i.e.*

$$HSC(X) = -\sum_{a,b,c,d} R_{z,a\bar{b}|c\bar{d}}^{(1)}(p) v^a \bar{v}^b v^c \bar{v}^d / g^{(1)}(X, \bar{X})^2,$$

where z is a coordinate around p and $X = (\partial_v^z)_p$. Then it holds that

$$\mu_{0,2} = (2 - HSC)(\mu_{0,1})^2 \ \text{on}\ T(M)\text{-}\{\text{the zero section}\}.$$

Remark 3.3. Theorem 3.1, combined with (3.2), says that when M satisfies conditions $(C_0), \cdots, (C_n)$ with $n\in \boldsymbol{N}$ every component of the fibre (pseudo-) metrics $g^{(2)}, \cdots, g^{(n+1)}$ is written as a rational function of the derivatives of the components of the Bergman metric $g^{(1)}$.

The remainder of this section is devoted to prove Theorem 3.1. From now on, we suppose that M satisfies conditions $(C_0), \cdots, (C_n)$ for some fixed $n\in \boldsymbol{N}$. We also fix a coordinate z in $U\subset M$, and suppress the dependence on z, i.e. $\partial_A = \partial_A^z$, $k = k_z$, $L(j_1, \cdots, j_u) = L_z(j_1, \cdots, j_u)$, $g_{A\bar{B}}^{(n)} = g_{z,A\bar{B}}^{(n)}$, etc.

For every pair of multi-indices A and B, we shall inductively define functions $L_{A\bar{B}}^{(j)}$ on U $(j=0, 1, \cdots, n+1)$ as follows:

$$\begin{cases} L^{(0)}_{A\bar{B}}=\partial_A\overline{\partial_B}\,.k \\ L^{(j+1)}_{A\bar{B}}=L^{(j)}_{A\bar{B}}-\sum_{C,D\in \mathrm{MII}(j)} L^{(j)\bar{D}C}L^{(j)}_{C\bar{B}}L^{(j)}_{A\bar{D}}\,, \end{cases}$$

where $(L^{(j)\bar{D}C})$ is the inverse matrix of $(L^{(j)}_{A\bar{B}})_{A,B\in \mathrm{MII}(j)}$ in the same sense as in (3.1). Non-singularity of the latter matrix is guaranteed by Lemma 3.4 below. Notice that

$$L^{(j+1)}_{A\bar{B}}=0 \text{ when } A \text{ or } B \text{ belongs to } \mathrm{MI}(j)\,. \tag{3.3}$$

For a sequence $(j_1, \cdots, j_u, s, t)$ of positive integers, set

$$\begin{cases} \mathcal{L}^{(j)}(j_1, \cdots, j_u)=[L^{(j)}_{\Phi(i)\overline{\Phi(l)}}]^{i=j_1,\cdots,j_u}_{l=j_1,\cdots,j_u} \\ L^{(j)}(j_1, \cdots, j_u)=\det \mathcal{L}^{(j)}(j_1, \cdots, j_u) \qquad (L^{(j)}(\phi)=1) \\ L^{(j)}(j_1, \cdots, j_u\,;\, s,\, t)=\det[L^{(j)}_{\Phi(i)\overline{\Phi(l)}}]^{i=j_1,\cdots,j_u,s}_{l=j_1,\cdots,j_u,t}\,, \end{cases}$$

where Φ is the numbering of $\bigcup_{j=0}^{\infty}\mathrm{MII}(j)$ given in § 1. By (1.3) we have

$$\begin{cases} \mathcal{L}^{(0)}(j_1, \cdots, j_u)=\mathcal{L}(j_1, \cdots, j_u) \\ L^{(0)}(j_1, \cdots, j_u)=L(j_1, \cdots, j_u) \\ L^{(0)}(j_1, \cdots, j_u\,;\, s,\, t)=L(j_1, \cdots, j_u\,;\, s,\, t)\,. \end{cases} \tag{3.4}$$

LEMMA 3.4. *If $l\in\{1, \cdots, n+1\}$; $s, t\in\{\varphi_{l-1}+1, \cdots, \varphi_l\}$ and $\varphi_{-1}=0$, the nthe following hold:*

(i) $\mathcal{L}^{(j)}(\varphi_{j-1}+1, \cdots, \varphi_j)$ *is positive-definite for every* $j\in\{0, \cdots, l-1\}$.
(ii) $L(1, 2, \cdots, \varphi_{l-1})=\prod_{j=0}^{l-1}L^{(j)}(\varphi_{j-1}+1, \cdots, \varphi_j)$.
(iii) $L(1, 2, \cdots, \varphi_{l-1}\,;\, s,\, t)=L(1, 2, \cdots, \varphi_{l-1})L^{(l)}_{\Phi(s)\overline{\Phi(t)}}$.

Proof. We first recall the following well-known fact: If A, B, C, and D are complex matrices of type (i, i), (i, j), (j, i), and (j, j), respectively, and if A is non-singular, then it holds that

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\det A \det(D-CA^{-1}B)\,. \tag{3.5}$$

By induction on $j\in\{0, 1, \cdots, l-1\}$, we can show the triple assertions

$(3.6)_j$ $\qquad L^{(j)}(\varphi_{j-1}+1, \cdots, r)>0 \quad$ for every $r\in\{\varphi_{j-1}+1, \cdots, \varphi_{l-1}\}$,

$(3.7)_j$ $\qquad L(1, 2, \cdots, \varphi_{l-1})=L^{(j)}(\varphi_{j-1}+1, \cdots, \varphi_{l-1}) \times\prod_{\nu=0}^{j-1}L^{(\nu)}(\varphi_{\nu-1}+1, \cdots, \varphi_\nu)$, and

$(3.8)_j$ $\qquad L(1, 2, \cdots, \varphi_{l-1}\,;\, s,\, t)=L^{(j)}(\varphi_{j-1}+1, \cdots, \varphi_{l-1}\,;\, s,\, t) \times\prod_{\nu=0}^{j-1}L^{(\nu)}(\varphi_{\nu-1}+1, \cdots, \varphi_\nu)\,.$

In fact, assertions $(3.6)_0$, $(3.7)_0$, and $(3.8)_0$ follow from (3.4). Next, assume $(3.6)_j$,

$(3.7)_j$, and $(3.8)_j$ hold for some $j\in\{0, \cdots, l-2\}$. Assumption $(3.6)_j$ implies that $L^{(j)}(\varphi_{j-1}+1, \cdots, \varphi_j)>0$; therefore $L^{(j+1)}_{A\bar{B}}$ can be defined. So, by (3.5) we have

$$L^{(j)}(\varphi_{j-1}+1, \cdots, r)=L^{(j)}(\varphi_{j-1}+1, \cdots, \varphi_j)L^{(j+1)}(\varphi_j+1, \cdots, r).$$

Thus, $(3.6)_{j+1}$ and $(3.7)_{j+1}$ hold. Furthermore, if we apply (3.5) to the first matrix in the right hand side of $(3.8)_j$, we obtain $(3.8)_{j+1}$. The assertion (i) of Lemma 3.4 follows from $(3.6)_j$ for $j=0, 1, \cdots, l-1$, while the assertion (ii) coincides with $(3.7)_{l-1}$. Since $L^{(l-1)}(\varphi_{l-2}+1, \cdots, \varphi_{l-1})>0$, the assertion (iii) follows from $(3.8)_{l-1}$ and (3.5).

PROPOSITION 3.5. *For $j\in\{1, 2, \cdots, n+1\}$, and $A, B\in \mathrm{MI}(j)$, it holds that*

$$g^{(j)}_{A\bar{B}}=L^{(j)}_{A\bar{B}}/\{(j\,!)^2 k\}.$$

Proof. Lemma 3.4 (iii) and Theorem 2.5 imply the assertion.

LEMMA 3.6. *If $j\in\{1, \cdots, n\}$, A, B are multi-indices, and $c\in \mathrm{MI}(1)$, then the following identities hold:*

(i) $\partial_c . L^{(j)}_{A\bar{B}}=L^{(j)}_{Ac\bar{B}}-\sum_{P, Q\in \mathrm{MII}(j-1)} L^{(j-1)\bar{Q}P} L^{(j-1)}_{A\bar{Q}} L^{(j)}_{Pc\bar{B}}$,

(ii) $\overline{\partial_c} . L^{(j)}_{A\bar{B}}=L^{(j)}_{A\bar{B}\bar{c}}-\sum_{P, Q\in \mathrm{MII}(j-1)} L^{(j-1)\bar{Q}P} L^{(j)}_{A\bar{Q}\bar{c}} L^{(j-1)}_{P\bar{B}}$.

Proof. Identity (i) is easily shown by the definition and by induction on j. By taking the complex conjugation of (i), we get (ii).

Proof of Theorem 3.1. Let $A, B\in \mathrm{MI}(n)$ and $c, d\in \mathrm{MI}(1)$. By applying Proposition 3.5 to the right hand side of (3.2), we get

$$(n\,!)^2 R^{(n)}_{A\bar{B}|c\bar{d}}=-\frac{1}{k^2} L^{(n)}_{A\bar{B}} L^{(1)}_{c\bar{d}} +\frac{1}{k}\{\partial_c\overline{\partial_d} . L^{(n)}_{A\bar{B}}-\sum_{P, Q\in \mathrm{MII}(n)} L^{(n)\bar{Q}P}(\partial_c . L^{(n)}_{A\bar{Q}})(\overline{\partial_d} . L^{(n)}_{P\bar{B}})\}.$$

Lemma 3.6, together with (3.3), implies that the term in the braces coincides with

$$L^{(n+1)}_{Ac\bar{B}\bar{d}}-\sum_{P, Q\in \mathrm{MII}(n-1)} L^{(n-1)\bar{Q}P} L^{(n)}_{A\bar{Q}\bar{d}} L^{(n)}_{Pc\bar{B}}.$$

So, the desired formula follows again from Proposition 3.5.

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