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# ON A FREE BOUNDARY PROBLEM OF PLASMA EQUILIBRIA
## —ASYMPTOTIC BEHAVIOR AND SYMMETRIC PROPERTY OF A SOLUTION—

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### §1. Introduction.

A simple model of a confused plasma in tokomak machine can be described by the following system:

$$
\text{(E)}\quad \begin{cases} -\Delta u=\lambda g(x, u) \quad \text{in} \quad \Omega_p=\{x\in\Omega \mid u(x)>0\}, & (1.1)\\ -\Delta u=0 \quad \text{in} \quad \Omega\setminus\Omega_p, & (1.2)\\ u|_{\partial\Omega}=\text{unknown constant}, & (1.3)\\ -\int_{\partial\Omega}\frac{\partial u}{\partial\nu}ds=I \ \text{(given positive constant)}, & (1.4)\end{cases}
$$

where $\Omega$ is a bounded domain in $\boldsymbol{R}^n$ with a smooth boundary and $\lambda$ is a given positive parameter. $\Omega_p$ is called a plasma domain and $\Omega\setminus\Omega_p$ is called a vacuum domain. We consider the free boundary problem of the following type:

(P) *Find*: $u\in H^2(\Omega)$ and $\Omega_p\subset\Omega$ s.t. $u$ and $\Omega_p$ satisfy (E).

We call $\partial\Omega_p$ a free boundary.

We consider this problem under the assumptions:

$$
\text{(A1)}\quad \begin{cases} g(x, s)=0 \quad \text{if} \quad s\leqq 0, & (1.5)\\ g(x, s)>0 \quad \text{if} \quad s>0, & (1.6)\\ g(x, s) \text{ is continuous in } \Omega\times R, & \\ \lim_{s\to\infty}\frac{g(x, s)}{s^p}=0 \quad \text{uniformly in } \bar{\Omega}, & \end{cases}
$$

where $p=n/(n-2)$ (if $n>2$) and $p={}^{\exists}p_0>1$ (if $n=2$). By using (1.5) and (1.6), we can rewrite (1.1) and (1.2) as follows.

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$$-\Delta u=\lambda g(x, u) \quad \text{in} \quad \Omega .$$

The problem (P) can be formulated as the following variational problem: (See Berestycki and Brezis [7])

$$\text{(V)} \quad \begin{cases} V\equiv\left\{\{u\in H^1(\Omega)\,;\ u|_{\partial\Omega}=\text{constant},\ \lambda\int_\Omega g(x, u)=I\right\} \\ \Phi(u)\equiv\frac{1}{2}\int_\Omega|\nabla u|^2dx-\lambda\int_\Omega\int_0^{u(x)}g(x, s)dsdx+Iu(\partial\Omega) \\ \textit{Find}:\ u\in V \text{ s.t. } \Phi(u)=\inf_{v\in V}\Phi(v) . \end{cases}$$

The problem (P) has been considered by several authers. First Temam [14] proved the existence of the solution in case of $g(x, s)=s_+$, where $s_+=\max\{0, s\}$ by using the variational method. We call this case " $g(x, s)$ is linear ", and the other case " $g(x, s)$ is non-linear". In linear case, several methods of proof of the existence of the solution of (P) are known. (See: Freedman [2], Sermange [10], K. C. Chang [3]) In non-linear case with $n=2$ or 3, Temam [13] proved the existence of the solution. In non-linear case with $n\geqq2$, Berestycki and Brezis [7] proved the existence of the solution of (V) in $W^{3,\alpha}(\Omega)(^\forall\alpha>1)$ under the assumption that $g(x)$ is convex and $g(x)$ satisfies (A1) by the variational method which is different from Temam's method [13]. In [7], they gave two other proofs, which are the method of successive approximation and the method of Leray-Schauder degree. In non-linear case Ambrosetti and Mancini [1] proved that the free boundary exists if $\lambda$ is sufficiently large under the assumption (A1) by using the method of Leray-Schauder degree and the bifurcation theory. And geometric property of $\Omega_p$ is studied by several authers. Berestycki and Brezis [7] showed $\Omega_p$ is connected. Kinderlehrer and Spruck [5] showed that $\partial\Omega_p$ is $C^{2,\alpha}$ $(0\leqq\alpha<1)$ in linear case with $n=2$. Moreover Spruck and Caffarelli [9] showed that the level line of the solution $u$ is convex if $\Omega$ is convex.

Caffarelli and Freedman [8] studied the problem (V) in case when $g(x, s)$ is linear and $n=2$. They showed that

$$\operatorname{diam}(\Omega_p)\to0 \quad \text{as} \quad \lambda\to\infty .$$

In this paper, we extend the result of Caffarelli and Freedman to non-linear case with $n\geqq2$.

We obtain the following result. We consider the solution of (V) under the following conditions:

$$\text{(A2)} \quad \begin{cases} g(x, s)\geqq Ks^\alpha & \text{for } ^\forall x\in\Omega,\ ^\exists K>0,\ 1\leqq{}^\exists\alpha<p,\ ^\forall s\geqq0 \\ g(x, \cdot) \text{ is convex} & \text{for all fixed } x\in\Omega \end{cases}$$

We assume (A1) and (A2). Then we obtain

$$d(\Omega_p)\to0 \quad \text{as} \quad \lambda\to\infty , \tag{1.7}$$

where $d(\Omega_p)$ is the maximum of the measure of the cross section of $\partial\Omega_p$ by an $(n-1)$-dimensional hyperplane.

The last section is devoted to the result about the symmetricity. We assume that $\Omega$ and $g(x, s)$ have symmetric property. i. e. $\Omega$ is symmetric for the $(n-1)$-dimensional hyperplane: $x_n=0$ and that $g(x, s)$ satisfies

$$g(x_1, x_2, \cdots, x_n, s)=g(x_1, x_2, \cdots, -x_n, s). \tag{A3}$$

Previously Sermange [11] showed the uniqueness of the symmetric solution for some $\lambda$ in linear case with $n=2$. We extend this result to non-linear case with $n\geqq 2$. We assume that $g(x, s)$ satisfies the following conditions:

$$\sup_{x\in\Omega}\sup_{s,s'\in\boldsymbol{R}}\frac{g(x, s)-g(x, s')}{s-s'}=M<\infty, \tag{A4}$$

We show the existence of the symmetric solution under the assumptions (A1)$\sim$(A3), and the uniqueness of the symmetric solution under the assumptions (A1)$\sim$(A4) for some $\lambda$.

In sections 2$\sim$4, we prove (1.7) by using the method introduced by Caffarelli and Friedman [8]. In section 2, we estimate $\Phi(u)$ in special case. Next we extend this estimate to general case in section 3. In section 4, we estimate the size of plasma domain of the solution of (V) by using an estimate of $\Phi(u)$. In section 5, we consider the existence and uniqueness of the symmetric solution of (P) by the method of Sermange [11].

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## § 2. Estimate of $\Phi(u)$ in special case.

In this section we assume that

$$\begin{cases} \Omega=B_R\subset\boldsymbol{R}^n, \\ g(x, u)=u^\alpha \quad (1\leqq\alpha<p), \end{cases} \tag{A5}$$

where $B_R$ is a ball with its radius $R$ in $\boldsymbol{R}^n$. In the next section, we will extend this estimate in this section to general case.

Let $u_0$ be the solution of the following problem:

$$\begin{cases} -\Delta u_0=u_0^\alpha & \text{in } B_1 \\ u_0|_{\partial B_1}=0 & \\ u_0>0 & \text{in } B_1 \end{cases} \tag{2.1}$$

Lions [12] and Amann [6] guaranteed the existence and uniqueness of the solution of (2.1) for $0<\alpha<(n+2)/(n-2)$ and $\alpha\neq 1$. By Freedman [2] (pp. 531), $u_0(x)$ depends only on $|x|$. In the rest of this paper, $\Lambda$ denote $|\nabla u_0(x)|=$

$\sup_{|x|=1}|\nabla u_0(x)|$ for $|x|=1$. $\Lambda$ depends only on $\alpha$ and $n$.

The following lemma gives us one method of an explicit construction of a solution of (P).

LEMMA 2.1. *Assume* (A5) *and* $\alpha\neq 1$. *Then the solution of* (P) *exists uniquely, and a plasma domain* $\Omega_p$ *is a ball with its radius given by*

$$\varepsilon=\left(\frac{\lambda I^{\alpha-1}}{\Lambda^{\alpha-1}|S_n|^{\alpha-1}}\right)^{-1/(n-(n-2)\alpha)}, \tag{2.2}$$

*and then* $u(x)$ *is given as follows*: *In case of* $n=2$,

$$u(x)=\begin{cases} \dfrac{I}{2\pi\Lambda}u_0\left(\dfrac{x}{\varepsilon}\right) & \text{in } B_\varepsilon, \quad (2.3)\\ \dfrac{I}{2\pi}(\log\varepsilon-\log|x|) & \text{in } B_R\setminus B_\varepsilon. \quad (2.4)\end{cases}$$

*In case of* $n>2$,

$$u(x)=\begin{cases} \dfrac{I}{\varepsilon^{n-2}\Lambda|S_n|}u_0\left(\dfrac{x}{\varepsilon}\right) & \text{in } B_\varepsilon, \quad (2.5)\\ \dfrac{I}{(n-2)\varepsilon^{n-2}|S_n|}\left(\dfrac{\varepsilon^{n-2}}{|x|^{n-2}}-1\right) & \text{in } B_R\setminus B_\varepsilon, \quad (2.6)\end{cases}$$

*where* $|S_n|$ *is area of surface of unit ball in* $\boldsymbol{R}^n$.

*Proof*. The spherical symmetric property of $u(x)$ is guaranteed in [2] and [6] (§.3). Therefore it suffices to consider this problem only in case when the free boundary is a ball. $u(x)$ satisfies (1.3). By (2.4) and (2.6), $u(x)$ satisfies $\Delta u=0$ in $B_R\setminus B_\varepsilon$ since $\log|x|$ (or $|x|^{2-n}$) is an elementary solution of Laplacian in case of $n=2$ (or $n>2$, respectively). By using (2.2), (2.3) and (2.5), we have

$$\Delta u=\lambda u^\alpha \quad \text{in } B_\varepsilon.$$

Thus $u$ satisfies (1.1) and (1.2). By using (2.4) and (2.6), we have $-\int_{\partial B_R}\frac{\partial u}{\partial r}dl=I$. By using

$$\lim_{\delta\to\varepsilon+0}\nabla u|_{\partial B_\delta}=\lim_{\delta\to\varepsilon-0}\nabla u|_{\partial B_\delta}\left(=-\frac{I}{\varepsilon^{n-1}|S_n|}\right)$$

and Theorem 7.8 in Gilbarg and Trudinger [4], we obtain $u\in H^2(\Omega)$. So an easy calculation give us the uniqueness of the solution of (P). Then $u(x)$ defined in the statemant in Lemma 2.1 is the solution of (P) and (V) by the uniqueness of the solution of (P). (Q. E. D.)

*Remark* 1. In the above lemma, the restriction $\alpha\neq 1$ is not essential. In case of $\alpha=1$, we can construct an explicit solution by using the first eigenfunction $u_1$ and eigenvalue $\lambda_1$, and replacing $u_0$ by $\lambda_1 u_1$ (See: Caffarelli and Freedman [8]).

The purpose of the rest of this section is to estimate $\Phi(u)$. To this end, we discuss the property of a solution of (2.1) in the following lemma.

LEMMA 2.2. *Let* $u_0$ *be a solution of* (2.1). *Then it follows that*

$$\int_{B_1} |\nabla u_0|^2 dx \leqq \frac{|S_n|}{n} \Lambda^2.$$

*Proof.* In $B_\mu$ $(0<\mu\leqq 1)$, it follows that $\Delta u_0<0$ and $u_0$ is not a constant. By Theorem 3.5 in Gilbarg and Trudinger [4], we have

$$u_0(x) > u_0(x)|_{\partial B_\mu} = c_\mu \quad \text{in} \quad B_\mu,$$

where $c_\mu$ is a constant depending only on $\mu$. So by Lemma 3.4 in Gilbarg and Trudinger [4], we obtain

$$\left.\frac{\partial u_0}{\partial \nu}\right|_{|x|=\mu} < 0.$$

Since $\mu$ is arbitrary in (0, 1], we have

$$\frac{\partial u_0}{\partial \nu} \leqq 0 \quad \text{in} \quad B_1. \tag{2.7}$$

On the other hand $\Delta u_0 = \frac{1}{r^{n-1}} \cdot \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u_0}{\partial r}\right)$ since $u_0(x)$ is depend only on $r$ $(=|x|)$. Thus we have

$$-\frac{d}{dr}\left(r^{n-1}\frac{du_0}{dr}\right) = r^{n-1} u_0^\alpha.$$

It follows that

$$\begin{aligned} -\frac{d^2 u_0}{dr^2} &= (1-n) r^{-n} \int_0^r s^{n-1} u_0(s)^\alpha ds + u_0(r)^\alpha \\ &\geqq (1-n) r^{-n} u_0(r)^\alpha \int_0^r s^{n-1} ds + u_0(r)^\alpha \\ &= \frac{1}{n} u_0(r)^\alpha \geqq 0. \end{aligned}$$

Thus $du_0/dr$ is decreasing for $r \in [0, 1]$. By this fact and (2.7), we have

$$0 \geqq \frac{\partial u_0}{\partial r} \geqq \left.\frac{\partial u_0}{\partial r}\right|_{r=1} = \text{const.} \quad \text{in} \quad B_1.$$

Since $\left(\frac{\partial u_0}{\partial r}\right)^2 = |\nabla u_0(x)|^2$, we obtain

$$|\nabla u_0(x)|^2 \leqq \Lambda^2.$$

Hence it follows that

$$\int_{B_1} |\nabla u_0|^2 dx \leqq (\text{volume of } B_1) \times \Lambda^2 = \frac{|S_n|}{n} \Lambda^2. \qquad \text{(Q.E.D.)}$$

In the following lemma, $\Phi(u)$ is calculated for special case.

LEMMA 2.3. *Assume* (A5). *Then it follows that*

$$\Phi(u)=\begin{cases} -C_1\lambda^{(n-2)/(n-(n-2)\alpha)}+C_2 & (\text{if } n>2), \\ -\dfrac{I^2}{8\pi}\log\lambda+C_3 & (\text{if } n=2), \end{cases}$$

*where* $C_j$ $(j=1, 2, 3)$ *are constants such that* $C_1=C_1(n, I, \alpha)>0$, $C_2=C_2(n, I, R)$, *and* $C_3=C_3(I, \alpha, R)$.

*Proof.* By Lemma 1.2, $u$ is determined by (2.2)$\sim$(2.6). Let us define $\Phi_i(u)$ $(i=1, 2, 3, 4)$ by

$$\Phi_1(u)\equiv\frac{1}{2}\int_{\Omega_p}|\nabla u|^2 dx\,,$$

$$\Phi_2(u)\equiv\frac{1}{2}\int_{\Omega_v}|\nabla u|^2 dx\,,$$

$$\Phi_3(u)\equiv-\lambda\int_{\Omega}\int_0^{u(x)} s^\alpha ds dx\,,$$

$$\Phi_4(u)\equiv Iu(\partial\Omega)\,,$$

where $\Omega_v$ is $\Omega\setminus\Omega_p$. By using (2.2), (2.5) and (2.6), we have

$$\Phi_1(u)=\frac{I^2}{2\Lambda^2|S_n|^2\varepsilon^{n-2}}\int_{B_1}|\nabla u_0(x)|^2 dx\,,$$

$$\Phi_2(u)=\begin{cases} \dfrac{I^2}{2(n-2)|S_n|}\Big(\dfrac{1}{\varepsilon^{n-2}}-\dfrac{1}{R^{n-2}}\Big) & (\text{if } n>2), \\ \dfrac{I^2}{4\pi}(\log R-\log\varepsilon) & (\text{if } n=2), \end{cases}$$

$$\Phi_3(u)=-\frac{I^2}{(\alpha+1)\Lambda^2|S_n|^2\varepsilon^{n-2}}\int_{B_1}|\nabla u_0(x)|^2 dx\,,$$

$$\Phi_4(u)=\begin{cases} \dfrac{I^2}{(n-2)|S_n|}\Big(\dfrac{1}{R^{n-2}}-\dfrac{1}{\varepsilon^{n-2}}\Big) & (\text{if } n>2), \\ \dfrac{I^2}{2\pi}(\log\varepsilon-\log R) & (\text{if } n=2), \end{cases}$$

where $\varepsilon$ is defined by (2.2). First we consider our lemma in case of $n>2$.

Since $\Phi(u)=\Phi_1(u)+\Phi_2(u)+\Phi_3(u)+\Phi_4(u)$, we obtain the following:

$$\Phi(u)=-C_4\times\Big(\frac{1}{\varepsilon^{n-2}}\Big)+\frac{I^2}{2(n-2)|S_n|R^{n-2}},$$

where

$$-C_4\equiv\Big(\frac{1}{2}-\frac{1}{\alpha+1}\Big)\frac{I^2}{\Lambda^2|S_n|^2}\int_{B_1}|\nabla u_0(x)|^2 dx-\frac{I^2}{2(n-2)|S_n|}.$$

Since $\alpha<(n+2)/(n-2)$, we have

$$-C_4 \leqq \frac{I^2}{n\Lambda^2|S_n|^2}\int_{B_1}|\nabla u_0(x)|^2 dx - \frac{I^2}{2(n-2)|S_n|}$$

By using lemma 2.2, we obtain

$$-C_4 < \frac{I^2}{n^2|S_n|} - \frac{I^2}{2(n-2)|S_n|} < 0 .$$

So we have $C_4>0$, which depend only on $n$, $I$, $\alpha$. By using the definition of $\varepsilon$, we can express $\Phi(u)$ as follows:

$$\Phi(u) = -C_1\lambda^{(n-2)/(n-(n-2)\alpha)} + C_2 ,$$

where

$$C_1 \equiv C_4 \times \left(\frac{I}{\Lambda|S_n|}\right)^{(n-2)(\alpha-1)/(n-(n-2)\alpha)},$$

$$C_2 \equiv \frac{I^2}{2(n-2)|S_n|R^{n-2}} .$$

In case of $n=2$, we obtain the following by using (2.2)~(2.4):

$$\Phi(u) = -\frac{I^2}{8\pi}\log\lambda + C_3 ,$$

where

$$C_3 \equiv \frac{I^2(\alpha-1)}{8\pi}\log\left(\frac{I}{2\pi\Lambda^2}\right) + \left(\frac{1}{2} - \frac{1}{\alpha+1}\right)\frac{I^2}{4\pi\Lambda^2}\int_{B_1}|\nabla u_0(x)|^2 dx - \frac{I^2}{4\pi}\log R .$$

Here $C_3$ is depends only on $I$, $\alpha$, $R$. Thus we have proved our this lemma. (Q. E. D.)

*Remark* 2. The lemmas in this section are valid for $0<\alpha<(n+2)/(n-2)$.

## § 3. The estimate of $\Phi(u)$ in general case.

In this section, we extend the result of the preceding section. When we emphasis that $\Phi(u)$ or $V$ depend on $g(x, s)$ or $\Omega$, we write $\Phi_g$, $\Phi_\Omega$, $V_g$ or $V_\Omega$. The next lemma is concerned with the relation between $\Omega$ and $\Phi$ when we fix $g(x, u)$.

LEMMA 3.1. *Let $\Omega_1$ and $\Omega_2$ be any domains in $\boldsymbol{R}^n$ such that $\Omega_1 \subset \Omega_2$. Assume* (A1) *and* (A2) *and that $\lambda$ is a sufficiently large number. Then it follows that*

$$\inf_{v\in V_{\Omega_1}} \Phi_{\Omega_1}(v) \geqq \inf_{v\in V_{\Omega_2}} \Phi_{\Omega_2}(v) .$$

*Proof.* Let $u_1$ be a minimizer of $\Phi_{\Omega_1}$. The existence of $u_1$ is guaranteed in Brezis [7]. Let us define $u_2$ by the following formula.

$$u_2(x) \equiv \begin{cases} u(x) & x \in \Omega_1 , \\ u(\partial \Omega_1) & x \in \Omega_2 \setminus \Omega_1 . \end{cases}$$

Since $\lambda$ is a sufficiently large number, we have $u(\partial\Omega_1)<0$. By the definition of $\Phi$ and $V$, we obtain

$$u_2(x) \in V_{\Omega_2}$$

and

$$\Phi_{\Omega_1}(u_1) = \Phi_{\Omega_2}(u_2) .$$

Thus it follows that

$$\inf_{v \in V_{\Omega_1}} \Phi_{\Omega_1}(v) \geqq \inf_{v \in V_{\Omega_2}} \Phi_{\Omega_2}(v) . \qquad \text{(Q. E. D.)}$$

We need the next lemma to prove lemma 3.3.

LEMMA 3.2. *Assume that $u'$ is the solution of* (V) *for* $g(x, s)=g_0(x, s) \equiv Ks_+^\alpha$. *Then it follows that*

$$\Phi_0(u'+\gamma) \leqq \Phi_0(u') \qquad \textit{for } {}^\forall\gamma \in R ,$$

*where* $\Phi_0 \equiv \Phi_{g_0}$.

*Proof.* By the definition of $\Phi_0(u)$, we have

$$\Phi_0(u'+\gamma) = \frac{1}{2}\int_{\Omega_p} |\nabla u'|^2 dx - \frac{\lambda}{\alpha+1}\int_\Omega K(u'+\gamma)^{\alpha+1} dx + I(u(\partial\Omega)+\gamma) .$$

So we obtain

$$\frac{\partial}{\partial\gamma}\Phi_0(u'+\gamma) = -\lambda\int_\Omega K(u'+\gamma)^\alpha dx + I .$$

Since $u'$ is the solution of (V), $u'$ is a solution of (P). So $u'$ satisfies

$$\lambda\int_\Omega g_0(x, u')dx = I .$$

Then we have

$$\frac{\partial}{\partial\gamma}\Phi_0(u'+\gamma) = 0 \qquad (\text{if } \gamma=0) .$$

Moreover since $g_0(x, s)$ is monotonically increasing, we have

$$\frac{\partial}{\partial\gamma}\Phi_0(u'+\gamma) > 0 \qquad (\text{if } \gamma<0) ,$$

$$\frac{\partial}{\partial\gamma}\Phi_0(u'+\gamma) < 0 \qquad (\text{if } \gamma>0) .$$

Thus we have proved the this lemma. (Q. E. D.)

In the next lemma, we consider the relation between $g$ and $\Phi$ when we fix $\Omega$.

LEMMA 3.3. *Assume that* $g(x, s)$ *satisfies* (A1) *and* (A2). *Then it follows that*

$$\inf_{v\in V_g}\Phi_g(v)\leqq\inf_{v\in V_{g_0}}\Phi_{g_0}(v),$$

*where* $g_0(x, s)=Ks_+^\alpha$.

*Proof.* Let $u'$ be the solution of (V) for $g_0(x, s)$. Since $g(x, s)$ grows to infinity as $s$ grows to infinity, there exist $\gamma\in\boldsymbol{R}^n$ which satisfies

$$\lambda\int_\Omega g(x, u'+\gamma)dx=I.$$

This implies that

$$u'+\gamma\in V_g.$$

By this fact, it follows that

$$\inf_{v\in V_g}\Phi_g(v)\leqq\Phi_g(u'+\gamma)=\frac{1}{2}\int_\Omega|\nabla u'|^2dx-\lambda\int_\Omega\int_0^{u'+\gamma}g(x, s)dxds+I(u'(\partial\Omega)+\gamma)$$

(by using our assumption (A2))

$$\leqq\frac{1}{2}\int_\Omega|\nabla u'|^2dx-\lambda\int_\Omega\int_0^{u'+\gamma}g_0(x, s)dxds+I(u'(\partial\Omega)+\gamma)$$

$$=\Phi_{g_0}(u'+\gamma)$$

(by lemma 3.2)

$$\leqq\Phi_{g_0}(u')$$

$$=\inf_{v\in V_{g_0}}\Phi_{g_0}(v).$$

(Q. E. D.)

By using lemma 3.3, we extend lemma 2.3 to the following form:

LEMMA 3.4. *Assume* (A1) *and* (A2). *Let* $R$ *be the maximum of radius of balls contained in* $\Omega$ *and* $u$ *be the solution of* $V$. *Then it follows that*

$$\Phi(u)\leqq\begin{cases}-\tilde{C}_1\lambda^{(n-2)/(n-(n-2)\alpha)}+\tilde{C}_2 & (\text{if } n>2),\\ -\dfrac{I^2}{8\pi}\log\lambda+\tilde{C}_3 & (\text{if } n=2),\end{cases}$$

*where* $\tilde{C}_1$, $\tilde{C}_2$ *and* $\tilde{C}_3$ *are constants, which depend on* $n$, $I$, $K$, $\alpha$, $R$.

*Proof.* By using lemma 3.1 and 3.3, we can estimate $\Phi(u)$ as follows:

$$\Phi(u)=\inf_{v\in V_{g,\Omega}}\Phi_{g,\Omega}(v)$$

$$\leqq\inf_{v\in V_{g,B_R}}\Phi_{g,B_R}(v)$$

$$\leqq\inf_{v\in V_{g_0,B_R}}\Phi_{g_0,B_R}(v).$$

Since lemma 2.3 gives an estimate for $\inf_{v\in V_{g_0,B_R}}\Phi_{g_0,B_R}(v)$, then we obtain this lemma. (Q. E. D.)

*Remark* 3. Further we can show the following strengthend form of lemma 3.4 under some assumption. Let $R_0$ be the maximum of the radius of a ball contained in $\Omega$ and let $R_1$ be the minimum of the radius of a ball which contains $\Omega$. Assume (A1), (A2) and that $\lambda$ is sufficiently large. If there exist $K_0$, $K_1$, $\alpha_0$, $\alpha_1$ such that $1\leqq\alpha_0\leqq\alpha_1<p$, $0<K_0$, $K_1$ and $K_0 s^{\alpha_0}\leqq g(x, s)\leqq K_1 s^{\alpha_1}$ for all $x\in\Omega$ and all $s>0$, then it follows that

$$-C_1'\lambda^{(n-2)/(n-(n-2)\alpha_1)}+C_2'\leqq\Phi(u)\leqq-C_3'\lambda^{(n-2)/(n-(n-2)\alpha_0)}+C_4' \qquad (\text{if } n>2),$$

$$\Phi(u)=-\frac{I^2}{8\pi}\log\lambda+O(1) \qquad (\text{if } n=2).$$

Here $C_1'>0$, $C_2'$ are constants which depend on $n$, $I$, $K_1$, $\alpha_1$, $R_1$; and $C_3'>0$, $C_4'$ are constants which depend on $n$, $I$, $K_0$, $\alpha_0$, $R_0$. Moreover $0(1)$ denote the quantity which remain bounded for $\lambda$ and depends on $n$, $I$, $K_0$, $K_1$, $\alpha_0$, $\alpha_1$, $R_0$, $R_1$.

*Remark* 4. The result of this section is valid if we replace (A2) by

(A2′) $\qquad g(x, s)\geqq Ks^{\alpha} \quad$ for $\ ^{\forall}x\in\Omega,\ ^{\exists}K>0,\ 0<{}^{\exists}\alpha<p,\ ^{\forall}s\geqq 0.$

## 4. An asymptotic property of a variational solution of (V).

We need the next lemma to estimate the size of $\Omega_p$ in Theorem 4.2.

LEMMA 4.1. *If $u$ is the solution of* (V), *then it follows that*

$$u(\partial\Omega)\leqq\frac{2}{I}\Phi(u).$$

*Remark* 5. In case when $\lambda$ is sufficiently large,

$$|u(\partial\Omega)|\geqq\frac{2}{I}|\Phi(u)|$$

since $\Phi(u)<0$.

*Proof*. Let $u$ be a solution of (V). Integrating by parts, we have

$$\begin{aligned}&\int_{\Omega_p}|\nabla u|^2dx=\lambda\int_{\Omega_p}ug(x, u)dx,\\&\int_{\Omega_v}|\nabla u|^2dx=-Iu(\partial\Omega).\end{aligned}\tag{4.1}$$

Since $g(x, \cdot)$ is a convex function, we obtain

$$\frac{g(x, s)}{s}\leqq\frac{g(x, t)}{t} \qquad \text{for} \quad 0\leqq s\leqq t.\tag{4.2}$$

And we have

$$\frac{1}{2}ug(x, u)-\int_0^u g(x, s)ds=\int_0^u\Big(\frac{g(x, u)}{u}s-g(x, s)\Big)ds\,. \tag{4.3}$$

So by using (4.1)~(4.3), we obtain

$$\begin{aligned}\Phi(u)&=\frac{\lambda}{2}\int_{\Omega_p}ug(x, u)dx-\frac{I}{2}u(\partial\Omega)\\ &\quad-\lambda\int_\Omega\int_0^{u(x)}g(x, s)dsdx+Iu(\partial\Omega)\\ &=\lambda\int_\Omega\int_0^{u(x)}\Big(\frac{g(x, u)}{u}s-g(x, s)\Big)dsdx+\frac{1}{2}Iu(\partial\Omega)\\ &\geqq\frac{1}{2}Iu(\partial\Omega)\,.\end{aligned}$$

(Q. E. D.)

The next theorem is the main theorem in this paper.

THEOREM 4.2. *Let $u$ and $\Omega_p$ be the solution of* (V). *If $g(x, s)$ satisfies* (A1) *and* (A2) *and $\lambda$ is a sufficiently large number, then it follows that*

$$\operatorname{diam}(\Omega_p)\leqq\frac{C}{\log\lambda} \qquad (if\ n=2),$$

$$d(\Omega_p)\leqq C\lambda^{-(n-2)/2(n-(n-2)\alpha)} \qquad (if\ n>2),$$

*where $d(\Omega_p)$ is the maximum of the measure of the cross section of $\partial\Omega_p$ by an $(n-1)$-dimensional hyperplane and $C$ depends on $n$, $I$, $K$, $\alpha$, $\Omega$.*

Proof of case of $n=2$. In this proof we use the method of Caffarelli and Freedman [8]. We choose $A$ and $B$ such that $|A-B|=\operatorname{diam}(\Omega_p)$ and $A$, $B\in\partial\Omega_p$. Consider the family of straight lines $\gamma_x$ passing throught $x$ and orthogonal to $\overline{AB}$ when $x$ varies on $\overline{AB}$. Denote by $\delta_x=\overline{y_xz_x}$ a segment lying in $\gamma_x$ such that $y_x\in\partial\Omega$, $z_x\in\partial\Omega_p$ and $\delta_x\subset\overline{\Omega_v}$. Then we have

$$u(y_x)-u(z_x)=\int_{\delta_x}\frac{\partial u}{\partial\delta_x}dl\,.$$

By using the identities $u(y_x)=u(\partial\Omega)$, $u(z_x)=0$, we obtain

$$|u(\partial\Omega)|\leqq\int_{\delta_x}|\nabla u|\,dl\,. \tag{4.3}$$

If we integrate this with respect to $x$ from $A$ to $B$, then we have

$$\begin{aligned}|B-A|\,|u(\partial\Omega)|&\leqq\int_A^B\int_{\delta_x}|\nabla u|\,dldl\,,\\ &\leqq\Big(\int_A^B\Big(\int_{\delta_x}|\nabla u|\,dl\Big)^2dl'\Big)^{1/2}\times|B-A|^{1/2},\end{aligned}$$

$$\leqq\left(\int_{\Omega_v}|\nabla u|^2 dx\right)^{1/2}\times|B-A|^{1/2}\times(\mathrm{diam}(\Omega))^{1/2}.$$

By using (4.1), we obtain

$$|B-A|\,|u(\partial\Omega)|\leqq I^{1/2}|u(\partial\Omega)|^{1/2}|B-A|^{1/2}\times(\mathrm{diam}(\Omega))^{1/2}.$$

Then it follows that

$$|B-A|\leqq I|u(\partial\Omega)|^{-1}\times\mathrm{diam}(\Omega).$$

By using Lemma 3.4 and Lemma 4.1, we obtain the theorem in case of $n=2$. (Q. E. D.)

Proof of case of $n>2$. Choose $S$ and an $(n-1)$-dimensional hypersurface $S'$ such that $|S|=d(\Omega_p)$ and $S=S'\cap\Omega_p$, where $|S|$ is the $(n-1)$-dimensional Lebesgue measure of $S$. Let $x$ be an arbitrary point contained in $S$. When $x$ varies in $S$, we consider the family of straight lines $l_x^i$ and of points $P_i$, $Q_i$, which satisfy the following condition. $l_x^1$ is a line that contains $x$ and orthogonal to $S$. $l_x^i$ is a line contained in $S$ such that $l_x^i$ $(i=2, \cdots, n)$ is orthogonal to $l_x^j$ $(1\leqq j<i)$ and passing through $x$. Let $P_i\in\partial\Omega_p$, $Q_i\in\partial\Omega_p$ be points such that $P_i$, $Q_i\in l_x^i$. If there are more than three points in $\partial\Omega_p\cap l_x^i$, we choose $P_i$, $Q_i$ in such a way that the distance from $P_i$ to $Q_i$ is the longest of all. Choose $T_1$ in $l_x^1\cap\partial\Omega$ such that $\overline{P_1T_1}$ belongs to $\Omega_v$. Of course $P_i$, $Q_i$ and $T_1$ depend on $x$. Let $\pi_i'$ be an $i$-dimensional hyperplane which contains $l_x^j$ $(1\leqq j\leqq i)$. And let $\pi_i$ be an intersection of $\Omega_v$ with $\pi_i'$. In particular, $\pi_1$ contains $\overline{P_1T_1}$ and $\pi_n$ is equal to $\Omega_v$. We can assume that $l_x^i$ is orthogonal to the $(n-1)$-dimensional hyperplane: $x_i=0$. By using the identities $u(Q_1)=0$, $u(T_1)=u(\partial\Omega)$, we have

$$u(\partial\Omega)=\int_{T_1}^{P_1}D_{x_1}u\,dx_1.$$

Then we obtain

$$|u(\partial\Omega)|\leqq\int_{\pi_1}|\nabla u|\,dx_1.$$

By integrating both side of the above formula from $P_2$ to $Q_2$, it follows that

$$|u(\partial\Omega)|\int_{Q_2}^{P_2}dx_2\leqq\int_{Q_2}^{P_2}\int_{\pi_1}|\nabla u|\,dx_1dx_2$$

$$\leqq\int_{\pi_2}|\nabla u|\,dx_2.$$

By repeating this process, we obtain

$$|u\partial(\Omega)|\int_{Q_n}^{P_n}\cdots\int_{Q_2}^{P_2}dx_2\cdots dx_n\leqq\int_{\pi_n}|\nabla u|\,dx_1\cdots dx_n$$

$$=\int_{\Omega_v}|\nabla u|\,dx$$

Since $\int_{Q_n}^{P_n}\cdots\int_{Q_2}^{P_2} dx_2\cdots dx_n \geqq |S|$, we have

$$|u(\partial\Omega)||S| \leqq \int_{\Omega_v} |\nabla u|\, dx$$
$$\leqq |\Omega|^{1/2}\left(\int_{\Omega_v} |\nabla u|^2 dx\right)^{1/2}$$
$$= |\Omega|^{1/2} |u(\partial\Omega)|^{1/2} \times I^{1/2}.$$

Then we obtain

$$(d(\Omega_p)=)|S| \leqq |\Omega|^{1/2} |u(\partial\Omega)|^{-1/2} \times I^{1/2}.$$

By using lemma 3.4 and lemma 4.1, we obtain this theorem. (Q. E. D.)

*Remark* 6. Even if we use the type of the estimates of Remark 3 in place of lemma 3.4 in the proof of the above theorem, we can not improve the estimate of $\Omega_p$.

*Remark* 7. In case of $n=2$, we can extend this result to

$$\mathrm{diam}(\Omega_p) \leqq C\lambda^{-1/2}$$

by using the method of Freedman [2] (lemma 13.5~lemma 13.7). But we can not apply this method in case of $n>2$.

The next corollary is an estimate of the size of the level curves of the solution of (V). We define $\Omega_t$ by $\Omega_t \equiv \{x \in \Omega\,;\, u(x) \geqq -t\}$. In particular $\Omega_0$ is equal to $\Omega_p$.

COROLLARY 4.3. *Let u be a solution of* (V). *If* $g(x, s)$ *satisfies* (A1) *and* (A2) *and* $\lambda$ *is sufficiently large, then it follows that*

$$\mathrm{diam}(\Omega_t) \leqq \frac{C}{\log \lambda} \qquad (\textit{if } n=2),$$
$$d(\Omega_t) \leqq C\lambda^{-(n-2)/2(n-(n-2)\alpha)} \qquad (\textit{if } n>2),$$

*where C depends on* $I, n, K, \alpha, \Omega, t$.

*Proof*. This corollary is trivial in case of $t<0$ since $\Omega_p \supseteqq \Omega_t$. Thus we consider the case of $t>0$. We use notations as in Theorem 4.2 with replacing $\Omega_p$ by $\Omega_t$. In case of $n=2$, we have

$$|u(\partial\Omega)+t| \leqq \int_{\delta_x} |\nabla u|\, dl\,.$$

Applying the process of the proof of Theorem 4.2, it follows that

$$|B-A| \leqq \frac{I|u(\partial\Omega)|}{|u(\partial\Omega)+t|^2} \tag{4.4}$$

On the other hand we obtain

$$|u(\partial\Omega)+t| \geqq \frac{1}{2}|u(\partial\Omega)| \tag{4.5}$$

for sufficiently large $\lambda$ since $u(\partial\Omega)<0$ and $u(\partial\Omega)\to-\infty$ as $\lambda\to\infty$ (by using lemma 3.4 and lemma 4.1). (4.4) and (4.5) implies that

$$|B-A| \leqq 4I|u(\partial\Omega)|^{-1}.$$

Then we obtain this corollary by using lemma 3.4 and lemma 4.1. By using this process, we can show this corollary in case of $n>2$. (Q. E. D.)

## §5. A symmetric property of a solution of (P).

In this section we discuss a symmetric property of solution of (P). We say a function is "symmetric" if it is symmetric with respect to the $(n-1)$-dimensional hyperplane: $x_n=0$. The symmetricity with respect to the $(n-1)$-dimensional hyperplane: $x_n=0$ is not essentially. Our argument is possible under a transformation $\tau$ such that $\tau\circ\tau=$identity and $\Delta$ is invariant under $\tau$. In this section we assume that $\Omega$ is symmetric.

Let $\{\lambda_n\}$ denote the eigenvalues of the equation:

$$\begin{cases} -\Delta\varphi=\lambda\varphi & \text{in } \Omega, \\ \varphi\in H_0^1(\Omega), \end{cases}$$

where $\lambda_{n+1}\geqq\lambda_n$. And $\{\lambda_n^*\}$ are the eigenvalues whose eigenfunctions are symmetric, provided that $\lambda_{n+1}^*\geqq\lambda_n^*$.

The next theorem is concerned with the existence and the uniqueness of the symmetric solution of (P).

THEOREM 5.1. *If $g(x, s)$ satisfies* (A1), (A2) *and* (A3), *there exists a symmetric solution of* (P). *Moreover if $g(x, s)$ satisfies* (A1), (A2), (A3) *and* (A4), *and*

*[$I$ and $\lambda$ are constants such that a free boundary exists for any solution of* (P).], (5.1)

*then it follows that a symmetric solution of* (P) *is uniquely determined for $\lambda<\lambda_2^*/M$.*

*Remark* 8. Under what conditions the statement (5.1) is satisfied?

By the proposition 7 and proposition 8 in Ambrosetti and Mancini [1], a free boundary exists under either of the following conditions.

$$^{\exists}\inf_{x\in\Omega}\lim_{s\to0}\frac{g(x, s)}{s}=m_0>0,$$

$I$: sufficiently small,

$$^{\forall}\lambda > \frac{\lambda_1}{m_0},$$

or

$$^{\exists}\inf_{x\in\Omega} \lim_{s\to\infty} \frac{g(x, s)}{s} = m_\infty > 0,$$

$I$: sufficiently large,

$$^{\forall}\lambda > \frac{\lambda_1}{m_\infty}.$$

Then by using our theorem, we obtain the uniqueness of the symmetric solution under the following condition:

$$\frac{\lambda_1}{m_0} < {}^{\forall}\lambda < \frac{\lambda_2^*}{M}, \qquad I\text{: sufficiently small,}$$

or

$$\frac{\lambda_1}{m_\infty} < {}^{\forall}\lambda < \frac{\lambda_2^*}{M}, \qquad I\text{: sufficiently large.}$$

Since $M \geqq m_0, m_\infty$, this interval may be empty in some cases. But the interval is not empty in the following simple example. We choose a domain $\Omega$ such that $\lambda_1 < \lambda_2^*$. We define $g(x, s)$ as follows:

$$g(x, s) = \begin{cases} 0 & (s<0) \\ as & (0 \leqq s \leqq s_0) \\ bs + (a-b)s_0 & (s_0 < s), \end{cases}$$

where $a$ and $b$ are constants such that $0<a<b$ and $b/a < \lambda_2^*/\lambda_1$. Then the symmetric solution is uniquely determined for $\lambda \in [\lambda_1/a, \lambda_2^*/b]$ if $\lambda$ is sufficiently small. If $\lambda$ is sufficiently large, the unique symmetric solution exists for $\lambda \in (\lambda_1/b, \lambda_2^*/b)$ without our assumption $b/a < \lambda_2^*/\lambda_1$.

*Proof of the existence.* We define successively $\{u_n\}$ as follows. Let $u_0$ be an element in $W \equiv$ {a symmetric function in $V$} and $u_n$ be a solution of the following system:

$$\begin{cases} -\Delta u_n = \lambda g(x, u_{n-1}) & \text{in } \Omega, \\ u_n|_{\partial\Omega} = \text{unknown constant}, \\ \lambda\int_\Omega g(x, u_n)dx = I. \end{cases} \tag{5.2}$$

By a proposition (p. 424) in Berestycki and Brezis [7], $\{u_n\}$ converges to the solution of (P) in $W$ under the assumption (A1) and (A2). Here we choose $u_0$ which is symmetric. We will show that $u_n$ is symmetric if $u_{n-1}$ is symmetric. Let us consider the following Dirichlet problem:

$$\begin{cases} -\Delta\varphi=\lambda g(x,\ u_{n-1}) & \text{in} \quad \Omega, \\ \varphi|_{\partial\Omega}=0 \end{cases} \tag{5.3}$$

This Dirichlet problem is uniquely solvable (See: Ch. 4 in Gilbarg and Trudinger [4]). Since

$$\lim_{s\to-\infty}\lambda\int_\Omega g(x,\ s)dx=0$$

and

$$\lim_{s\to\infty}\lambda\int_\Omega g(x,\ s)dx=\infty\,,$$

$g(x,\ s)$ is continuous and $\varphi\in L^\infty(\Omega)$, there exist a constant $c\in R$ such that

$$\lambda\int_\Omega g(x,\ \varphi+c)dx=I$$

Then $\varphi+c$ satisfies (5.2). We define $u_n=\varphi+c$. Then $u_n$ is the solution of (5.2). We assume that $u_n$ is not symmetric. i. e. $\varphi$ is not symmetric. We define $\varphi'$ by the following.

$$\varphi'(x_1,\ \cdots,\ x_{n-1},\ x_n)=\varphi(x_1,\ \cdots,\ x_{n-1},\ -x_n)$$

$\varphi\neq\varphi'$ and $\varphi'$ is a solution of (5.3). This contradicts the uniqueness of (5.3). So $u_n$ is symmetric. Then $u_n$ converge to a function $\in W$ and $u$ is the symmetric solution of (P). (Q. E. D.)

*Proof of uniqueness.* In this proof we use the method of Sermange [11]. Let $u_1$ and $u_2$ be two symmetric solutions of the problem (P). And let $\omega_i$ be a plasma domain of $u_i$. We can assume $u_1(\partial\Omega)\geqq u_2(\partial\Omega)$. We define $\tilde{u}_1(x)$ as follows. In case of $u_1(\partial\Omega)=u_2(\partial\Omega)$, we set

$$\tilde{u}_1(x)\equiv u_1(x)\,.$$

In case of $u_1(\partial\Omega)>u_2(\partial\Omega)$, we set $\tilde{u}_1(x)\in H^1(\Omega)$ such that

$$\tilde{u}_1(x)\equiv\begin{cases} u_1(x) & (\text{if } x\in\bar{\omega}_1), \\ 0 & (\text{if } x\in\partial\bar{\omega}_1), \\ \text{harmonic} & (\text{if } x\in\Omega\setminus\bar{\omega}_1), \\ u_2(\partial\Omega) & (\text{if } x\in\partial\Omega). \end{cases}$$

Then $\tilde{u}_1(x)$ satisfies

$$\begin{cases} -\Delta\tilde{u}_1(x)=\lambda g(x,\ \tilde{u}_1(x)) & \text{in} \quad \Omega, \\ \lambda\int_\Omega g(x,\ \tilde{u}_1(x))dx=I\,, \end{cases}$$

in the sense of $H^1(\Omega)$. But $\tilde{u}_1(x)$ is not the solution of (P) since $u$ does not belong to $H^2(\Omega)$. We set $w(x)=\tilde{u}_1(x)-u_2(x)$, then $w(x)$ satisfies the following.

$$\begin{cases} -\Delta w=\lambda(g(x,\ \tilde{u}_1(x))-g(x,\ u_2(x))\,, \\ w\in H_0^1(\Omega)\,. \end{cases} \tag{5.4}$$

On the other hand if we set

$$h(x)=\begin{cases} 0 & (\text{if}\quad \tilde{u}_1(x)=u_2(x)), \\ \dfrac{g(x,\ \tilde{u}_1(x))-g(x,\ u_2(x))}{\tilde{u}_1(x)-u_2(x)} & (\text{if}\quad \tilde{u}_1(x)\neq u_2(x)), \end{cases}$$

then $h(x)$ is a measurable symmetric function. And we have

$$0\leqq h(x)\leqq M \tag{5.5}$$

by our assumption (A4) and the monotone increasing property of $g(x,\ \cdot)$. By using the definition of $h(x)$, we can rewrite (5.4) as follows.

$$\begin{cases} -\Delta w=\lambda hw & \text{in}\quad \Omega, \\ w\in H_0^1(\Omega)\,. \end{cases} \tag{5.6}$$

Thus $w$ is an eigenfunction and $\lambda$ is an eigenvalue in (5.6).

We compare the following two eigenvalue problems:

$$\begin{cases} -\Delta\varphi=\mu^* h\varphi & \text{in}\quad \Omega, \\ \varphi\in H_0^1(\Omega)\,, \end{cases} \tag{5.7}$$

$$\begin{cases} -\Delta\phi=\left(\dfrac{\lambda^*}{M}\right)M\phi & \text{in}\quad \Omega, \\ \phi\in H_0^1(\Omega)\,, \end{cases} \tag{5.8}$$

where $\mu^*$ is an eigenvalue whose eigenfunction $\varphi$ is symmetric. (5.8) is an ordinary eigenvalue problem. By (5.5) we have

$$\mu_i^*\geqq\frac{\lambda_i^*}{M}.$$

And by this fact and our assumption, we obtain

$$\lambda<\frac{\lambda_2^*}{M}\leqq\mu_2^*.$$

Since $\lambda$ is an eigenvalue of (5.7), it follows that

$$\lambda=\mu_1^*,$$

i.e. $w(x)$ is the first eigenfunction of (5.7). Since $w(x)$ is symmetric, we have

$$w>0\quad(\text{or}\ w<0)\qquad\text{in}\quad \Omega,$$

$$\text{i.e.}\quad \tilde{u}_1(x)>u_2\quad(\text{or}\ \tilde{u}_1(x)<u_2)\qquad\text{in}\quad \Omega.$$

By this fact and the monotone increasing property of $g(x, \cdot)$, it follows that

$$I=\lambda\int_{\Omega}g(x, \tilde{u}_1(x))dx>\lambda\int_{\Omega}g(x, u_2(x))dx=I\,.$$

This is a contradiction. Thus we have proved the uniqueness of symmetric solutions. (Q. E. D.)

*Remark* 9. We can rewrite (5.1) as follows. "$I$ and $\lambda$ are constants such that a free boundary exists for any symmetric solution of (P)."

## REFERENCES


[1] AMBROSETTI, A. AND G. MANCINI, A free boundary problem and a related semilinear equation, Nonlinear Analysis T. M. A., **4**(5) (1980), 909-915.

[2] FREEDMAN, A., Variational principles and free boundary problems, Wiley-Interscience (1982).

[3] CHANG, K. C., Remarks on free boundary problem for flux equation in plasma physics, Comm. in P. D. E., **5**(9) (1980), 741-751.

[4] GILBARG, D. AND N. S. TRUDINGER, Elliptic partial differential equations of second order, Springer-Verlag (1977).

[5] KINDERLEHRER, D. AND J. SPRUCK, The shape and smoothness of stable plasma physics, Annali. Scu. Norm. Sup. Pisa., **5** (1978), 131-148.

[6] AMANN, H., Fixed point equation and nonlinear eigenvalue problems in ordered banach spaces, SIAM Review, **18**(4) (1976), 620-709.

[7] BERESTYCKI, H. AND H. BREZIS, On a free boundary problem arising in plasma physics, Nonlinear Analysis T. M. A., **4**(3) (1980), 415-436.

[8] CAFFARELLI, L. A. AND A. FREEDMAN, Asymptotic estimate for the plasma problems, Duke Math. J., **47**(3) (1980), 705-742.

[9] CAFFARELLI, L. A. AND J. SPRUCK, Convexity properties of solution to some classical variational problem, Comm. in P. D. E., 7(11) (1982), 1337-1379.

[10] SERMANGE, M., Un méthode numérique en bifurcation application á frontié de la physique des plasmas, Appl. Math. Optim., **5** (1979), 127-151.

[11] SERMANGE, M., Bifurcation of free boundary plasma equilibria, Duke Math. J., **47**(4) (1980), 923-942.

[12] LIONS, P. L., On the existence of positive solutions of semilinear elliptic equations, SIAM Review, **24**(2) (1982), 441-467.

[13] TEMAM, R., A non linear eigenvalue problem: The shape at equilibrium of a confused plasma, Arch. Rat. Mech. Anal., **60** (1975), 51-73.

[14] TEMAM, R., Remarks on a free boundary problem arising in plasma physics, Comm. in P. D. E., **2**(6) (1977), 563-585.



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