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A METHOD TO A PROBLEM OF R. NEVANLINNA, II

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§1. **Introduction.** This paper is a continuation of our earlier paper [3]. In this paper we shall prove the following theorems.

THEOREM 1. *Let $f(z)$ be a meromorphic function of regular growth of order ρ. Then*

$$K(f)\geqq L(\rho)\liminf_{t\to\infty} S(t, E)/T(t, f).$$

THEOREM 2. *Let $f(z)$ be a meromorphic function defined by a quotient of two canonical products of genus q*

$$f(z)=\Pi E\left(\frac{z}{a_n}, q\right)\Big/\Pi E\left(\frac{z}{b_n}, q\right).$$

Suppose that the order λ and the lower order μ of $f(z)$ satisfies $q\leqq\mu<\lambda<q+1$. Let β be a number satisfying $\mu<\beta<\lambda$. Them for any E

$$\sup_{\mu<\beta<\lambda} L(\beta)\liminf_{t\to\infty} S(t, E)/T(t, f)\leqq K(f).$$

Theorem 2 was already stated without proof in [3].

In order to prove Theorem 1 we make use of the notion of proximate order of $T(t, f)$. The proximate order $l(t)$ is defined by the following conditions:

(i) $l(t)$ is real continuous and piecewise differentiable for $t>t_0$,

(ii) $l(t)\to\rho$ as $t\to\infty$,

(iii) $tl'(t)\log t\to 0$ as $t\to\infty$,

(iv) $\limsup_{t\to\infty}\dfrac{T(t, f)}{t^{l(t)}}=1$.

Let us put

$$\mu(t)=t^{\rho-l(t)},$$

then $\mu(t)$ is a slowly varying function in the sense of Karamata, that is, $\mu(t)$ satisfies $\mu(ct)/\mu(t)\to 1$ as $t\to\infty$ for any positive c. It is known that the above convergence is uniform in the wider sense in $(0, \infty)$. See Seneta [5]. Then it is easy to prove that

$$\int_{t_0}^{\infty} T(t, f)t^{-1-l(t)}dt=\infty$$

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for any finite t_0. Further it is also easy to prove the following result: Let $\mu(t)$ be slowly varying for $0<t<\infty$ and let $\phi(s)$ be absolutely integrable over $(0, \infty)$ and such that

$$|\phi(s)|=O(s^{\gamma-1}) \qquad \gamma>0 \ (s\to 0),$$

$$|\phi(s)|=O(s^{-\alpha-1}) \qquad \alpha>0 \ (s\to\infty).$$

Then

$$\int_0^\infty \frac{\mu(st)}{\mu(t)}\phi(s)ds=\int_0^\infty \phi(s)ds+o(1)$$

for $t\to\infty$. Hence for an arbitrary $\varepsilon_1>0$ there exists an s_0 such that for any $s\geqq s_0$

$$\int_0^\infty \frac{\mu(st)}{\mu(t)}\cdot\frac{x^{q-\rho-\varepsilon}}{x+e^{-i\theta}}dx=\int_0^\infty \frac{x^{q-\rho-\varepsilon}}{x+e^{-i\theta}}dx+m(s)$$

with

$$|m(s)|<\varepsilon_1.$$

Here $q=[\rho]<\rho<q+1$, $\varepsilon>0$. Further for any $\varepsilon_2>0$

$$|tl'(t)\log t|<\varepsilon_2, \quad 0<|\rho-l(t)|<\varepsilon_2$$

for $t\geqq s_0$.

In order to prove Theorem 2 we need the following Lemma, which was stated in Edrei and Fuchs [2] and Edrei [1].

LEMMA A. *Let $f(z)$ be defined as in Theorem 2. Then for $|z|\leqq R$*

$$\log|f(z)|=\sum_{|a_\mu|\leqq 2R}\log\left|E\left(\frac{z}{a_\mu}, q\right)\right|-\sum_{|b_\nu|\leqq 2R}\log\left|E\left(\frac{z}{b_\nu}, q\right)\right|+S(z, R),$$

where

$$|S(z, R)|\leqq 14(r/2R)^{q+1}T(4R, f), \qquad r=|z|$$

for $q\geqq 1$.

§2. **Proof of Theorem 1.** Let us put

$$f(z)=Az^pe^{P(z)}\Pi_1/\Pi_2,$$

where

$$\Pi_1=\Pi E\left(\frac{z}{a_n}, q\right), \quad \Pi_2=\Pi E\left(\frac{z}{b_n}, q\right)$$

and $P(z)$ is a polynomial of degree at most q and p is an integer, A a constant. Put

$$g(z)=Az^pe^{P(z)}\Pi_1^*/\Pi_2^*,$$

where

$$\Pi_1^*=\prod_{|a_n|\leqq s_0}E\left(\frac{z}{a_n}, q\right), \quad \Pi_2^*=\prod_{|b_n|\leqq s_0}E\left(\frac{z}{b_n}, q\right).$$

Here s_0 is a constant defined in §1. Let $F(z)$ be $f(z)/g(z)$. We shall consider

$$\int_0^\infty \log|F(te^{i\theta})|t^{-1-l(t)-\varepsilon}dt.$$

Here $l(t)$ is a proximate order of $T(t, f)$ and ε is an arbitrary positive constant. It is convenient to consider the following integral

$$\int_0^\infty \log E\Big(-\frac{t}{|a_n|}e^{i(\theta-\varphi_n)},\ q\Big)t^{-1-l(t)-\varepsilon}dt$$

$$=(-1)^q e^{i(\theta-\varphi_n)q}\int_{|a_n|}^\infty s^{-1-q}ds\int_0^\infty \frac{t^{q-l(t)-\varepsilon}}{t+se^{-i(\theta-\varphi_n)}}dt.$$

The inner integral of the above is equal to

$$\int_0^\infty \mu(t)\frac{t^{q-\rho-\varepsilon}}{t+se^{-i(\theta-\varphi_n)}}dt=\int_0^\infty \mu(sx)\frac{x^{q-\rho-\varepsilon}}{x+e^{-i(\theta-\varphi_n)}}dx\ s^{q-\rho-\varepsilon}$$

$$=\mu(s)s^{q-\rho-\varepsilon}\int_0^\infty \frac{\mu(sx)}{\mu(s)}\cdot\frac{x^{q-\rho-\varepsilon}}{x+e^{-i(\theta-\varphi_n)}}dx$$

$$=s^{q-l(s)-\varepsilon}\Big(\int_0^\infty \frac{x^{q-\rho-\varepsilon}}{x+e^{-i(\theta-\varphi_n)}}dx+m(s)\Big)$$

$$=-s^{q-l(s)-\varepsilon}\Big(\pi\frac{\exp(\rho+\varepsilon-q)i(\theta-\varphi_n)}{\sin\pi(q-\rho-\varepsilon)}+m(s)\Big),$$

where $|m(s)|\leqq\varepsilon_1$ for $s\geqq s_0$ as in § 1. Hence

$$\int_0^\infty \log E\Big(-\frac{t}{|a_n|}e^{i(\theta-\varphi_n)},\ q\Big)\frac{dt}{t^{1+l(t)+\varepsilon}}$$

$$=\frac{\pi}{\sin\pi(\rho+\varepsilon)}e^{i(\rho+\varepsilon)(\theta-\varphi_n)}\int_{|a_n|}^\infty \frac{ds}{s^{1+l(s)+\varepsilon}}+c\int_{|a_n|}^\infty m(s)\frac{ds}{s^{1+l(s)+\varepsilon}}.$$

Since $l(t)\to\rho$ for $t\to\infty$,

$$n(t,\ 0,\ F)\int_t^\infty s^{-1-l(s)-\varepsilon}ds\to 0,$$

$$N(t,\ 0,\ F)t^{-l(t)-\varepsilon}\to 0$$

for $t\to\infty$. Hence

$$\sum_{|a_n|>s_0}\int_{|a_n|}^\infty s^{-1-l(s)-\varepsilon}ds=\int_{s_0}^\infty d\ n(t,\ 0,\ F)\int_t^\infty s^{-1-l(s)-\varepsilon}ds$$

$$=\int_{s_0}^\infty \frac{N(t,\ 0,\ F)}{t^{1+l(t)+\varepsilon}}(t\ l'(t)\log t+\varepsilon+l(t))dt$$

$$=\rho\int_{s_0}^\infty \frac{N(t,\ 0,\ F)}{t^{1+l(t)+\varepsilon}}dt+\int_{s_0}^\infty m_2(t)\frac{N(t,\ 0,\ F)}{t^{1+l(t)+\varepsilon}}dt,$$

where $|m_2(t)|=|l(t)-\rho+\varepsilon+tl'(t)\log t|\leqq 2\varepsilon_2+\varepsilon$. Hence we have

$$\int_0^\infty \log|F(te^{i\theta})|t^{-1-l(t)-\varepsilon}dt$$

$$=\frac{\pi}{\sin\pi(\rho+\varepsilon)}\sum_{|a_n|>s_0}\cos(\rho+\varepsilon)(\theta-\varphi_n)\int_{|a_n|}^\infty s^{-1-l(s)-\varepsilon}ds$$

$$-\frac{\pi}{\sin\pi(\rho+\varepsilon)}\sum_{|b_n|>s_0}\cos(\rho+\varepsilon)(\theta-\psi_n)\int_{|b_n|}^\infty s^{-1-l(s)-\varepsilon}ds$$

$$+\sum_{|a_n|>s_0}\int_{|a_n|}^\infty \Re(cm(t))t^{-1-l(t)-\varepsilon}dt-\sum_{|b_n|>s_0}\int_{|b_n|}^\infty \Re(cm(t))t^{-1-l(t)-\varepsilon}dt.$$

Let E be a measurable subset of $[-\pi, \pi]$. Then

$$\int_0^\infty \frac{1}{2\pi}\int_E \log|F(te^{i\theta})|d\theta\, t^{-1-l(t)-\varepsilon}dt$$

$$=\frac{1}{\sin\pi(\rho+\varepsilon)}\sum_{|a_n|>s_0}\frac{\rho+\varepsilon}{2}\int_E\cos(\rho+\varepsilon)(\theta-\varphi_n)d\theta\frac{1}{\rho+\varepsilon}\int_{|a_n|}^\infty s^{-1-l(s)-\varepsilon}ds$$

$$-\frac{1}{\sin\pi(\rho+\varepsilon)}\sum_{|b_n|>s_0}\frac{\rho+\varepsilon}{2}\int_E\cos(\rho+\varepsilon)(\theta-\psi_n)d\theta\frac{1}{\rho+\varepsilon}\int_{|b_n|}^\infty s^{-1-l(s)-\varepsilon}ds$$

$$+S_1,$$

where

$$|S_1|\leqq|c|\varepsilon_1\Big[\sum_{|a_n|>s_0}\int_{|a_n|}^\infty t^{-1-l(t)-\varepsilon}dt+\sum_{|b_n|>s_0}\int_{|b_n|}^\infty t^{-1-l(t)-\varepsilon}dt\Big]$$

$$\leqq|c|\varepsilon_1[(\rho+\varepsilon+2\varepsilon_2)]\int_{s_0}^\infty\frac{N(t, 0, F)+N(t, \infty, F)}{t^{1+l(t)+\varepsilon}}dt.$$

Let us put

$$S(t, E, F)=\frac{1}{2\pi}\int_E\log|F(te^{i\theta})|d\theta+N(t, \infty, F).$$

Hence

$$\int_0^\infty S(t, E, F)t^{-1-l(t)-\varepsilon}dt$$

$$\leqq\frac{1}{(\rho+\varepsilon)L(\rho+\varepsilon)}\Big\{\sum_{|a_n|>s_0}\int_{|a_n|}^\infty s^{-1-l(s)-\varepsilon}ds+\sum_{|b_n|>s_0}\int_{|b_n|}^\infty s^{-1-l(s)-\varepsilon}ds\Big\}$$

$$+S_2,$$

where

$$S_2=S_1+\frac{\varepsilon}{\rho(\rho+\varepsilon)}\sum_{|b_n|>s_0}\int_{|b_n|}^\infty s^{-1-l(s)-\varepsilon}ds.$$

Thus

$$\int_0^\infty S(t, E, F)t^{-1-l(s)-\varepsilon}dt\leqq L(\rho+\varepsilon)^{-1}\frac{\rho}{\rho+\varepsilon}\int_{s_0}^\infty\frac{N(t, 0, F)+N(t, \infty, F)}{t^{1+l(t)+\varepsilon}}dt+S_3,$$

where S_3 can be estimated as

$$|S_3|\leqq H(\rho,\ \varepsilon,\ \varepsilon_1,\ \varepsilon_2)\int_{s_0}^{\infty}\frac{T(t,\ F)}{t^{1+l(t)+\varepsilon}}dt\ .$$

Here $H(\rho,\ \varepsilon,\ \varepsilon_1,\ \varepsilon_2)$ is a constant satisfying

$$\lim_{\varepsilon,\varepsilon_1,\varepsilon_2\to 0} H(\rho,\ \varepsilon,\ \varepsilon_1,\ \varepsilon_2)=0\ .$$

Since s_0 is sufficiently large, $N(t,\ 0,\ F)+N(t,\ \infty,\ F)\leqq(K(F)+\varepsilon_3)T(t,\ F)$ for any $t\geqq s_0$. Hence

$$\int_0^{\infty}S(t,\ E,\ F)t^{-1-l(t)-\varepsilon}dt\leqq\frac{K(F)+\varepsilon_3}{L(\rho+\varepsilon)}\cdot\frac{\rho}{\rho+\varepsilon}\int_{s_0}^{\infty}\frac{T(t,\ F)}{t^{-1-l(t)-\varepsilon}}dt(1+H(\rho,\ \varepsilon,\ \varepsilon_1,\ \varepsilon_2))\ .$$

Since $T(t,\ F)\geqq T(t,\ f)-ct^q$ for any sufficiently large t,

$$\int_{s_0}^{\infty}T(t,\ F)t^{-1-l(t)-\varepsilon}dt\to\infty$$

as $\varepsilon\to 0$. Therefore

$$L(\rho)\liminf_{t\to\infty} S(t,\ E,\ F)/T(t,\ F)\leqq K(F)+\varepsilon_3+H(\rho,\ 0,\ \varepsilon_1,\ \varepsilon_2)L(\rho)\ .$$

Here ε_3, ε_1, ε_2 are arbitrary. Hence we have the desired result for F. It is easy to prove the following relations:

$$|T(t,\ f)-T(t,\ F)|\leqq At^q,$$
$$0\leqq N(t,\ 0,\ f)-N(t,\ 0,\ F)\leqq A\log t\ ,$$
$$0\leqq N(t,\ \infty,\ f)-N(t,\ \infty,\ F)\leqq A\log t\ ,$$
$$|S(t,\ E,\ f)-S(t,\ E,\ F)|\leqq At^q$$

for any sufficiently large t. Therefore $K(f)=K(F)$ and

$$L(\rho)\liminf_{t\to\infty} S(t,\ E,\ f)/T(t,\ f)\leqq K(f)\ ,$$

which is just our desired result.

§3. **Proof of Theorem 2.** Let ρ_0 be any positive number satisfying $\rho_0\leqq\min(|a_1|,\ |b_1|)$. Let us consider

$$\int_{\rho_0}^{R}\log|f(te^{i\theta})|t^{-1-\beta}dt=\sum_{|a_\mu|\leqq 2R}\log\left|E\left(-\frac{t}{|a_\mu|}e^{i(\theta-\varphi_\mu)},\ q\right)\right|\frac{dt}{t^{1+\beta}}$$
$$-\sum_{|b_\nu|\leqq 2R}\log\left|E\left(-\frac{t}{|b_\nu|}e^{i(\theta-\psi_\nu)},\ q\right)\right|\frac{dt}{t^{1+\beta}}$$
$$+\frac{14}{q+1-\beta}\cdot\frac{T(4R)}{2^{q+1}R^{q+1}}(R^{q+1-\beta}-\rho_0^{q+1-\beta})\ ,$$

where β is a constant satisfying $q<\beta<q+1$. We shall compute

$$I=\int_{\rho_0}^{R}\log\left|E\left(-\frac{t}{|a|}e^{i(\theta-\varphi)},\ q\right)\right|t^{-1-\beta}dt$$

for $R<|a|\leqq 2R$ or for $|a|\leqq R$.

As in the proof of Theorem 2 in [3] we have

$$I=O(R^{-\beta})$$

for the first case and hence

$$\sum_{R<|a|\leqq 2R} I=n(2R)O(R^{-\beta}).$$

For the second case we need the method of contour integration and have

$$I=(-1)^q\frac{\pi\cos\beta(\theta-\varphi)}{\beta\sin\pi(\beta-q)}\cdot\frac{1}{|a|^{\beta}}+L_1+L_2+L_3,$$

where

$$L_3=O(\rho_0^{q-\beta+1}|a|^{-q-1}),$$

$$L_2=O(R^{-\beta})$$

and

$$L_1=A_0R^{q-\beta}(|a|^{-q}-R^{-q})+A_1R^{q-\beta-1}(|a|^{1-q}-R^{1-q})+\cdots$$
$$+A_{q-1}R^{1-\beta}(|a|^{-1}-R^{-1})+A_qR^{-\beta}\log(R/|a|)+A_{q+1}R^{-\beta}$$

with positive constants $A_0, \cdots, A_{q+1}$. Hence

$$\sum_{|a|\leqq R}|L_3|=O(\rho_0^{q+1-\beta})\int_{\rho_0}^{R}t^{-q-1}dn(t,\ 0),$$

$$\sum_{|a|\leqq R}|L_2|=n(R,\ 0)O(R^{-\beta})$$

and

$$\sum_{|a|\leqq R}|L_1|=O(n(R,\ 0)R^{-\beta})+O(N(R,\ 0)R^{-\beta})$$
$$+\sum_{j=1}^{q}O\left(R^{j-\beta}\int_{\rho_0}^{R}N(t,\ 0)t^{-j-1}dt\right).$$

Similar results hold for poles. Therefore we have

$$\int_{\rho_0}^{R}\log|f(te^{i\theta})|\frac{dt}{t^{1+\beta}}=\frac{\pi}{\beta\sin\pi\beta}\sum_{|a_\mu|\leqq R}\frac{\cos\beta(\theta-\varphi_\mu)}{|a_\mu|^{\beta}}$$
$$-\frac{\pi}{\beta\sin\pi\beta}\sum_{|b_\nu|\leqq R}\frac{\cos\beta(\theta-\psi_\nu)}{|b_\nu|^{\beta}}+S^*,$$

where

$$S^*=O(T(4R)/R^{\beta})+\sum_{j=1}^{q}O\left(R^{j-\beta}\int_{\rho_0}^{R}T(t)t^{-j-1}dt\right)+S_1(\rho_0).$$

Here $S_1(\rho_0)\to 0$ as $\rho_0\to 0$. Let E be a measurable subset of $[-\pi,\ \pi]$. Then as in our earlier paper

$$\int_{\rho_0}^{R} S(t,\ E)\frac{dt}{t^{1+\beta}}=\frac{1}{\beta^2\sin\pi\beta}\sum_{|a_\mu|\leq R}\frac{\beta}{2}\int_E\cos\beta(\theta-\varphi_\mu)d\theta\,|a_\mu|^{-\beta}$$

$$+\frac{1}{\beta^2\sin\pi\beta}\sum_{|b_\nu|\leq R}\left\{\sin\pi\beta-\frac{\beta}{2}\int_E\cos\beta(\theta-\psi_\nu)d\theta\right\}|b_\nu|^{-\beta}+S_1^*,$$

where S_1^* behaves like S^*. Then we can prove that

$$\int_{\rho_0}^{R} S(t,\ E)t^{-1-\beta}dt\leqq L(\beta)^{-1}\int_{\rho_0}^{R}(N(t,\ 0)+N(t,\ \infty))t^{-1-\beta}dt+S_2^*,$$

where S_2^* behaves like S_1^*. The right hand side term can be estimated by

$$L(\beta)^{-1}(K(f)+\varepsilon_1)\int_{\rho_0}^{R} T(t)t^{-1-\beta}dt+S_2^*+O(1)\int_{\rho_0}^{t_0}T(t)t^{-1-\beta}dt\,.$$

Firstly we put $\rho_0\to 0$. Let $\{r_n\}$ be a sequence of Pólya peaks of the first kind and of order β_1 for $T(t)$. Here we take $\max(\mu,\ q)<\beta_1<\beta<\lambda<q+1$. Further we put $4R=2A_n r_n$ with $A_n\to\infty$ as $n\to\infty$. Existence of such a sequence $\{r_n\}$ is well-known. Now suppose

$$L(\beta)\liminf_{t\to\infty}\frac{S(t,\ E)}{T(t)}\leqq K(f)+\varepsilon_1$$

is false. Then for $t\geqq t_1\geqq t_0$

$$S(t,\ E)>CT(t)\,,$$

$$C>L(\beta)^{-1}(K(f)+\varepsilon_1)\,.$$

Then

$$\{C-L(\beta)^{-1}(K(f)+\varepsilon_1)\}\int_0^R T(t)t^{-1-\beta}dt\leqq S_2^*+O(1)\,.$$

Evidently with $n(t)=n(t,\ 0)+n(t,\ \infty)$ and $N(t)=N(t,\ 0)+N(t,\ \infty)$

$$\int_0^R T(t)t^{-1-\beta}dt\geqq\frac{1}{2}\int_0^R N(t)t^{-1-\beta}dt$$

$$=\sum_{|a_n|\leq R}|a_n|^{-\beta}/2\beta^2+\sum_{|b_n|\leq R}|b_n|^{-\beta}/2\beta^2$$

$$-O(n(R)/R^\beta)-O(N(R)/R^\beta)\,.$$

Further

$$\int_0^R T(t)t^{-1-\beta}dt\geqq\int_{r_n}^{2r_n}T(t)t^{-1-\beta}dt$$

$$\geqq T(r_n)\int_{r_n}^{2r_n}t^{-1-\beta}dt=\frac{2^\beta-1}{\beta 2^\beta}\cdot T(r_n)/r_n^\beta\,.$$

If $T(r_n)/r_n^\beta\to\infty$ as $n\to\infty$, then

$$\int_0^R T(t)t^{-1-\beta}dt\to\infty$$

for $R\to\infty$. If $T(r_n)/r_n^\beta$ is bounded for $n\to\infty$, then by

$$\frac{T(2R)}{(2R)^\beta}\leqq\frac{T(r_n)}{r_n^{\beta_1}}\cdot\frac{1}{(2R)^{\beta-\beta_1}}=\frac{1}{A_n^{\beta-\beta_1}}\cdot\frac{T(r_n)}{r_n^\beta}=o\left(\frac{T(r_n)}{r_n^\beta}\right)$$

and by

$$\sum_{|a_n|\leq R}|a_n|^{-\beta}+\sum_{|b_n|\leq R}|b_n|^{-\beta}\to\infty \qquad (R\to\infty)$$

for $\beta<\lambda$ we have again

$$\int_0^R T(t)t^{-1-\beta}dt\to\infty \qquad (R\to\infty).$$

We now consider the residual terms. For example

$$R^{j-\beta}\int_0^R T(t)t^{-j-1}dt\leqq R^{j-\beta}T(r_n)r_n^{-\beta_1}\int_0^R t^{\beta_1-j-1}dt$$

$$=\frac{1}{(\beta_1-j)A_n^{\beta-\beta_1}}T(r_n)r_n^{-\beta}=o(T(r_n)/r_n^\beta).$$

Hence

$$\{C-L(\beta)^{-1}(K(f)+\varepsilon_1)\}\int_0^R T(t)t^{-1-\beta}dt=o(1)\int_0^R T(t)t^{-1-\beta}dt.$$

This is clearly a contradiction. Therefore we have the desired result.

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