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A METHOD TO A PROBLEM OF R. NEVANLINNA

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§1. **Introduction.** This paper is concerned with a problem posed by R. Nevanlinna in his monumental paper [6] and successively in his treatise on meromorphic functions [7]. He proved the following theorem.

THEOREM A. *Let $f(z)$ be a meromorphic function in $|z|<\infty$ and let*

$$K(f)=\limsup_{t\to\infty}\frac{N(t,0)+N(t,\infty)}{T(t,f)}.$$

Then there is a constant $C(\rho)$ such that for a non-integral order ρ of f

$$K(f)\geqq C(\rho)>0.$$

Simultaneously he made the following conjecture:

$$\kappa(\rho)\equiv\inf K(f)=\begin{cases}\dfrac{|\sin\pi\rho|}{q+|\sin\pi\rho|} & (q\leqq\rho<q+1/2),\\[2mm] \dfrac{|\sin\pi\rho|}{q+1} & (q+1/2\leqq\rho\leqq q+1),\end{cases}$$

where inf is taken over all meromorphic functions f of order ρ.

Edrei and Fuchs [2] proved

$$\kappa(\rho)=\begin{cases}1 & (0\leqq\rho<1/2),\\ \sin\pi\rho & (1/2\leqq\rho\leqq1).\end{cases}$$

Goldberg's lemma played the decisive role in their paper. Hellerstein and Williamson [5] proved that the conjecture is true for entire functions of order ρ with only negative zeros. They made use of Shea's representation and of a very precise analysis of the given function.

Through this paper we shall restrict to the following meromorphic function $f(z)$ defined by a quotient of two canonical products

$$f(z)=f_1(z)/f_2(z),$$

$$f_1(z)=\prod E(z/a_n, q),\qquad f_2(z)=\prod E(z/b_n, q).$$

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Here $E(x, q)$ means the Weierstrass primary factor of genus q. Further $f(z)$ is of order ρ $(q<\rho<q+1)$.

We now list up two conditions (A) and (B).

(A) $\int_a^\infty T(t)t^{-1-\alpha}dt\to\infty$ as $\alpha\to\rho$ decreasingly. This is equivalent to

$$\int_a^\infty T(t)t^{-1-\rho}dt=\infty.$$

(B) For any positive ε there is a sequence $\{r_n(\varepsilon)\}$ such that for any t in $[r_n(\varepsilon), R_n(\varepsilon)]$ with $R_n(\varepsilon)=r_n(\varepsilon)\log 1/\varepsilon$

$$T(t)t^{-\rho}\leqq k\ T(r_n(\varepsilon))r_n(\varepsilon)^{-\rho} \qquad (k: \text{bounded}),$$

$$T(r_n(\varepsilon))r_n(\varepsilon)^{-\rho+\varepsilon}\leqq T(t)t^{-\rho+\varepsilon}$$

and $r_n(\varepsilon)\to\infty$ as $n\to\infty$.

For simplicity's sake we abbreviate

$$\frac{1}{2\pi}\int_E \log|f(te^{i\theta})|\,d\theta+N(t, \infty)$$

as $S(t, E)$, where E is a measurable subset of $[-\pi, \pi]$. Our results are the following theorems. Let $L(\rho)$ be the constant defined by

$$L(\rho)=\begin{cases}\dfrac{|\sin\pi\rho|}{q+|\sin\pi\rho|} & (q<\rho<q+1/2)\,,\\[2ex] \dfrac{|\sin\pi\rho|}{q+1} & (q+1/2\leqq\rho<q+1)\,.\end{cases}$$

This does not mean $\inf K(f)$.

THEOREM 1. *Under the condition* (A)

$$L(\rho)\liminf_{t\to\infty} S(t, E)/T(t, f)\leqq K(f)$$

for any measurable subset E *of* $[-\pi, \pi]$.

THEOREM 2. *Under the condition* (B)

$$L(\rho)\liminf_{t\to\infty} S(t, E)/T(t, f)\leqq K(f)$$

for any measurable subset E *of* $[-\pi, \pi]$.

There were several papers in which the problem was attacked in the most general setting. However all of them did not succeed to gain the precise constant $L(\rho)$ in any form. In our method the constant $L(\rho)$ appears quite easily and naturally. So there may be a hope of giving a new light to the problem in our method, although our present results are not decisive.

Several extremal problems in value-distribution theory are formulated and solved by making use of the concept of the lower order. We can obtain the following result.

THEOREM 3. *Suppose the lower order* μ *and the order* ρ *satisfy* $q\leqq\mu<\rho<q+1$. *Let* λ *be a number in* (μ, ρ). *Then*

$$\sup_{\mu<\lambda<\rho} L(\lambda) \liminf_{t\to\infty} S(t, E)/T(t, f)\leqq K(f)$$

for any measurable subset E *in* $[-\pi, \pi]$.

We shall not discuss this theorem 3 in this paper.

§ 2. Lemmas.

We need several lemmas. The first one was stated in Edrei and Fuchs [1].

LEMMA 1. *For* $t\in(2r, R/2)$

$$\log|f(te^{i\theta})|=\sum_{r<|a_n|\leq R}\log|E(te^{i\theta}/a_n, q)|-\sum_{r<|b_n|\leq R}\log|E(te^{i\theta}/b_n, q)|+S,$$

$$|S|\leqq(t/r)^q 20T(\alpha r, f)+(t/R)^{q+1}12T(\alpha R, f) \qquad (q\geqq1),$$

where

$$\alpha=\exp(1/(q+1)).$$

LEMMA 2. *Let* $h_1(x)$ *and* $h_2(x)$ *be real functions defined on* $[0, \infty)$ *such that* $h_2(x)\geqq0$,

$$\int^\infty |h_1(x)|x^{-1-\alpha}dx<\infty, \quad \int^\infty h_2(x)x^{-1-\alpha}dx<\infty$$

for any $\alpha>\rho$ *and*

$$\int_{x*}^\infty h_2(x)x^{-1-\alpha}dx\to\infty$$

for any fixed $x*$ *as* α *tends to* ρ *decreasingly. Assume that*

$$\int_0^\infty h_1(x)x^{-1-\alpha}dx\leqq C(\alpha)\int_0^\infty h_2(x)x^{-1-\alpha}dx$$

for $\alpha>\rho$, *where* $C(\alpha)$ *is a positive constant depending on* α *continuously around* ρ. *Then*

$$\liminf_{x\to\infty} h_1(x)/h_2(x)\leqq C(\rho).$$

Proof. Suppose this is false. Then there are a constant C and x_0 such that

$$h_1(x)/h_2(x)\geqq C>C(\rho)$$

for $x \geqq x_0$. Taking α sufficiently near ρ but $\alpha > \rho$, we have

$$(C-C(\alpha))\int_{x_0}^{\infty} h_2(x)x^{-1-\alpha}dx \leqq C(\alpha)\int_0^{x_0} h_2(x)x^{-1-\alpha}dx - \int_0^{x_0} h_1(x)x^{-1-\alpha}dx .$$

Now $C-C(\alpha) \to C-C(\rho) > 0$ for $\alpha \to \rho$. Then

$$\int_{x_0}^{\infty} h_2(x)x^{-1-\alpha}dx \to \infty$$

as $\alpha \to \rho$ decreasingly implies clearly a contradiction.

The following was stated in [4].

LEMMA 3. *Let q be an arbitrary integer >0, and let α be a constant in $(q, q+1)$. Then*

$$\int_0^{\infty} \log|E(-te^{i\theta}, q)| \frac{dt}{t^{1+\alpha}} = \frac{\pi \cos \theta\alpha}{\alpha \sin \pi\alpha}$$

is valid for all $\theta \in [-\pi, \pi]$. Further, this integral is absolutely convergent for each value of θ.

§3. Proof of Theorem 1. By Lemma 3 we have

$$\int_0^{\infty} \log|f(te^{i\theta})| \frac{dt}{t^{1+\alpha}} = \frac{\pi}{\alpha \sin \pi\alpha}\left\{\sum_1^{\infty} \frac{\Phi_\alpha(\theta-\varphi_n)}{|a_n|^\alpha} - \sum_1^{\infty} \frac{\Phi_\alpha(\theta-\psi_n)}{|b_n|^\alpha}\right\}$$

for $\alpha > \rho$, where φ_n, $\psi_n \in [-\pi, \pi]$ are the arguments of $-a_n$, $-b_n$, respectively and

$$\Phi_\alpha(\theta) = \begin{cases} \cos \theta\alpha & -\pi \leqq \theta \leqq \pi , \\ \cos(2\pi-\theta)\alpha & \pi < \theta \leqq 2\pi , \\ \cos(-2\pi-\theta)\alpha & -2\pi \leqq \theta < -\pi . \end{cases}$$

Let E be an arbitrary measurable subset of $[-\pi, \pi]$. Then

$$\int_0^{\infty} \frac{1}{2\pi}\int_E \log|f(te^{i\theta})| d\theta \frac{dt}{t^{1+\alpha}}$$

$$= \frac{1}{\alpha^2 \sin \pi\alpha}\left\{\sum_1^{\infty} \frac{\alpha\int_E \Phi_\alpha(\theta-\varphi_n)d\theta}{2|a_n|^\alpha} - \sum_1^{\infty} \frac{\alpha\int_E \Phi_\alpha(\theta-\psi_n)d\theta}{2|b_n|^\alpha}\right\} .$$

By the identity

$$\frac{1}{\alpha^2}\sum_1^{\infty} \frac{1}{|b_n|^\alpha} = \int_0^{\infty} \frac{N(t, \infty)}{t^{1+\alpha}} dt ,$$

we have

$$\int_0^{\infty} \frac{S(t, E)}{t^{1+\alpha}} dt = \frac{1}{\alpha^2 \sin \pi\alpha}\left\{\sum_1^{\infty} \frac{U_n}{|a_n|^\alpha} + \sum_1^{\infty} \frac{V_n}{|b_n|^\alpha}\right\}$$

with

$$U_n=\frac{\alpha}{2}\int_E \Phi_\alpha(\theta-\varphi_n)d\theta ,$$

$$V_n=\sin \pi\alpha-\frac{\alpha}{2}\int_E \Phi_\alpha(\theta-\psi_n)d\theta .$$

If $\sin \pi\alpha>0$, then for any n, $U_n\leqq A(\alpha)$, $V_n\leqq A(\alpha)$, where

$$A(\alpha)=\begin{cases} q+|\sin \pi\alpha| & (q<\alpha<q+1/2) , \\ q+1 & (q+1/2\leqq\alpha<q+1) . \end{cases}$$

If $\sin \pi\alpha<0$, then for any n, $U_n\geqq -A(\alpha)$, $V_n\geqq -A(\alpha)$. Hence

$$\int_0^\infty \frac{S(t, E)}{t^{1+\alpha}}dt\leqq \frac{A(\alpha)}{|\sin \pi\alpha|}\left\{\sum_1^\infty \frac{1}{\alpha^2|a_n|^\alpha}+\sum_1^\infty \frac{1}{\alpha^2|b_n|^\alpha}\right\}$$

$$=C(\alpha)\int_0^\infty \frac{N(t, 0)+N(t, \infty)}{t^{1+\alpha}}dt .$$

By the definition of $K(f)$ there exists a t_0 such that

$$N(t, 0)+N(t, \infty)\leqq(K(f)+\varepsilon)T(t, f)$$

for $t\geqq t_0$. Thus

$$\int_0^\infty \frac{S(t, E)}{t^{1+\alpha}}dt\leqq C(\alpha)(K(f)+\varepsilon)\int_{t_0}^\infty \frac{T(t, f)}{t^{1+\alpha}}dt+2C(\alpha)\int_0^{t_0}\frac{T(t, f)}{t^{1+\alpha}}dt .$$

Since

$$\int_{t_0}^\infty \frac{T(t, f)}{t^{1+\alpha}}dt\rightarrow\infty$$

for $\alpha\rightarrow\rho$ decreasingly,

$$\int_0^\infty \frac{S(t, E)}{t^{1+\alpha}}dt\leqq C(\alpha)(K(f)+\varepsilon+o(1))\int_0^\infty \frac{T(t, f)}{t^{1+\alpha}}dt$$

if α is sufficiently near ρ. By Lemma 2 we have

$$\liminf_{t\rightarrow\infty} S(t, E)/T(t, f)\leqq K(f)L(\rho)^{-1},$$

which is the desired result.

§4. **Proof of Theorem 2.** In the first place we should remark that it is enough to consider the case

$$\int^\infty \frac{T(t)}{t^{1+\rho}}dt<\infty$$

by Theorem 1. In this case $T(t)/t^\rho\rightarrow 0$ for $t\rightarrow\infty$. Let us compute

$$I=\int_{2n}^{R/2}\log E\Big(-\frac{t}{|a|}e^{i(\theta-\varphi)},\ q\Big)\frac{dt}{t^{1+\rho}}.$$

Case 1). $R/2<|a|\leqq R$. In this case

$$I=(-1)^{q}e^{iq(\theta-\varphi)}\int_{|a|}^{\infty}\frac{dx}{x^{q+1}}\int_{2r}^{R/2}\frac{t^{q-\rho}}{t+xe^{-i(\theta-\varphi)}}dt$$

$$=(-1)^{q}e^{i(q+1)(\theta-\varphi)}\sum_{s=0}^{\infty}(-1)^{s}e^{si(\theta-\varphi)}\int_{|a|}^{\infty}\frac{dx}{x^{s+q+2}}\int_{2r}^{R/2}t^{s+q-\rho}dt$$

$$=(-1)^{q}e^{i(q+1)(\theta-\varphi)}\sum_{s=0}^{\infty}(-1)^{s}e^{si(\theta-\varphi)}\frac{1}{(q-\rho+s+1)(q+s+1)}\cdot\frac{1}{|a|^{q+s+1}}\Big\{\Big(\frac{R}{2}\Big)^{q-\rho+s+1}-(2r)^{q-\rho+s+1}\Big\}.$$

Therefore

$$|I|\leqq\sum_{s=0}^{\infty}\frac{1}{(q-\rho+s+1)(q+s+1)}\Big(\frac{R}{2}\Big)^{q-\rho+s+1}\Big/|a|^{q+s+1}$$
$$=O((R/2)^{-\rho}).$$

Case 2). $2r<|a|\leqq R/2$. In this case

$$I=(-1)^{q}e^{iq(\theta-\varphi)}\int_{|a|}^{R/2}\frac{dx}{x^{q+1}}\int_{2r}^{R/2}\frac{t^{q-\rho}}{t+xe^{-i(\theta-\varphi)}}dt$$
$$+(-1)^{q}e^{iq(\theta-\varphi)}\int_{R/2}^{\infty}\frac{dx}{x^{q+1}}\int_{2r}^{R/2}\frac{t^{q-\rho}}{t+xe^{-i(\theta-\varphi)}}dt$$
$$\equiv L_2+L_1.$$

As in case 1)

$$|L_1|=O((R/2)^{-\rho}).$$

Let us put D as the domain defined by $\{2r<|z|<R/2\}$-the segment $(2r,\ R/2)$. By the contour integration along ∂D we have

$$\int_{2r}^{R/2}\frac{t^{q-\rho}}{t+xe^{-i(\theta-\varphi)}}dt=\frac{2\pi i(-1)^{q-\rho}x^{q-\rho}}{1-e^{2\pi i(q-\rho)}}e^{-i(q-\rho)(\theta-\varphi)}$$
$$-\frac{i(R/2)^{q-\rho+1}}{1-e^{2\pi i(q-\rho)}}\int_{0}^{2\pi}\frac{e^{i(q-\rho)\phi}}{(R/2)e^{i\phi}+xe^{-i(\theta-\varphi)}}e^{i\phi}d\phi$$
$$+\frac{i(2r)^{q-\rho+1}}{1-e^{2\pi i(q-\rho)}}\int_{0}^{2\pi}\frac{e^{i(q-\rho)\phi}}{2r\,e^{i\phi}+xe^{-i(\theta-\varphi)}}e^{i\phi}d\phi,$$

if x satisfies $2r<x<R/2$. Three terms in the right hand side are denoted by U_1, U_2 and U_3. Then $L_2=V_1+V_2+V_3$, where

$$V_j=(-1)^{q}e^{iq(\theta-\varphi)}\int_{|a|}^{R/2}U_jx^{-q-1}dx\qquad(j=1,\ 2,\ 3).$$

Now we have

$$\mathfrak{R}V_1=\frac{\pi\cos\rho(\theta-\varphi)}{\rho\sin\pi\rho}\{|a|^{-\rho}-(R/2)^{-\rho}\}.$$

By the power series expansion we can prove that

$$|V_2|=O((R/2)^{-\rho})+O((\log R/2-\log|a|)/(R/2)^{\rho})$$
$$+\sum_{j=0}^{q-1}O((|a|^{-q+j}-(R/2)^{-q+j})/(R/2)^{\rho-q+j})$$

and

$$|V_3|=O((2r)^{q+1-\rho}/|a|^{q+1}).$$

Case 3). $r<|a|<2r$. In this case

$$I=(-1)^q e^{iq(\theta-\varphi)}\left[\int_{|a|}^{2r}+\int_{2r}^{R/2}+\int_{R/2}^{\infty}\right]\frac{dx}{x^{q+1}}\int_{2r}^{R/2}\frac{t^{q-\rho}}{t+xe^{-i(\theta-\varphi)}}dt$$
$$\equiv Y_1+Y_2+Y_3.$$

It is easy to prove $|Y_3|=O((R/2)^{-\rho})$. For Y_1 we have by the power series expansion

$$|Y_1|=O((2r)^{q-\rho}|a|^{-q})+O((2r)^{-\rho}\log(2r/|a|))+O((2r)^{-\rho}).$$

In order to estimate Y_2 we need the contour integration as in case 2) and have

$$|Y_2|=O((2r)^{-\rho})+\sum_{j=0}^{q-1}O((R/2)^{q-\rho-j}/(2r)^{q-j})$$
$$+O((R/2)^{-\rho}(\log R/2-\log 2r))+O((R/2)^{-\rho}).$$

What we really need in the sequel is $\mathfrak{R}I$ for various a_ν and b_ν. Now we make use of Lemma 1. Then

$$\int_{2r}^{R/2}\log|f(te^{i\theta})|\frac{dt}{t^{1+\rho}}=\sum_{r<|a_\nu|\le R}\mathfrak{R}I(a_\nu)-\sum_{r<|b_\nu|\le R}\mathfrak{R}I(b_\nu)+\int_{2r}^{R/2}S\,t^{-1-\rho}dt,$$

$$I(x)=\int_{2r}^{R/2}\log E\left(-\frac{t}{|x|}e^{i(\theta-\beta)},\,q\right)t^{-1-\rho}dt,$$

where β is the argument of $-x$. Let us denote $n(t)=n(t,0)+n(t,\infty)$ and $N(t)=N(t,0)+N(t,\infty)$. Then

$$\int_{2r}^{R/2}\log|f(te^{i\theta})|\frac{dt}{t^{1+\rho}}$$
$$=\sum_{2r<|a_\nu|\le R/2}\frac{\pi\cos\rho(\theta-\varphi_\nu)}{\rho\sin\pi\rho}\cdot\frac{1}{|a_\nu|^{\rho}}-\sum_{2r<|b_\nu|\le R/2}\frac{\pi\cos\rho(\theta-\phi_\nu)}{\rho\sin\pi\rho}\cdot\frac{1}{|b_\nu|^{\rho}}+S_1,$$

where

$$S_1=O(T(\alpha r,\ f)/(2r)^\rho)+O(T(\alpha R,\ f)/R^\rho)+O(n(2r)/(2r)^\rho)$$
$$+O(N(2r)/(2r)^\rho)+O\Big(\frac{n(2r)}{(R/2)^\rho}\log\frac{R/2}{2r}\Big)+O(n(R/2)/(R/2)^\rho)$$
$$+O(N(R/2)/(R/2)^\rho)+O((R/2)^{q-\rho}\int_{2r}^{R/2}N(t)t^{-q-1}dt)$$
$$+O(n(R/2)(2r)^{q+1-\rho}/(R/2)^{q+1})+O(N(R/2)(2r)^{q+1-\rho}/(R/2)^{q+1})$$
$$+O\Big((2r)^{q+1-\rho}\int_{2r}^{R/2}N(t)t^{-q-2}dt\Big)+O(n(R)/(R/2)^\rho)\,.$$

Let E be a measurable subset of $[-\pi,\ \pi]$. Then as in Theorem 1

$$\int_{2r}^{R/2}S(t,\ E)\frac{dt}{t^{1+\rho}}\leqq L(\rho)^{-1}\int_{2r}^{R/2}\frac{N(t,\ 0)+N(t,\ \infty)}{t^{1+\rho}}dt+S_2\,,$$

where

$$S_2=S_1+O(T(R,\ f)/R^\rho)\,.$$

By our assumption (B) we can choose $\{2r_n(\varepsilon)\}$, $\{2R_n(\varepsilon)\}$ for any positive ε such that for any $t\in[2r_n(\varepsilon),\ 2R_n(\varepsilon)]$

$$T(t)/t^\rho\leqq kT(2r_n(\varepsilon))/(2r_n(\varepsilon))^\rho,$$
$$T(2r_n(\varepsilon))/(2r_n(\varepsilon))^{\rho-\varepsilon}\leqq T(t)/t^{\rho-\varepsilon}$$

and $R_n(\varepsilon)=r_n(\varepsilon)\log\varepsilon^{-1}$. We simply write $2r$, $2R$ and $T(r)$ instead of $2r_n(\varepsilon)$, $2R_n(\varepsilon)$ and $T(r,\ f)$.

Next we shall estimate the residual term S_2 in comparison with

$$\int_{2r}^{R/2}T(t)t^{-1-\rho}dt\,.$$

This integral is not less than

$$\frac{T(2r)}{(2r)^{\rho-\varepsilon}}\int_{2r}^{R/2}t^{-1-\varepsilon}dt=\frac{T(2r)}{(2r)^\rho}\varepsilon^{-1}\Big(1-\Big(\frac{1}{4}\log\varepsilon^{-1}\Big)^{-\varepsilon}\Big)\,.$$

Hence we have

$$\frac{T(2r)}{(2r)^\rho}=o\Big(\int_{2r}^{R/2}T(t)t^{-1-\rho}dt\Big)\,.$$

Further

$$T(2R)/(2R)^\rho\leqq kT(2r)/(2r)^\rho,$$
$$\Big(\frac{R}{2}\Big)^{q-\rho}\int_{2r}^{R/2}T(t)t^{-q-1}dt\leqq\frac{k}{\rho-q}\cdot\frac{T(2r)}{(2r)^\rho}\Big(1-\Big(\frac{4r}{R}\Big)^{\rho-q}\Big)=O(T(2r)/(2r)^\rho)\,,$$
$$(2r)^{q+1-\rho}\int_{2r}^{R/2}T(t)t^{-q-2}dt\leqq kT(2r)/(2r)^{q+1-2\rho}\int_{2r}^{R/2}t^{-q-2+\rho}dt=O(T(2r)/(2r)^\rho)\,,$$

$$(2r)^{q+1-\rho}T(R)/R^{q+1}=o(T(2r)/(2r)^{\rho})$$

and

$$\frac{T(3r)}{R^{\rho}}\log\frac{R/2}{2r}\leqq 3^{\rho}k\frac{T(2r)}{(2r)^{\rho}}\cdot\frac{\log(4^{-1}\log\varepsilon^{-1})}{(\log\varepsilon^{-1})^{\rho}}=o(T(2r)/(2r)^{\rho}).$$

Therefore

$$\int_{2r}^{R/2}S(t,\ E)t^{-1-\rho}dt\leqq L(\rho)^{-1}(K(f)+o(1))\int_{2r}^{R/2}T(t)t^{-1-\rho}dt.$$

As in the proof of Lemma 2 we have the desired result.

§5. **An application.** Let $E(t)$ be the set of intervals $I_1(t), \cdots, I_l(t)$ of θ on which $|f(te^{i\theta})|\geqq 1$. We assume that (B) holds. So for any $\varepsilon>0$ there are intervals $X_n(\varepsilon)=[2r_n(\varepsilon), 2R_n(\varepsilon)]$. We now introduce another assumption (C) in the following manner: Let $\alpha_j(t)$, $\beta_j(t)$ be two ends of I_j and let $\alpha_j(t)<\beta_j(t)$. Then for all j there exist

$$\lim_{\substack{t\to\infty\\ t\in X_n(\varepsilon)}}\alpha_j(t)=\alpha_j\,,\quad \lim_{\substack{t\to\infty\\ t\in X_n(\varepsilon)}}\beta_j(t)=\beta_j\,.$$

Let I_j be $[\alpha_j, \beta_j]$ and let E be $I_1\cup\cdots\cup I_l$.

The following lemma was proved by Edrei and Fuchs [3].

LEMMA 4. *Let $g(z)$ be meromorphic. Let $\mu(r)$ be the measure of $I(r)$. Then for* $1<r<R'$

$$\frac{1}{2\pi}\int_{I(r)}\log^+|g(re^{i\theta})|\,d\theta\leqq\frac{11R'}{R'-r}T(R',\ g)\mu(r)\Big[1+\log^+\frac{1}{\mu(r)}\Big].$$

We now consider

$$\left|\frac{1}{2\pi}\int_{E}\log|f(te^{i\theta})|\,d\theta-\frac{1}{2\pi}\int_{E(t)}\log|f(te^{i\theta})|\,d\theta\right|$$

for $t\in X_n(\varepsilon)$. This is equal to

$$\frac{1}{2\pi}\int_{E(t)-E\cap E(t)}\log^+|f(te^{i\theta})|\,d\theta+\frac{1}{2\pi}\int_{E-E\cap E(t)}\log^+\frac{1}{|f(te^{i\theta})|}\,d\theta.$$

By making use of Lemma 4 this is not greater than

$$\frac{22R'}{R'-t}T(R',\ f)\mu(t)\Big[1+\log^+\frac{1}{\mu(t)}\Big],$$

where $\mu(t)$ is the sum of measures of $E(t)-E\cap E(t)$ and $E-E\cap E(t)$. Let μ_n be $\max\mu(t)$ for $t\in X_n(\varepsilon)$. Then μ_n tends to zero as n tends to ∞. Let us put $R'=\gamma t$ and $\gamma=1+\sqrt{\mu_n}$. Then the last expression is not greater than

$$22(1+\sqrt{\mu_n})T((1+\sqrt{\mu_n})t,\ f)\sqrt{\mu_n}[1+\log^+\mu_n^{-1}].$$

Hence

$$S(t, E) \geqq \frac{1}{2\pi}\int_{E(t)} \log^+|f(te^{i\theta})|\,d\theta + N(t, \infty)$$

$$-22(1+\sqrt{\mu_n})T((1+\sqrt{\mu_n})t, f)\sqrt{\mu_n}(1+\log^+ 1/\mu_n)$$

$$= T(t, f) - 22(1+\sqrt{\mu_n})T((1+\sqrt{\mu_n})t, f)\sqrt{\mu_n}(1+\log^+ 1/\mu_n).$$

Hence

$$\int_{2r}^{R/2} T(t)t^{-1-\rho}dt \leqq L(\rho)^{-1}\{K(f)+\varepsilon+o(1)\}\int_{2r}^{R/2} T(t)t^{-1-\rho}dt$$

$$+22(1+\sqrt{\mu_n})\sqrt{\mu_n}\left(1+\log^+\frac{1}{\mu_n}\right)\int_{2r}^{R/2} T((1+\sqrt{\mu_n})t)t^{-1-\rho}dt.$$

Since the last integral is not greater than

$$\leqq (1+\sqrt{\mu_n})^{\rho}\left\{\int_{2r}^{R/2} T(s)s^{-1-\rho}ds + \int_{R/2}^{(1+\sqrt{\mu_n})R/2} T(s)s^{-1-\rho}ds\right\}$$

$$\leqq (1+\sqrt{\mu_n})^{\rho}\left\{\int_{2r}^{R/2} T(s)s^{-1-\rho}ds + k\log(1+\sqrt{\mu_n})T(2r)/(2r)^{\rho}\right\}$$

$$= (1+\sqrt{\mu_n})^{\rho}(1+o(1))\int_{2r}^{R/2} T(t)t^{-1-\rho}dt,$$

we have

$$\int_{2r}^{R/2} T(t)t^{-1-\rho}dt \leqq L(\rho)^{-1}\{K(f)+\varepsilon+o(1)\}\int_{2r}^{R/2} T(t)t^{-1-\rho}dt.$$

Thus we have

$$K(f) \geqq L(\rho).$$

This gives

$$\inf K(f) = L(\rho),$$

that is,

$$L(\rho) = \kappa(\rho).$$

§ 6. **Remarks.** (1). If either $T(t)/t^{\rho} \to 0$ decreasingly or $T(t)/t^{\rho-\varepsilon} \to \infty$ increasingly for any positive ε, then (B) holds. Hence there are lots of such functions.

(2). We do not know when or under what condition on $T(t)$ or so the condition (C) is satisfied. This seems to be a very important problem in future.

(3). Still there is another open problem, for which our method is applicable, that is, the following conjecture

$$\limsup_{t\to\infty} \frac{N(t, 0)}{\log M(t, f)} \geqq \frac{|\sin \pi\rho|}{\pi\rho}$$

for entire functions of order ρ. As in our theorems our final result on this conjecture is not definite either. So we shall not discuss this problem.

REFERENCES

[1] EDREI, A. AND W.H.J. FUCHS, On the growth of meromorphic functions with several deficient values, Trans. Amer. Math. Soc., **93** (1959), 292–328.

[2] EDREI, A. AND W. H. J. FUCHS, The deficiencies of meromorphic functions of order less than one, Duke Math. J. **27** (1960), 233-249.

[3] EDREI, A. AND W. H. J. FUCHS, Bounds for the number of deficient values of certain classes of functions, Proc. London Math. Soc., **12** (1962), 315-344.

[4] HELLERSTEIN, S. AND D. F. SHEA, Bounds for the deficiencies of meromorphic functions of finite order, Proc. Symposia Pure Math., **11**, Entire functions and related parts analysis (1968), 214-239.

[5] HELLERSTEIN, S. AND J. WILLIAMSON, Entire functions with negative zeros and a problem of R. Navanlinna, J. Analyse Math., **22** (1969), 233-267.

[6] NEVANLINNA, R., Zur Theorie der meromorphen Funktionen, Acta Math., **46** (1925), 1-99.

[7] NEVANLINNA, R., Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthier-Villars, Paris, 1929.

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