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KODAI MATH. J.
8 (1985), 1—4

ON THE SUMMABILITY ALMOST EVERYWHERE OF THE MULTIPLE FOURIER SERIES AT THE CRITICAL INDEX

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Recently Stein [5] shows the existence of an H^1 function on the k-torus, whose Fourier series is almost everywhere non-summable with respect to the Bochner-Riesz means of the critical index $(k-1)/2$.

On the background of this example, he says as follows. In the case $k=1$, It is known that there exists an H^1 function whose Fourier series diverges almost everywhere (see [6], [7]). In the other direction the theorem of Carleson-Hunt-Sjölin guarantees the convergence almost everywhere whenever $f\in L\log L(\log\log L)$; see [3].

For the multiple Fourier series, S. Bochner pointed out that summability at the critical index $(k-1)/2$ is the correct analogue of convergence for phenomena near L. In this sense Stein's example is a version in the case of general k. On the other hand, as another version he says that whenever $f\in L(\log L)^2$, the multiple Fourier series of f is summable almost everywhere at the critical index $(k-1)/2$. However this version is slightly different from the one dimensional case. The purpose of this note is to show that we can replace the last condition by $f\in L(\log L)(\log\log L)$.

Let $f(x)=f(x_1, x_2, \cdots, x_k)\in L$ on $Q_k: -\pi<x_i\leqq\pi$ $(i=1, 2, \cdots, k)$ and its Fourier series be

$$f(x)\sim\sum a_n e^{in\cdot x}$$

where

$$a_n=(2\pi)^{-k}\int_{Q_k} f(x)e^{-in\cdot x}dx\,.$$

Then the δ-th Bochner-Riesz means of the series is

$$(S_R^\delta f)(x)=\sum_{|n|<R}(1-|n|^2/R^2)^\delta a_n e^{in\cdot x}.$$

THEOREM. *If* $\int_{Q_k}|f(x)|(\log^+|f(x)|)(\log^+\log^+|f(x)|)dx<\infty$, *then,*

$$\lim_{R\to\infty}(S_R^\alpha f)(x)=f(x) \qquad \text{a.e.},$$

where $\alpha=(k-1)/2$ $(k>1)$.

Received January 30, 1984

For the proof we generalize the index δ to complex number and set $\delta=\sigma+i\tau$. The estimate of Stein [4, p. 128] is, for $\sigma>\alpha$,

$$(S_*^\delta f)(x)\leqq A_\sigma e^{\pi|\tau|}(\sigma-\alpha)^{-1}f^*(x)$$

where

$$(S_*^\delta f)(x)=\sup_{0<R<\infty}|(S_R^\delta f)(x)|$$

and $f^*(x)$ is the spherical maximal function, that is

$$f^*(x)=\sup_{0<N<\infty} kN^{-k}\omega_k^{-1}\int_{|y|<N}|f(x+y)|dy ,$$

$$\omega_k=2(\pi)^{k/2}/\Gamma(k/2) .$$

The constant A_σ remains bounded as $\sigma\to\alpha$. Since $f^*(x)$ is week type (1, 1), that is, μ being the Lebesgue measure,

$$\mu\{x\,|\,f^*(x)>y\}\leqq\frac{A}{y}\|f\|_1 ,$$

where A is a constant which depends on the dimension k only, we have for $\sigma>\alpha$

$$(1)\qquad \mu\{x\,|\,(S_*^\delta f)(x)>y\}\leqq A_\sigma e^{\pi|\tau|}(\sigma-\alpha)^{-1}\frac{\|f\|_1}{y} .$$

On the other hand it is routine for $f\in L^2$ to have

$$\|(S_*^\delta f)(x)\|_2\leqq B_\sigma e^{\pi|\tau|}\|f\|_2 ,$$

for $\sigma>0$ and $B_\sigma=0(\sigma^{-1})$ as $\sigma\to0$. Hence we have naturally for $\sigma>0$

$$(2)\qquad \mu\{x\,|\,(S_*^\delta f)(x)>y\}\leqq\left(B_\sigma e^{\pi|\tau|}\frac{\|f\|_2}{y}\right)^2$$

From (1) and (2) we shall prove the following proposition.

PROPOSITION. *Let $f(x)\in L^p(Q_k)$ and α be the critical index, then*

$$\mu\{x\,|\,(S_*^\alpha f)(x)>y\}\leqq\left(\frac{C}{p-1}\cdot\frac{\|f\|_p}{y}\right)^p$$

for every $1<p<2$.

Proof. We apply to (1) and (2) the interpolation theorem on an analytic family of linear operators on $L(p, q)$ space (see [2]). The $L(p, q)$ space notation of (1) and (2) are

$$(3)\qquad \|(S_*^\delta f)(x)\|_{1,\infty}^*\leqq A_\sigma e^{\pi|\tau|}(\sigma-\alpha)^{-1}\|f\|_{1,1}^*$$

for $\sigma>\alpha$, and

$$(4)\qquad \|(S_*^\delta f)(x)\|_{2,\infty}^*\leqq B_\sigma e^{\pi|\tau|}\|f\|_{2,2}^*$$

for $\sigma>0$.

Let $R(x)$ be a measurable function on Q_k such as $0\leqq R(x)\leqq R_0<\infty$ and $\delta(z)=\varepsilon_0(1-z)+(\alpha+\varepsilon_1)z$, where ε_0, $\varepsilon_1>0$ will be decided soon and we define an analytic family of linear operators

$$(T_z f)(x)=(S_{R(x)}^{\delta(z)} f)(x)$$

where $0\leqq \operatorname{Re} z\leqq 1$.

For a given $p(1<p<2)$, we choose $\varepsilon_0=\alpha-(2-p)/4\geqq 1/2-(2-p)/4=p/4>1/4>0$, and $\varepsilon_1=(p-1)/2>0$. Set $(1-t)/2+t/1=1/p$, that is $1/p=(1+t)/2$ $(0<t<1)$, then $\delta(t)=\varepsilon_0(1-t)+(\alpha+\varepsilon_1)t=\alpha$.

On the line $z=iy$, we have $\delta(iy)=\alpha-(2-p)/4+ipy/4$, that is

$$\operatorname{Re}\delta(iy)=\varepsilon_0=\alpha-(2-p)/4>1/4$$

and

$$|\operatorname{Im}\delta(iy)|=p|y|/4\leqq |y|/2 .$$

From (4) we have

$$(5)\qquad \|T_{iy}f\|_{2,\infty}^{*}\leqq \|(S_{*}^{\delta(iy)}f)(x)\|_{2,\infty}\leqq B_{\varepsilon_0}e^{\pi|y|/2}\|f\|_{2,2}^{*} .$$

Secondly on the line $z=1+iy$, we have $\delta(1+iy)=\alpha+(p-1)/2+ipy/4$, that is,

$$\operatorname{Re}\delta(1+iy)=\alpha+(p-1)/2>\alpha$$

and

$$|\operatorname{Im}\delta(1+iy)|=p|y|/4\leqq |y|/2 .$$

Hence from (3),

$$(6)\qquad \|T_{1+iy}f\|_{1,\infty}^{*}\leqq \|(S_{*}^{\delta(1+iy)}f)(x)\|_{1,\infty}^{*}$$
$$\leqq A_{\alpha+\varepsilon_1}((p-1)/2)^{-1}e^{\pi|y|/2}\|f\|_{1,1}^{*} .$$

Here we can write (5) and (6) to

$$(7)\qquad \|T_{iy}f\|_{2,\infty}^{*}\leqq K_0(y)\|f\|_{2,2}^{*}$$

$$(8)\qquad \|T_{1+iy}f\|_{1,\infty}^{*}\leqq K_1(y)\|f\|_{1,1}^{*}$$

where $K_0(y)\leqq K_0e^{\pi|y|/2}$ and $K_1(y)\leqq K_1(p-1)^{-1}e^{\pi|y|/2}$. Therefore by the interpolation, we get

$$(9)\qquad \|(S_{*}^{\alpha}f)(x)\|_{p,\infty}^{*}\leqq A_t\|f\|_{p,p}^{*}$$

where A_t is given by

$$\log A_t=\int_{-\infty}^{+\infty}\omega(1-t,\ y)\log K_0(y)dy+\int_{-\infty}^{\infty}\omega(t,\ y)\log K_1(y)dy ,$$

and $\omega(t, y)$ is the Poisson kernel for the strip $0\leqq t\leqq 1$, $-\infty<y<\infty$. By the properties of Poisson kernel, we have

$$\omega(t,\ y)\geqq 0\,,\quad \int_{-\infty}^{\infty}\omega(1-t,\ y)dy\leqq 1\,,\quad \int_{-\infty}^{\infty}\omega(t,\ y)\leqq 1\,,$$

$$\int_{-\infty}^{\infty}\omega(1-t,\ y)|y|dy\leqq K \quad\text{and}\quad \int_{-\infty}^{\infty}\omega(t,\ y)|y|dy\leqq K\,.$$

So we get $\log A_t\leqq\log C+\log(p-1)^{-1}$. Hence (9) becomes

$$\|(S_*^{\delta}f)(x)\|_{p,\infty}^*<C(p-1)^{-1}\|f\|_{p,p}^*\,,$$

this means for $1<p<2$

$$\mu\{x\,|\,(S_*^{\alpha}f)(x)>y\}<\left(\frac{C}{p-1}\cdot\frac{\|f\|_p}{y}\right)^p.$$

Thus we get the proposition.

From this proposition, we can derive our theorem by a lemma of Carleson-Sjölin [3]. There is also a different type of proof of the lemma in [1, p. 481].

Remark. We reported this result before (Oct. 29, 1981) at the seminar on real analysis in Kanazawa (see Reports of the seminar on real analysis 1981 in Japanese). But the method of proof was more complicated. The proof of here was suggested by Professor M. Kaneko.

LITERATURE

[1] CALDERON, C. P., Lacunary spherical means, Illinois Journ. Math., **23** (1979), 476-484.
[2] SAGHER, Y., On analytic families of operators, Israel Journ. Math., **7** (1969), 350-356.
[3] SJÖLIN, P., An inequality of Paley and convergence a. e. of Walsh Fourier series, Arkiv för Mat., **7** (1968), 551-570.
[4] STEIN, E. M., Localization and summability of multiple Fourier series, Acta Math., **100** (1958), 93-147.
[5] STEIN, E. M., An H^1 function with non-summable Fourier expansion, Lecture notes in Math., No. **992**, Springer, 1983, Harmonic analysis, Proc. Conf. Cortona, Italy, 193-200.
[6] SUNOUCHI, G., A Fourier series which belongs to the class H diverges almost everywhere, Kodai Math. Seminar Rep., **5** (1953), 27-28.
[7] ZYGMUND, A., Trigonometric series, Cambridge Univ. Press, 1959.

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