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A CERTAIN PROPERTY OF GEODESICS OF THE FAMILY OF RIEMANNIAN MANIFOLDS O_n^2 (VI)

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§0. Introduction.

This is exactly a continuation of Part (V) ([15]) with the same title written by the present author which proved the following conjecture is true for $16 \leqq n \leqq 84$. On the methods used in it, the lower bound 16 of this effective interval is crucial from the argument in it. We shall show that this conjecture is also true for $9.7 \leqq n \leqq 16$ in the present paper by improving them. We shall use the same notation in the previous ones Parts (I)~(V).

The period T of any non-trivial solution $x(t)$ of the non-linear differential equation of order 2:

$$\text{(E)} \qquad nx(1-x^2)\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+(1-x^2)(nx^2-1)=0$$

with a constant $n>1$ such that $x^2+x'^2<1$ is given by the integral:

$$\text{(0.1)} \qquad T=\sqrt{nc}\int_{x_0}^{x_1}\frac{dx}{x\sqrt{(n-x)\{x(n-x)^{n-1}-c\}}},$$

where $x_0=n\{\min x(t)\}^2$, $x_1=n\{\max x(t)\}^2$, $0<x_0<1<x_1<n$ and $c=x_0(n-x_0)^{n-1}=x_1(n-x_1)^{n-1}$.

CONJECTURE C. The period T as a function of $\tau=(x_1-1)/(n-1)$ and n is monotone decreasing with respect to $n(>2)$ for any fixed $\tau(0<\tau<1)$.

Here the author thanks heartily Professor Naoto Abe for his cooperation in the numerical computations by computors.

§1. The fundamental principle to attain the purpose.

Setting $T=\Omega(\tau, n)$, we have the formulas

$$\text{(1.1)} \qquad \frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{\sqrt{c}}{2b^2\sqrt{n}}\int_{x_0}^{1}\frac{(1-x)W(x, x_1)dx}{x(n-x)\sqrt{x(n-x)^{n-1}-c}}$$

(Lemma 3.1 in (III)) and

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$$\frac{\partial \Omega(\tau, n)}{\partial n} = -\frac{\sqrt{c}}{2b^2 n\sqrt{n}} \int_{x_0}^{1} \frac{(1-x)\sqrt{x(n-x)^{n-1}-c}\,V(x, x_1)dx}{x^2(n-x)^n} \tag{1.2}$$

((7.4) and Proposition 3 in (III)), where $b=\sqrt{B-c}$, $B=(n-1)^{n-1}$ and $W(x, x_1)$ and $V(x, x_1)$ are defined as follow:

$$\begin{aligned} W(x, x_1) :=& \frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\{\lambda(X)-\lambda(x)\} \\ &+\left[\frac{X\sqrt{n-X}\,f_0(X)}{(X-1)^3} - \frac{x\sqrt{n-x}\,f_0(x)}{(x-1)^3}\right]\left\{\frac{n}{n-1}\frac{(x_1-1)^2}{x_1(n-x_1)} - \lambda(x_1)+\lambda(X)\right\} \\ &-\left[\frac{X^2 f_1(X)}{(1-X)^3\sqrt{n-X}} - \frac{x^2 f_1(x)}{(1-x)^3\sqrt{n-x}}\right]\frac{\phi(x)-\phi(x_1)}{n\phi(x)}, \end{aligned} \tag{1.3}$$

and

$$V(x, x_1) := \frac{x^2 N(x, x_1)}{(1-x)^5\sqrt{n-x}} + \frac{X^2 N(X, x_1)}{(X-1)^5\sqrt{n-X}}, \tag{1.4}$$

where

$$f_0(z) := (2n-1-z)B-(n-z)^{n-1}\{n-z+(n-1)z^2\}, \tag{1.5}$$

$$f_1(z) := \{4n-1-(2n+1)z\}B-(n-z)^{n-1}\{n+(2n-1)z-(n+1)z^2\}, \tag{1.6}$$

$$\begin{aligned} F_2(z) :=& -\{(2n+1)z^2-2(2n^2+5n-4)z+16n^2-16n+3)\}B \\ &+(n-z)^{n-1}\{-(n-1)z^3+(2n^2-7n+8)z^2+(n-3)(4n-1)z+3n(2n-1)\}. \end{aligned} \tag{1.7}$$

$$\lambda(z) := \log(n-z)+\frac{n-1}{n-z}, \qquad \phi(z) := z(n-z)^{n-1}, \tag{1.8}$$

$$\tilde{\lambda}(z) := \lambda(z) - \frac{n}{n-1}\frac{(z-1)^2}{z(n-z)} = \log(n-z)+\frac{nz-1}{(n-1)z}, \tag{1.9}$$

$$\begin{aligned} N(z, x_1) :=& (n-z)F_2(z)_3\{\lambda(z)-\tilde{\lambda}(x_1)\}+3(z-1)^2 f_0(z) \\ &-2n(z-1)^3\{B-z(n-z)^{n-1}\}, \end{aligned} \tag{1.10}$$

and $X=X_n(x)$, $0<x<1<X<n$, defined by $\phi(x)=\phi(X)$.

Now taking a constant b $(1<b\leqq 2)$, if we can prove that

$$V(x, X(x))\geqq 0 \quad \text{for} \quad 0<x\leqq X^{-1}(b),$$

and

$$V(x, b)>0 \quad \text{for} \quad X^{-1}(b)\leqq x<1,$$

then we have

$$V(x, x_1)>0 \quad \text{for} \quad b\leqq x_1<n,$$

since $V(x, x_1)$ is increasing with respect to x_1 for $0<x<1$ by Lemma 8.1 in (III), and hence we obtain

$$\frac{\partial \Omega(\tau,\ n)}{\partial n}<0 \qquad \text{for} \quad b\leqq x_1<n$$

by means of (1.2). Furthermore, if we can prove that

$$W(x,\ x_1)>0 \qquad \text{for} \quad X^{-1}(b)\leqq x<1\,, \quad X(x)<x_1\leqq b\,,$$

then we obtain

$$\frac{\partial \Omega(\tau,\ n)}{\partial n}<0 \qquad \text{for} \quad 1<x_1\leqq b$$

by means of (1.1). We shall take a suitable value b for each $n(9.7\leqq n\leqq 16)$ and show that this principle is appropriate for our purpose.

§ 2. Introduction of an auxiliary function $H(x)$.

First of all we notice that when $n\geqq 6$ we have

(2.1) $$V(x,\ x_1)>0 \qquad \text{for} \quad 0<x\leqq\alpha_0(n)=X^{-1}(2)\,, \quad X(x)\leqq x_1<n$$

(Lemma 2.5 and Lemma 3.1 in (IV) or Lemma 11.1 in (V)). By (8.1), (8.2) and (8.3) in (III) $V(x,\ X(x))$ can be represented as

$$V(x,\ X(x))=U_0(x)-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x)\,,$$

where

$$U_0(x):=\frac{x^2\sqrt{n-x}}{(1-x)^5}F_2(x)\{\lambda(x)-\tilde{\lambda}(X(x))\}\,,$$

$$U_1(x):=\frac{3x^2 f_0(x)}{(x-1)^3\sqrt{n-x}}\,, \qquad U_2(x):=\frac{2nx^2\{B-\psi(x)\}}{(1-x)^2\sqrt{n-x}}\,,$$

$$U_3(x):=\frac{nxF_2(x)}{(n-1)(x-1)^3\sqrt{n-x}}$$

and

$$U_4(x):=U_3(X(x))\,, \quad U_5(x):=U_1(X(x))\,, \quad U_6(x):=U_2(X(x))$$

and $U_i(x)$, $i=0, 1, 2, 4, 5, 6$, are all positive by Lemma 2.1 and Proposition 1 in (II) and Lemma 7.1 in (III). Furthermore, when $n>2$ we have

(2.2) $$-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x)$$

$$>\frac{nX}{(n-1)(X-1)^3\sqrt{n-X}}\{G_2(X)+3(X-1)f_0(X)\} \qquad \text{for} \quad 0<x<1$$

by Lemma 2.5 in (IV), where $X=X(x)$ and

$$G_2(x):=F_2(x)-2(n-1)x(x-1)\{B-\psi(x)\}\{1-\nu(x)\}\,,$$

$$\nu(x) := \frac{\sqrt{B}\,(x-1)^2(n-x)^{2n-3/2}}{x^2\{B-(n-x)^{n-1}\}^2\sqrt{nB-(n-x)^{n-1}}}\,.$$

By Lemma 3.1 in (IV), $G_2(x)+3(x-1)f_0(x)>0$ for $2\leqq x<n$, when $n\geqq 6$, which implies (2.1). We have

$$\begin{aligned}&G_2(x)+3(x-1)f_0(x)\\&=2(n-1)x(x-1)\{B-\phi(x)\}\,\nu(x)\\&\quad-2(n-1)x(x-1)\{B-\phi(x)\}+F_2(x)+3(x-1)f_0(x)\\&=\frac{2(n-1)(x-1)^3(n-x)^{2n-3/2}\{B-x(n-x)^{n-1}\}\sqrt{B}}{x\{B-(n-x)^{n-1}\}^2\sqrt{nB-(n-x)^{n-1}}}\\&\quad+\{F_2(x)+3(x-1)f_0(x)-2(n-1)x(x-1)B+2(n-1)x^2(x-1)(n-x)^{n-1}\}\end{aligned}$$

and by (1.5) and (1.7)

$$\begin{aligned}&F_2(x)+3(x-1)f_0(x)-2(n-1)x(x-1)B+2(n-1)x^2(x-1)(n-x)^{n-1}\\&=-\{(2n+1)x^2-2(2n^2+5n-4)x+16n^2-16n+3\}\,B\\&\quad+(n-x)^{n-1}\{-(n-1)x^3+(2n^2-7n+8)x^2+(n-3)(4n-1)x+3n(2n-1)\}\\&\quad+3(x-1)(2n-1-x)B-3(n-x)^{n-1}(x-1)\{n-x+(n-1)x^2\}\\&\quad-2(n-1)x(x-1)B+2(n-1)x^2(x-1)(n-x)^{n-1}\\&=2(n-x)\{(2n+1)x-(8n-5)\}\,B+2(n-x)^n\{(n-1)x^2+(2n-5)x+3n\}\,,\end{aligned}$$

hence

$$\begin{aligned}(2.3)\qquad G_2(x)+3(x-1)f_0(x)=&\frac{2(n-1)(x-1)^3(n-x)^{2n-3/2}\{B-x(n-x)^{n-1}\}\sqrt{B}}{x\{B-(n-x)^{n-1}\}^2\sqrt{nB-(n-x)^{n-1}}}\\&+2(n-x)\{(2n+1)x-(8n-5)\}\,B+2(n-x)^n\{(n-1)x^2+(2n-5)x+3n\}\,.\end{aligned}$$

Thus, we obtain the following inequality: when $n>2$

$$\begin{aligned}(2.4)\qquad &-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x)\\&>\frac{2n(n-X)^{2(n-1)}\{B-X(n-X)^{n-1}\}\sqrt{B}}{\{B-(n-X)^{n-1}\}^2\sqrt{nB-(n-X)^{n-1}}}+\frac{2nX\sqrt{n-X}\,B}{(n-1)(X-1)^3}H(X)\end{aligned}$$

for $0<x<1$, where $X=X(x)$ and

$$(2.5)\qquad H(x):=\left(\frac{n-x}{n-1}\right)^{n-1}\{(n-1)x^2+(2n-5)x+3n\}-\{8n-5-(2n+1)x\}\,.$$

§ 3. Certain properties of $H(x)$.

LEMMA 3.1. *When* $n\geqq 4$, $H(x)$ *is increasing for* $\frac{2n+10}{2n+1}\leqq x<n$.

Proof. From (2.5) we obtain

$$H'(x)=-\left(\frac{n-x}{n-1}\right)^{n-2}\{(n-1)x^2+(2n-5)x+3n\}$$

$$+\left(\frac{n-x}{n-1}\right)^{n-1}\{2(n-1)x+2n-5\}+2n+1$$

$$=-\left(\frac{n-x}{n-1}\right)^{n-2}\left\{(n+1)x^2-\frac{3n}{n-1}x+\frac{n(n+2)}{n-1}\right\}+2n+1,$$

i. e.

$$H'(x)=-\frac{1}{n-1}\left(\frac{n-x}{n-1}\right)^{n-2}\{(n^2-1)x^2-3nx+n(n+2)\}+2n+1, \tag{3.1}$$

from which we obtain

$$(n-1)H''(x)=\frac{n-2}{n-1}\left(\frac{n-x}{n-1}\right)^{n-3}\{(n^2-1)x^2-3nx+n(n+2)\}$$

$$-\left(\frac{n-x}{n-1}\right)^{n-2}\{2(n^2-1)x-3n\}$$

$$=\left(\frac{n-x}{n-1}\right)^{n-3}\left\{(n-2)(n+1)x^2-\frac{3n(n-2)x}{n-1}+\frac{n(n^2-4)}{n-1}\right.$$

$$\left.-2(n+1)x(n-x)+\frac{3n(n-x)}{n-1}\right\},$$

i. e.

$$H''(x)=\frac{n}{n-1}\left(\frac{n-x}{n-1}\right)^{n-3}(x-1)\{(n+1)x-(n+4)\}. \tag{3.2}$$

From (2.5), (3.1) and (3.2) we have easily

$$H(1)=H'(1)=H''(1)=0$$

and

$$\frac{2n+10}{2n+1}-\frac{n+4}{n+1}=\frac{3(n+2)}{(n+1)(2n+1)}>0.$$

From (3.2) we see easily that $H'(x)$ is increasing for $\frac{n+4}{n+1}\leqq x<n$. Hence we have

$$H'(x)>H'\left(\frac{2n+10}{2n+1}\right) \quad \text{for} \quad x>\frac{2n+10}{2n+1}. \tag{3.3}$$

From (3.1) we have

$$H'\Big(\frac{2n+10}{2n+1}\Big)=-\frac{1}{n-1}\Big(\frac{2n^2-n-10}{2n^2-n-1}\Big)^{n-2}\Big\{\frac{4(n^2-1)(n+5)^2}{(2n+1)^2}$$

$$-\frac{6n(n+5)}{2n+1}+n(n+2)\Big\}+2n+1$$

$$=-\Big(\frac{2n^2-n-10}{2n^2-n-1}\Big)^{n-2}\cdot\frac{8n^4+40n^3+39n^2-68n-100}{(n-1)(2n+1)^2}+2n+1\,.$$

hence the condition:

$$H'\Big(\frac{2n+10}{2n+1}\Big)>0$$

is equivalent to

$$\frac{8n^4+40n^3+39n^2-68n-100}{(n-1)(2n+1)^3}<\Big(\frac{2n^2-n-1}{2n^2-n-10}\Big)^{n-2}. \tag{3.4}$$

This inequality will be guaranteed by the next lemma. Assuming (3.4), we obtain $H'(x)>0$ for $x>\dfrac{2n+10}{2n+1}$ and hence this lemma.

LEMMA 3.2. *The inequality* (3.4) *holds for* $n\geqq 4$.

Proof. Since we have for $n=4$

$$\text{the left hand side of (3.4)}=\frac{4860}{3\cdot 9^3}=2.222\cdots,$$

$$\text{the right hand side of (3.4)}=\Big(\frac{27}{18}\Big)^2=\frac{9}{4}=2.25\,,$$

the inequality (3.4) holds at $n=4$. Then, we shall show that it is also true for sufficiently large n. In fact, setting $n=1/t$, we have

$$\frac{8n^4+40n^3+39n^2-68n-100}{(n-1)(2n+1)^3}=\frac{8+40t+39t^2-68t^3-100t^4}{(1-t)(2+t)^3}$$

$$=\Big(1+5t+\frac{39}{8}t^2-\frac{17}{2}t^3-\frac{25}{2}t^4\Big)(1+t+t^2+t^3+\cdots)$$

$$\times\Big(1-\frac{3}{2}t+\frac{3}{2}t^2-\frac{5}{4}t^3+\frac{15}{16}t^4-\cdots\Big)$$

$$=\Big(1+6t+\frac{87}{8}t^2+\frac{19}{8}t^3-\frac{81}{8}t^4-\cdots\Big)$$

$$\times\Big(1-\frac{3}{2}t+\frac{3}{2}t^2-\frac{5}{4}t^3+\frac{15}{16}t^4-\cdots\Big)$$

$$=1+\frac{9}{2}t+\frac{27}{8}t^2-\frac{99}{16}t^3-\frac{63}{16}t^4+\cdots$$

and

$$\left(\frac{2n^2-n-1}{2n^2-n-10}\right)^{n-2}=\exp\left[\frac{1-2t}{t}\log\frac{(1-t)(2+t)}{(1+2t)(2-5t)}\right]$$

$$=\exp\left[\frac{1-2t}{t}\left\{\log(1-t)+\log\left(1+\frac{t}{2}\right)-\log(1+2t)-\log\left(1-\frac{5}{2}t\right)\right\}\right]$$

$$=\exp\left[\frac{1-2t}{t}\left\{-t-\frac{1}{2}t^2-\frac{1}{3}t^3-\frac{1}{4}t^4-\frac{1}{5}t^5-\cdots\right.\right.$$

$$+\frac{1}{2}t-\frac{1}{8}t^2+\frac{1}{24}t^3-\frac{1}{64}t^4+\frac{1}{160}t^5-\cdots$$

$$-2t+2t^2-\frac{8}{3}t^3+4t^4-\frac{32}{5}t^5+\cdots$$

$$\left.\left.+\frac{5}{2}t+\frac{25}{8}t^2+\frac{125}{24}t^3+\frac{625}{64}t^4+\frac{3125}{160}t^5+\cdots\right\}\right]$$

$$=\exp\left[(1-2t)\left(\frac{9}{2}t+\frac{9}{4}t^2+\frac{27}{2}t^3+\frac{207}{16}t^4+\cdots\right)\right]$$

$$=\exp\left[\frac{9}{2}t-\frac{27}{4}t^2+9t^3-\frac{225}{16}t^4+\cdots\right]$$

$$=1+\frac{9}{2}t-\frac{27}{4}t^2+9t^3-\frac{225}{16}t^4+\cdots$$

$$+\frac{1}{2}t^2\left(\frac{81}{4}-\frac{243}{4}t+\frac{2025}{16}t^2+\cdots\right)$$

$$+\frac{1}{6}t^3\left(\frac{729}{8}-\frac{6561}{16}t+\cdots\right)+\frac{1}{24}t^4\left(\frac{6561}{16}+\cdots\right)$$

$$=1+\frac{9}{2}t+\frac{27}{8}t^2-\frac{99}{16}t^3-\frac{261}{128}t^4+\cdots,$$

which imply the above fact.

Now, we suppose that (3.4) does not hold for $n>4$. Then there exists a value ξ (>4) such that

$$\frac{8n^4+40n^3+39n^2-68n-100}{(n-1)(2n+1)^3}=\left(\frac{2n^2-n-1}{2n^2-n-10}\right)^{n-2} \tag{3.5}$$

and

$$\frac{d}{dn}\left(\frac{8n^4+40n^3+39n^2-68n-100}{(n-1)(2n+1)^3}\right)\leqq\frac{d}{dn}\left(\frac{2n^2-n-1}{2n^2-n-10}\right)^{n-2} \tag{3.6}$$

at $n=\xi$. By (3.5), (3.6) can be replaced by

$$\frac{d}{dn}\log\left(\frac{8n^4+40n^3+39n^2-68n-100}{(n-1)(2n+1)^3}\right)\leqq\frac{d}{dn}\log\left(\frac{2n^2-n-1}{2n^2-n-10}\right)^{n-2}. \tag{3.6'}$$

We have

$$\begin{aligned}&\frac{d}{dn}\log\left(\frac{8n^4+40n^3+39n^2-68n-100}{(n-1)(2n+1)^3}\right)\\ &=\frac{32n^3+120n^2+78n-68}{8n^4+40n^3+39n^2-68n-100}-\frac{1}{n-1}-\frac{6}{2n+1}\\ &=\frac{9(8n^4+12n^3-45n^2-50n+48)}{(n-1)(2n+1)(8n^4+40n^3+39n^2-68n-100)}\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dn}\log\left(\frac{2n^2-n-1}{2n^2-n-10}\right)^{n-2}=\log\frac{2n^2-n-1}{2n^2-n-10}\\ &\qquad+(n-2)(4n-1)\left(\frac{1}{2n^2-n-1}-\frac{1}{2n^2-n-10}\right)\\ &=\log\frac{2n^2-n-1}{2n^2-n-10}-\frac{9(n-2)(4n-1)}{(2n^2-n-1)(2n^2-n-10)},\end{aligned}$$

Therefore (3.6′) becomes

$$\begin{aligned}\frac{1}{9}\log\frac{2n^2-n-1}{2n^2-n-10}\geqq&\frac{(n-2)(4n-1)}{(2n^2-n-1)(2n^2-n-10)}\\ &-\frac{8n^4+12n^3-45n^2-50n+48}{(n-1)(2n+1)(8n^4+40n^3+39n^2-68n-100)}.\end{aligned}$$

Since we have

$$2n^2-n-1=(n-1)(2n+1),\qquad 2n^2-n-10=(n+2)(2n-5),$$
$$8n^4+40n^3+39n^2-68n-100=(n+2)^2(8n^2+8n-25)$$

and

$$\begin{aligned}&(n-2)(4n-1)(n+2)(8n^2+8n-25)-(2n-5)(8n^4+12n^3-45n^2-50n+48)\\ &=16n^5+40n^4-86n^3-196n^2+86n+140\\ &=2(n-1)(n+2)(8n^3+12n^2-39n-35),\end{aligned}$$

the above inequality can be written as

$$\frac{2n^2-n-10}{18}\log\frac{2n^2-n-1}{2n^2-n-10}\geqq\frac{8n^3+12n^2-39n-35}{(2n+1)(8n^2+8n-25)}. \tag{3.7}$$

Now, we have

$$\frac{2n^2-n-1}{2n^2-n-10}=1+\frac{9}{2n^2-n-10}$$

and

$$\frac{9}{2n^2-n-10} \leqq \frac{9}{2\cdot 4^2-4-10} = \frac{1}{2} \qquad \text{for} \quad n \geqq 4 .$$

Setting

$$\frac{9}{2n^2-n-10} = u \qquad (\text{for } n \geqq 4)$$

we have

$$\frac{2n^2-n-10}{18} \log \frac{2n^2-n-1}{2n^2-n-10} = \frac{1}{2u} \log(1+u)$$

$$= \frac{1}{2u}\left(u-\frac{1}{2}u^2+\frac{1}{3}u^3-\frac{1}{4}u^4+\cdots\right) = \frac{1}{2}-\frac{1}{4}u+\frac{1}{6}u^2-\frac{1}{8}u^3+\cdots .$$

Next, we have for the right hand side of (3.7)

$$\frac{8n^3+12n^2-39n-35}{(2n+1)(8n^2+8n-25)} = \frac{8n^3+12n^2-39n-35}{16n^3+24n^2-42n-25}$$

$$= \frac{1}{2} - \frac{9(4n+5)}{2(2n+1)(8n^2+8n-25)} .$$

From these expressions, we see that (3.7) can be replaced by

$$\frac{1}{4}u-\frac{1}{6}u^2+\frac{1}{8}u^3-\cdots \leqq \frac{9(4n+5)}{2(2n+1)(8n^2+8n-25)} ,$$

and divided the both sides by u

$$\frac{1}{4}-\frac{1}{6}u+\frac{1}{8}u^2-\cdots \leqq \frac{(n+2)(2n-5)(4n+5)}{2(2n+1)(8n^2+8n-25)} ,$$

i.e.

$$\frac{1}{6}u-\frac{1}{8}u^2+\frac{1}{10}u^3-\cdots \geqq \frac{1}{4} - \frac{(n+2)(2n-5)(4n+5)}{2(2n+1)(8n^2+8n-25)}$$

$$= \frac{3(4n^2+16n+25)}{4(2n+1)(8n^2+8n-25)} :$$

In conclusion, (3.7) can be replaced

$$\text{(3.8)} \qquad \frac{1}{6}-\frac{1}{8}u+\frac{1}{10}u^2-\frac{1}{12}u^3+\cdots \geqq \frac{(n+2)(2n-5)(4n^2+16n+25)}{12(2n+1)(8n^2+8n-25)} ,$$

where $u=\dfrac{9}{2n^2-n-10}$. (3.8) must hold at $n=\xi$.

We show in the following that the riget hand side of (3.8) is greater than 1/6 when $n \geqq 4$. In fact, we have

$$\frac{(n+2)(2n-5)(4n^2+16n+25)}{12(2n+1)(8n^2+8n-25)} - \frac{1}{6} = \frac{8n^4-4n^3-54n^2-101n-200}{12(2n+1)(8n^2+8n-25)} > 0 ,$$

because

$$8n^4-4n^3-54n^2-101n-200$$

$$\geqq(8\cdot4^2-4\cdot4-54)n^2-101n-200=58n^2-101n-200>0 \qquad \text{for} \quad n\geqq4,$$

On the other hand it is clear that the left hand side of (3.8) is less than 1/6. Thus, we obtain a contradiction. We conclude the proof of this lemma.

Q.E.D.

§4. Certain properties of constants b_n .

From the argument of the proof of Lemma 3.1 and

$$H(n)=(n-1)(2n-5), \qquad H'(n)=2n+1,$$

there exists a uniquely determined constant $b=b_n$ for each $n>5/2$ by the condition:

$$H(b)=0 \quad \text{and} \quad 1<b<n \tag{4.1}$$

and we have the following

LEMMA 4.1. $b_n>\dfrac{2n+10}{2n+1}$ *for* $n\geqq4$.

Proof. By means of Lemma 3.1 it is sufficient to prove the inequality

$$H\left(\frac{2n+10}{2n+1}\right)<0. \tag{4.2}$$

Since we have from (2.5)

$$H\left(\frac{2n+10}{2n+1}\right)=3\left\{\left(\frac{2n^2-n-10}{2n^2-n-1}\right)^{n-1}\cdot\frac{(n+2)(8n^2+8n-25)}{(2n+1)^2}-2n+5\right\},$$

the above condition is equivalent to

$$\frac{(n+2)(8n^2+8n-25)}{(2n+1)^2(2n-5)}<\left(\frac{2n^2-n-1}{2n^2-n-10}\right)^{n-1}$$

or

$$\frac{(n+2)^2(8n^2+8n-25)}{(n-1)(2n+1)^3}<\left(\frac{2n^2-n-1}{2n^2-n-10}\right)^{n-2}.$$

Since $(n+2)^2(8n^2+8n-25)=8n^4+40n^3+39n^2-68n-100$, the above inequality coincides with (3.4). Therefore (4.2) holds for $n\geqq4$. Q.E.D.

Here, we show the approximate values of b_n for the values of n as shown in Table 1 below such that

$$b'_n<b_n<b''_n=b'_n+10^{-4}$$

which are computed by (4.1) by means of a micro computor.

Table 1

n	b'_n	$2n+10/2n+1$	n	b'_n	$2n+10/2n+1$
4	2.2679	2.0000	9.5	1.5998	1.4500
5	2.0651	1.8182	10	1.5716	1.4286
5.3	2.0144	1.7759	11	1.5223	1.3913
5.4	1.9985	1.7627	12	1.4808	1.3600
6	1.9117	1.6923	13	1.4453	1.3333
7	1.7949	1.6000	14	1.4147	1.3103
8	1.7038	1.5294	15	1.3880	1.2903
9	1.6310	1.4737	16	1.3645	1.2727

LEMMA 4.2. $b_n<2.3=\frac{23}{10}$ *for* $n\geqq 4$.

Proof. By Lemma 3.1 the condition $b_n<\frac{23}{10}$ is equivalent to

$$\left(\frac{10n-23}{10n-10}\right)^{n-1}\cdot\frac{1289n-1679}{34n-73}>10. \tag{4.3}$$

We show that the left hand side of (4.3) is increasing for $n\geqq 4$. In fact we have

$$\frac{d}{dn}\log\left\{\left(\frac{10n-23}{10n-10}\right)^{n-1}\cdot\frac{1289n-1679}{34n-73}\right\}$$
$$=\log\frac{10n-23}{10(n-1)}+\frac{13}{10n-23}-\frac{37011}{(1289n-1679)(34n-73)},$$

which tends to 0 as $n\to\infty$ and

$$\frac{d^2}{dn^2}\log\left\{\left(\frac{10n-23}{10n-10}\right)^{n-1}\cdot\frac{1289n-1679}{34n-73}\right\}$$
$$=\frac{13}{(n-1)(10n-23)}-\frac{130}{(10n-23)^2}+\frac{37011(87652n-151183)}{(1289n-1679)^2(34n-73)^2}$$
$$=-169\left\{\frac{1}{(n-1)(10n-23)^2}-\frac{219(87652n-151183)}{(1289n-1679)^2(34n-73)^2}\right\}$$
$$=-\frac{169\Psi(n)}{(n-1)(10n-23)^2(1289n-1679)^2(34n-73)^2},$$

where

$$\Psi(n):=(1289n-1679)^2(34n-73)^2-219(n-1)(10n-23)^2(89652n-151183).$$

Since we have

$$(1289n-1679)^2(34n-73)^2=(43826n^2-151183n+122567)^2$$

$$=1920718276n^4-13251492316n^3+33599542173n^2$$

$$-37060093522n+15022669489$$

and

$$(n-1)(10n-23)^2(89652n-151183)$$

$$=8765200n^4-64203420n^3+171350308n^2-195887895n+79975807\,,$$

we can write $\Psi(n)$ as follows

$$\Psi(n)=1139476n^4+809056664n^3-3926175279n^2+5839355483n-2492032244\,.$$

Furthermore since we have

$$\frac{d}{dn}(1139476n^3+809056664n^2-3926175279n)$$

$$=3418428n^2+1618113328n-3926175279$$

$$\geqq 10^6(3n^2+1600n-4000)\geqq 10^6(3\cdot 4^2+1600\cdot 4-4000)$$

$$=10^6\cdot 2448>0 \qquad \text{for} \quad n\geqq 4$$

and hence

$$1139476n^3+809056664n^2-3926175279n+5839355483$$

$$\geqq 1139476\cdot 4^3+809056664\cdot 4^2-3926175279\cdot 4+5839355483$$

$$=3152487455 \qquad \text{for} \quad n\geqq 4\,,$$

we obtain

$$\Psi(n)\geqq 3152487455n-2492032244>0 \qquad \text{for} \quad n\geqq 4\,.$$

Thus we obtain the inequality

$$\frac{d}{dn}\log\left\{\left(\frac{10n-23}{10n-10}\right)^{n-1}\cdot\frac{1289n-1679}{34n-73}\right\}>0 \qquad \text{for} \quad n\geqq 4\,.$$

On the other hand we have

$$\left(\frac{10n-23}{10n-10}\right)^{n-1}\cdot\frac{1289n-1679}{34n-73}\bigg|_{n=4}=\left(\frac{17}{30}\right)^3\cdot\frac{1159}{21}=\frac{5694167}{567000}>10\,,$$

hence we obtain finally (4.3)

$$\left(\frac{10n-23}{10n-10}\right)^{n-1}\cdot\frac{1289n-1679}{34n-73}>10 \qquad \text{for} \quad n\geqq 4\,. \qquad \text{Q.E.D.}$$

LEMMA 4.3. *b_n is monotone decreasing with respect to $n(\geqq 4)$.*

Proof. Suppose $n\geqq 4$. For $1<b<n$ the condition $b_n<b$ is equivalent to

$$\left(\frac{n-b}{n-1}\right)^{n-1}\cdot\frac{(b^2+2b+3)n-b(b+5)}{2(4-b)n-(b+5)}>1\,. \tag{4.4}$$

If the left hand side of (4.4) is monotone increasing with respect to n for $1<b<2.3$, considering the fact

$$\frac{2n+10}{2n+1}<b_n<\frac{23}{10} \qquad \text{for} \quad n\geqq 4\,,$$

then we see that

$$b_{n_0}\leqq b \quad \text{implies} \quad b_n<b \qquad \text{for} \quad n>n_0$$

and hence

$$b_n<b_{n_0}\,.$$

First of all, we have

$$\frac{\partial}{\partial n}\log\left\{\left(\frac{n-b}{n-1}\right)^{n-1}\cdot\frac{(b^2+2b+3)n-b(b+5)}{2(4-b)n-(b+5)}\right\}$$

$$=\log\frac{n-b}{n-1}+\frac{b-1}{n-b}-\frac{3(b-1)^2(b+5)}{\{(b^2+2b+3)n-b(b+5)\}\{2(4-b)n-(b+5)\}}\,,$$

which tends to 0 as $n\to\infty$ and

$$\frac{\partial^2}{\partial n^2}\log\left\{\left(\frac{n-b}{n-1}\right)^{n-1}\cdot\frac{(b^2+2b+3)n-b(b+5)}{2(4-b)n-(b+5)}\right\}$$

$$=-\frac{(b-1)^2}{(n-1)(n-b)^2}$$

$$+3(b-1)^2(b+5)\frac{4(b^2+2b+3)(4-b)n-(b+5)(-b^2+10b+3)}{\{(b^2+2b+3)n-b(b+5)\}^2\{2(4-b)n-(b+5)\}^2}$$

$$=-\frac{(b-1)^2\cdot(☆)}{(n-1)(n-b)^2\{(b^2+2b+3)n-b(b+5)\}^2\{2(4-b)n-(b+5)\}^2}\,,$$

where

$$(☆)=\{(b^2+2b+3)n-b(b+5)\}^2\{2(4-b)n-(b+5)\}^2$$
$$-3(b+5)(n-1)(n-b)^2\{4(b^2+2b+3)(4-b)n-(b+5)(-b^2+10b+3)\}$$
$$=(b-1)\{4(4-b)(b^2+2b+3)(-b^2+b+3)n^4$$
$$+(b+5)(-4b^4+20b^3+5b^2-12b-45)n^3-3b(b+5)(4-b)(3b^2+8b+1)n^2$$
$$+b(b+5)(4-b)(13b^2+32b+15)n-4b^2(b+5)^2(4-b)\}\,.$$

In the following, we shall show the polynomial of $n:\Phi_b(n):=(☆)/(b-1)$ is positive for $n\geqq 4$. Second, noticing the positive root of the equation: $x^2-x-3=0$ is

$$x=\frac{1+\sqrt{13}}{2}=2.3027\cdots \qquad >\frac{23}{10}\,,$$

we show that

$$(4.5)\qquad -\frac{(b+5)(-4b^4+20b^3+5b^2-12b-45)}{8(4-b)(b^2+2b+3)(-b^2+b+3)}<4 \qquad \text{for} \quad 1<b<\frac{23}{10},$$

which is equivalent to

$$32(4-b)(b^2+2b+3)(-b^2+b+3)+(b+5)(-4b^4+20b^3+5b^2-12b-45)$$

$$=28b^5-96b^4-87b^3-19b^2+759b+927>0\,,$$

Since we have

$$\frac{d}{db}(28b^5-96b^4-87b^3-19b^2+759b)$$

$$=140b^4-384b^3-261b^2-38b+759$$

and

$$\frac{d^2}{db^2}(28b^5-96b^4-87b^3-19b^2+759b)$$

$$=560b^3-1152b^2-522b-38\leqq -209.2b-38<0\,,$$

$140b^4-384b^3-261b^2-38b+759$ is decreasing with respect to b in the interval $1\leqq b\leqq 2.3$ and it becomes 216 at $b=1$ and -1463.444 at $b=2.3$. Therefore, $28b^5-96b^4-87b^3-19b^2+759b+927$ is greater than the minimum of its values at $b=1$ and 2.3 in this interval $1<b<2.3$. In fact this polynomial of b becomes 1512 at $b=1$ and 629.36344 at $b=2.3$. Thus we have proved (4.5).

Next, using the fact (4.4) we see that

$$4(4-b)(b^2+2b+3)(-b^2+b+3)n^2+(b+5)(-b^4+20b^3+5b^2-12b-45)n$$

$$-3b(b+5)(4-b)(3b^2+8b+1)\geqq 64(4-b)(b^2+2b+3)(-b^2+b+3)$$

$$+4(b+5)(-4b^4+20b^3+5b^2-12b-45)-3b(b+5)(4-b)(3b^2+8b+1)$$

$$=57b^5-159b^4-117b^3-489b^2+1248b+1404 \qquad \text{for} \quad n\geqq 4$$

and hence

$$(4.6)\qquad \Phi_b(n)\geqq(57b^5-159b^4-117b^3-489b^2+1248b+1404)n^2$$

$$+b(b+5)(4-b)(13b^2+32b+15)n-4b^2(b+5)^2(4-b)$$

$$\text{for} \quad n\geqq 4 \quad \text{and} \quad 1<b<2.3\,.$$

Since we have

$$\frac{d}{db}(57b^5-159b^4-117b^3-489b^2+1248b)$$

$$=285b^4-636b^3-351b^2-978b+1248$$

$$<-306.15b^2-978b+1248<-36.15 \qquad \text{for} \quad 1<b<2.3\,.$$

$57b^5-159b^4-117b^3-489b^2+1248b+1404$ is decreasing in $1<b<2.3$ and hence we obtain

(4.7) $57b^5-159b^4-117b^3-489b^2+1248b+1404 \geqq 20.46267$ for $1<b\leqq 2.1$

and

(4.8) $57b^5-159b^4-117b^3-489b^2+1248b+1404 \leqq -250.07616$ for $2.2\leqq b<2.3$.

In the following, we shall divide our argument into two cases. First, we restrict the interval of b on $1<b\leqq 2.1$. Then, we have from (4.6) and (4.7)

$$\begin{aligned}\Phi_b(n) &> 20n^2+b(b+5)(4-b)(13b^2+32b+15)n-4b^2(b+5)^2(4-b)\\ &=20n^2+b(b+5)(4-b)\{(13b^2+32b+15)n-4b(b+5)\}\\ &\geqq 20n^2+b(b+5)(4-b)\{4(13b^2+32b+15)-4b(b+5)\}\\ &=20n^2+12b(b+5)(4-b)(4b^2+9b+5)>0 \quad \text{for } n\geqq 4 \text{ and } 1<b\leqq 2.1.\end{aligned}$$

Therefore, the left hand side of (4.4) is monotone increasing with respect to $n(\geqq 4)$ for $1<b\leqq 2.1$.

Next, we consider the case $2.1\leqq b<2.3$. Taking the first four terms regarding n of $\Phi_b(n)$, we have

$$\begin{aligned}&\frac{\partial}{\partial n}\{4(4-b)(b^2+2b+3)(-b^2+b+3)n^3+(b+5)(-4b^4+20b^3+5b^2-12b-45)n^2\\ &\quad -3b(b+5)(4-b)(3b^2+8b+1)n\}\\ &=12(4-b)(b^2+2b+3)(-b^2+b+3)n^2+2(b+5)(-4b^4+20b^3+5b^2-12b-45)n\\ &\quad -3b(b+5)(4-b)(3b^2+8b+1)\\ &\geqq 192(4-b)(b^2+2b+3)(-b^2+b+3)+8(b+5)(-4b^4+20b^3+5b^2-12b-45)\\ &\quad -3b(b+5)(4-b)(3b^2+8b+1)\\ &=3(b-4)(67b^4-87b^3-87b^2-571b-576)-8(4b^5-105b^3-13b^2+105b+225)\\ &=169b^5-543b^4-465b^3-565b^2+4284b+5112\\ &\geqq -860.01b^3-565b^2+4284b+5112>-1564.9529b+5112>0\\ &\qquad\qquad\qquad\text{for } n\geqq 4 \text{ and } 2.1\leqq b<2.3,\end{aligned}$$

by (4.5) and $543/(2\times 169)=1.606\cdots<2.1$. Hence, we see that

$$\begin{aligned}&4(4-b)(b^2+2b+3)(-b^2+b+3)n^3+(b+5)(-4b^4+20b^3+5b^2-12b-45)n^2\\ &-3b(b+5)(4-b)(3b^2+8b+1)n\end{aligned}$$

is increasing with respect to n $(\geqq 4)$, when $2.1\leqq b<2.3$. Now, we have

$$4(4-b)(b^2+2b+3)(-b^2+b+3)\cdot 4^3+(b+5)(-4b^4+20b^3+5b^2-12b-45)\cdot 4^2$$
$$-3b(b+5)(4-b)(3b^2+8b+1)\cdot 4+b(b+5)(4-b)(13b^2+32b+15)$$
$$=256(b^5-3b^4-6b^3-b^2+27b+36)-16(4b^5-105b^3-13b^2+105b+225)$$
$$+12(3b^5+11b^4-51b^3-159b^2-20b)-13b^5-45b^4+213b^3+625b^2+300b$$
$$=215b^5-681b^4-255b^3-1331b^2+5292b+5616$$
$$\geqq -736.95b^3-1331b^2+5292b+5616>-1667.7655b+5616>1780.13935$$
$$\text{for} \quad 2.1\leqq b<2.3$$

and hence we obtain

$$\Phi_b(n)\geqq(215b^5-681b^4-255b^3-1331b^2+5292b+5616)n-4b^2(b+5)^2(4-b)$$
$$>1780n-4b^2(b+5)^2(4-b)>4(1780-3^2\cdot 8^2\cdot 2)=1256>0\,.$$

Thus, we have proved that the left hand side of (4.4) is also monotone increasing with respect to $n(\geqq 4)$ for $2.1\leqq b<2.3$. Q.E.D.

Remark. From Lemma 3.1 and the properties of b_n stated in this section we can obtain immediately a more sharp result than Lemma 3.1 in (IV) as follows.

LEMMA 4.3. *When* $n\geqq 4$, $G_2(x)+3(x-1)f_0(x)$ *is positive for* $b_n\leqq x<n$. *Especially when* $n\geqq 5.4$, *it is so for* $2\leqq x<n$.

This lemma implies by means of (2.4) that when $n\geqq 4$ we have

$$V(x,\ X_n(x))>0 \qquad \text{for} \quad 0<x\leqq X_n^{-1}(b_n)\,. \tag{4.9}$$

§ 5. Evaluation of $V(x, b)$.

From (1.4)~(1.10) we obtain

$$V(x,\ x_1)=\frac{x^2\sqrt{n-x}F_2(x)\{\lambda(x)-\tilde{\lambda}(x_1)\}}{(1-x)^5}-\frac{3x^2f_0(x)}{(x-1)^3\sqrt{n-x}}+\frac{2nx^2\{B-\phi(x)\}}{(1-x)^2\sqrt{n-x}}$$
$$+\frac{X^2\sqrt{n-X}F_2(X)\{\lambda(X)-\tilde{\lambda}(x_1)\}}{(X-1)^5}+\frac{3X^2f_0(X)}{(X-1)^3\sqrt{n-X}}-\frac{2nX^2\{B-\phi(X)\}}{(X-1)^2\sqrt{n-X}}\,, \tag{5.1}$$

where $X=X_n(x)$, $0<x<1$ and $X_n(x)\leqq x_1<n$ (§ 11 in (V)). Taking a constant $b(1<b<n)$, we obtain from (2.4) and (5.1)

$$V(x,\ b)=V(x,\ X_n(x))+\frac{x^2\sqrt{n-x}F_2(x)}{(1-x)^5}+\frac{X^2\sqrt{n-X}F_2(X)}{(X-1)^5}\{\tilde{\lambda}(X)-\tilde{\lambda}(b)\} \tag{5.2}$$

and

$$(5.3)\qquad V(x,\ b)>U_0(x)+\frac{2n(n-X)^{2(n-1)}\{B-X(n-X)^{n-1}\}\sqrt{B}}{\{B-(n-X)^{n-1}\}^2\sqrt{nB-(n-X)^{n-1}}}$$

$$+\frac{2nBX\sqrt{n-X}H(X)}{(n-1)(X-1)^3}+\left\{\frac{x^2\sqrt{n-x}F_2(x)}{(1-x)^5}+\frac{X^2\sqrt{n-X}F_2(X)}{(X-1)^5}\right\}\{\tilde{\lambda}(X)-\tilde{\lambda}(b)\}$$

$$\text{for}\quad X_n^{-1}(b)\leqq x<1,$$

whose terms in the right hand side except the third one are all positive. Looking at the right hand side of (5.3), we see that if we can prove the following inequality:

$$(5.4)\qquad \frac{2nBX\sqrt{n-X}H(X)}{(n-1)(X-1)^3}+\frac{X^2\sqrt{n-X}F_2(X)}{(X-1)^5}\{\tilde{\lambda}(X)-\tilde{\lambda}(b)\}\geqq 0$$

$$\text{for}\quad 1<X\leqq b,$$

then we can obtain $V(x,\ b)>0$ for $X_n^{-1}(b)\leqq x<1$.

We proved the following fact in §5 of (IV) (Proposition 3).

LEMMA 5.1. *When $n\geqq 13$, we have*

$$G_2(X)+3(X-1)f_0(X)>0 \qquad \textit{for}\quad \frac{2n+10}{2n+1}\leqq X<n\,.$$

This lemma implies immediately by (2.2) the following

LEMMA 5.2. *When $n\geqq 13$, we have*

$$V(x,\ x_1)>0 \qquad \textit{for}\quad 0<x\leqq\alpha_1(n),\quad X_n(x)\leqq x_1<n\,,$$

where $\alpha_1(n)=X_n^{-1}\left(\frac{2n+10}{2n+1}\right)$.

In another coming paper [16], the present author proved that the fact in Lemma 5.2 is also true when $9.7\leqq n\leqq 13$, which is derived from

PROPOSITION 1. *When $9.7\leqq n\leqq 13$, we have*

$$\frac{2nBH(X)}{n-1}+\frac{XF_2(X)}{(X-1)^2}\{\tilde{\lambda}(X)-\tilde{\lambda}(b_n)\}>0 \qquad \textit{for}\quad \frac{2n+10}{2n+1}\leqq X\leqq b_n\,.$$

In fact, the proof of this proposition is complicated and too long to describe in this paper. So, the present author separated it from this paper, which was given by means of the argument using micro computors.

In the following, we shall evaluate $V(x,\ b)$ for $\alpha_1(n)\leqq x<1$.

LEMMA 5.3. *When $n\geqq 4$ and $1<b<2.3$, we have*

$$\tilde{\lambda}(x)-\tilde{\lambda}(b)>\frac{\{n+(n-1)b\}(b-x)(x+b-2)}{2b^2(n-1)(n-b)} \qquad \textit{for}\quad 1<x<b\,.$$

Proof. Since we get from (1.9)

$$\tilde{\lambda}'(x) = -\frac{(x-1)\{n+(n-1)x\}}{(n-1)x^2(n-x)},$$

we obtain

$$\tilde{\lambda}(x)-\tilde{\lambda}(b) = \frac{1}{n-1}\int_x^b \frac{(t-1)\{n+(n-1)t\}\,dt}{t^2(n-t)}.$$

Since we have

$$\frac{d}{dt}\frac{n+(n-1)t}{t^2(n-t)} = -\frac{1}{t^3(n-t)^2}\cdot\{2n^2+n(n-4)t-2(n-1)t^2\}$$

and

$$2n^2+n(n-4)t-2(n-1)t^2 \geqq 2\{n^2-(n-1)t^2\}.$$

$$\frac{n}{\sqrt{n-1}} \geqq \frac{4}{\sqrt{3}} = 2.3094\cdots > 2.3 \qquad \text{for} \quad n \geqq 4,$$

the function $\dfrac{n+(n-1)t}{t^2(n-t)}$ is decreasing with respect to t for $0<t<2.3$. Therefore we obtain

$$\tilde{\lambda}(x)-\tilde{\lambda}(b) > \frac{n+(n-1)b}{(n-1)b^2(n-b)}\int_x^b (t-1)dt$$

$$= \frac{n+(n-1)b}{2b^2(n-1)(n-b)}\cdot(b-x)(b+x-2) \qquad \text{for} \quad 1<x<b \quad \text{Q.E.D.}$$

LEMMA 5.4. *When* $n \geqq 4$, *we have*

$$H(X) > -\frac{n}{2(n-1)}(X-1)^3 \qquad \textit{for} \quad 1 < X \leqq \frac{2n+10}{2n+1}.$$

Proof. Regarding (3.2), we obtain easily

$$\frac{d}{dx}\left[\left(\frac{n-x}{n-1}\right)^{n-3}\{(n+1)x-(n+4)\}\right] = -\frac{n-2}{n-1}\left(\frac{n-x}{n-1}\right)^{n-4}\{(n+1)x-2(n+3)\}$$

and

$$\frac{2(n+3)}{n+1}-\frac{2n+10}{2n+1} = \frac{2(n^2+n-2)}{(n+1)(2n+1)} > 0.$$

Hence $\left(\dfrac{n-x}{n-1}\right)^{n-3}\{(n+1)x-(n+4)\}$ is increasing for $1<x\leqq\dfrac{2n+10}{2n+1}$ and so we obtain there

$$\left(\frac{n-x}{n-1}\right)^{n-3}\{(n+1)x-(n+4)\} > 3.$$

Thus we obtain

$$H''(X) > -\frac{3n}{n-1}(X-1) \qquad \text{for} \quad 1 < X \leqq \frac{2n+10}{2n+1}.$$

Using $H(1)=H'(1)=0$ and integrating both sides of the above inequality, we obtain this lemma. Q. E. D.

LEMMA 5.5. *When $n \geqq 4$, we have*

$$F_2(X) > \frac{n(8n^3-24n^2-18n+61)(8n^3-4n^2-32n-5)(2n^2-n-10)^{n-2}}{96(n-1)^2(2n+1)^n} \cdot (X-1)^4$$

$$\textit{for} \quad 1 < X \leqq \frac{2n+10}{2n+1}.$$

Proof. This inequality is equivalent to (7.8) in (IV), which is proved under the condition $n \geqq 10$ in Lemma 7.2 but this part is available for $n \geqq 4$. Q. E. D.

LEMMA 5.6. *When $n \geqq 4$, we have*

$$\frac{2nBX\sqrt{n-X}H(X)}{(n-1)(X-1)^3} + \frac{X^2\sqrt{n-X}F_2(X)}{(X-1)^5}\{\tilde{\lambda}(X)-\tilde{\lambda}(b)\}$$

$$> \frac{n^2B}{(n-1)^2} \cdot \frac{\{(n-1)b+n\}\{(2n+1)b-2n-4\}\{(2n+1)b-2n+2\}}{b^2(n-b)}$$

$$\cdot L(n, b) \cdot X\sqrt{n-X} \qquad \textit{for} \quad 1 < X \leqq \frac{2n+4}{2n+1},$$

where

$$L(n, b) := \left(\frac{2n^2-n-10}{2n^2-n-1}\right)^{n-1} \cdot \frac{(8n^3-24n^2-18n+61)(8n^3-4n^2-32n-5)}{288n(n-1)(2n-5)(2n+1)^3} - \frac{b^2(n-b)}{\{(n-1)b+n\}\{(2n+1)b-2n+2\}\{(2n+1)b-2n-4\}} \tag{5.5}$$

and $\dfrac{2n+10}{2n+1} < b < 2.3$.

Proof. From Lemma 5.3~5.5 we obtain

$$\frac{2nBX\sqrt{n-X}H(X)}{(n-1)(X-1)^3} + \frac{X^2\sqrt{n-X}F_2(X)}{(X-1)^5}\{\tilde{\lambda}(X)-\tilde{\lambda}(b)\} \tag{5.6}$$

$$> -\frac{n^2B}{(n-1)^2}X\sqrt{n-X} + \frac{X^2\sqrt{n-X}}{X-1} \frac{\{n+(n-1)b\}(b-X)(X+b-2)}{2b^2(n-1)(n-b)}$$

$$\times \frac{n(8n^3-24n^2-18n+61)(8n^3-4n^2-32n-5)(2n^2-n-10)^{n-2}}{96(n-1)^2(2n+1)^n}$$

$$= \frac{n^2B}{(n-1)^2}X\sqrt{n-X}\left[-1 + \frac{X(b-X)(X+b-2)}{X-1} \cdot \frac{n+(n-1)b}{b^2(n-b)}\right.$$

$$\times\left(\frac{2n^2-n-10}{2n^2-n-1}\right)^{n-2}\cdot\frac{(8n^3-24n^2-18n+61)(8n^3-4n^2-32n-5)}{192n(n-1)^2(2n+1)^2}\Bigg]$$

$$\text{for}\quad 1<X\leqq\frac{2n+10}{2n+1}.$$

Now, the function $\frac{X(b-X)(X+b-2)}{X-1}$ is decreasing with respect to X for $X>1$. In fact, we have

$$\frac{d}{dX}\frac{X(b-X)(X+b-2)}{X-1}=-\frac{2X^3-5X^2+4X+b(b-2)}{(X-1)^2}$$

and

$$2X^3-5X^2+4X+b(b-2)>1+b(b-2)=(b-1)^2>0\qquad\text{for}\quad X>1.$$

Thus, we obtain

$$\frac{X(b-X)(X+b-2)}{X-1}\geqq\frac{2(n+2)\{(2n+1)b-2n-4\}\{(2n+1)b-2n+2\}}{3(2n+1)^2}$$

$$\text{for}\quad 1<X\leqq\frac{2n+4}{2n+1}.$$

Hence we obtain

$$-1+\frac{X(b-X)(X+b-2)}{X-1}\cdot\frac{n+(n-1)b}{b^2(n-b)}\cdot\left(\frac{2n^2-n-10}{2n^2-n-1}\right)^{n-2}$$

$$\times\frac{(8n^3-24n^2-18n+61)(8n^3-4n^2-32n-5)}{192n(n-1)^2(2n+1)^2}$$

$$\geqq-1+\left(\frac{2n^2-n-10}{2n^2-n-1}\right)^{n-2}\cdot\frac{(n+2)(8n^3-24n^2-18n+61)(8n^3-4n^2-32n-5)}{288n(n-1)^2(2n+1)^4}$$

$$\times\frac{\{(n-1)b+n\}\{(2n+1)b-2n-4\}\{(2n+1)b-2n+2\}}{b^2(n-b)}$$

$$=\frac{\{(n-1)b+n\}\{(2n+1)b-2n-4\}\{(2n+1)b-2n+2\}}{b^2(n-b)}\cdot L(n,\,b)$$

$$\text{for}\quad 1<X\leqq\frac{2n+4}{2n+1}.$$

From this inequality and (5.6) we can obtain the desired one. Q. E. D.

In the following we shall show that $L(n,\,b_n)>0$ for $9.5\leqq n\leqq 16$.

LEMMA 5.7. $\frac{(8n^3-24n^2-18n+61)(8n^3-4n^2-32n-5)}{n(n-1)(2n-5)(2n+1)^3}$ *is increasing for* $n\geqq 4$.

Proof. Performing calculation carefully we obtain

$$\frac{d}{dn}\frac{(8n^3-24n^2-18n+61)(8n^3-4n^2-32n-5)}{n(n-1)(2n-5)(2n+1)^3}$$

$$=\frac{\Phi(n)}{n^2(n-1)^2(2n-5)^2(2n+1)^4},$$

where

$$\Phi(n):=384n^8+1024n^7-16256n^6+28608n^5+28864n^4$$
$$-85664n^3+25566n^2+7930n+1525.$$

Since we have

$$(384n^3+1024n^2-16256n)'=1152n^2+2048n-16256\geqq 10368 \qquad \text{for} \quad n\geqq 4,$$

we obtain

$$384n^3+1024n^2-16256n+28608\geqq 4544 \qquad \text{for} \quad n\geqq 4.$$

Hence we have

$$\Phi(n)\geqq 4544n^5+28864n^4-85664n^3+25566n^2+7930n+1525>0 \qquad \text{for} \quad n\geqq 4,$$

which implies the claim of this lemma. Q.E.D.

LEMMA 5.8. $\dfrac{b^2(n-b)}{\{(n-1)b+n\}\{(2n+1)b-2n+2\}\{(2n+1)b-2n-4\}}$ *is decreasing with respect to* n *and* b *respectively for* $4\leqq n$ *and* $\dfrac{2n+4}{2n+1}<b<2.3$.

Proof. 1) First we shall prove that the above function of n and b is increasing with respect to n. We obtain

$$\frac{\partial}{\partial n}\frac{b^2(n-b)}{\{(n-1)b+n\}\{(2n+1)b-2n+2\}\{(2n+1)b-2n-4\}}$$

$$=\frac{b^2\Phi(n,b)}{\{(n-1)b+n\}^2\{(2n+1)b-2n+2\}^2\{(2n+1)b-2n-4\}^2},$$

where

$$\Phi(n,b):=3(2n+1)(2n-1)b^4-2(2n+1)(2n^2+2n-3)b^3$$
$$+4(2n^3-4n^2-4n-3)b^2+4n(n+2)(2n+1)b-4n^2(2n+1).$$

Then we obtain easily

$$\frac{\partial\Phi(n,b)}{\partial b}=12(4n^2-1)b^3-6(4n^3+6n^2-4n-3)b^2+8(2n^3-4n^2-n-3)b$$
$$+4(2n^3+5n^2+2n)$$

and

$$\frac{\partial^2\Phi(n,b)}{\partial b^2}=36(4n^2-1)b^2-12(4n^3+6n^2-4n-3)b+8(2n^3-4n^2-4n-3),$$

from which we have

$$\left.\frac{\partial^2\Phi(n, b)}{\partial b^2}\right|_{b=1}=-32n^3+40n^2+16n-24=-8(n-1)^2(4n+3)<0$$

and

$$\left.\frac{\partial\Phi(n, b)}{\partial b}\right|_{b=1}=-18<0,$$

$$\left.\frac{\partial\Phi(n, b)}{\partial b}\right|_{b=2.3}=-100.56n^3+339.976n^2+61.36n-105.984$$

$$\leqq -187.696n-105.984<0 \qquad \text{for} \quad n\geqq 4$$

These imply that

$$\frac{\partial\Phi(n, b)}{\partial b}<0 \qquad \text{for} \quad 4\leqq n \quad \text{and} \quad 1<b<2.3.$$

Since we have $\Phi(n, 1)=-9<0$, we obtain

$$\Phi(n, b)<0 \qquad \text{for} \quad 4\leqq n \quad \text{and} \quad 1<b<2.3,$$

which implies the first statement.

2) Next, we obtain

$$\frac{\partial}{\partial b}\frac{b^2(n-b)}{\{(n-1)b+n\}\{(2n+1)b-2n+2\}\{(2n+1)b-2n-4\}}$$

$$=\frac{b\Psi(n, b)}{\{(n-1)b+n\}^2\{(2n+1)b-2n+2\}^2\{(2n+1)b-2n-4\}^2},$$

where

$$\Psi(n, b):=-(2n+1)^2(n^2-2n+2)b^3+4(2n^3+4n^2+7n-4)b^2$$

$$-2n(2n^3+10n^2+13n-16)b+8n^2(n-1)(n+2).$$

Then, we obtain

$$\frac{\partial\Psi(n, b)}{\partial b}=-3(4n^4-4n^3+n^2+6n+2)b^2+8(2n^3+4n^2+7n-4)b$$

$$-2(2n^4+10n^3+13n^2-16n).$$

Since we have

$$\frac{4(2n^3+4n^2+7n-4)}{3(4n^4-4n^3+n^2+6n+2)}<1$$

i.e.

$$12n^4-20n^3-13n^2-10n+22>0 \qquad \text{for} \quad n\geqq 3,$$

we see that

$$\frac{\partial\Psi(n, b)}{\partial b}<\left.\frac{\partial\Psi(n, b)}{\partial b}\right|_{b=1}=-16n^4+8n^3+3n^2+70n-38<0 \qquad \text{for} \quad n\geqq 3$$

Hence $\Psi(n, b)$ is decreasing with respect to b for $1<b$, when $n\geqq 3$. Now, since we have

$$\Psi(n, 1)=-27n^2+54n-18=-9(3n^2-6n+2)<0 \quad \text{for} \quad n\geqq 3,$$

we obtain

$$\Psi(n, b)<0 \quad \text{for} \quad b>1 \quad \text{and} \quad n\geqq 3,$$

which implies the second statement. Q.E.D.

LEMMA 5.9. $\left(\frac{2n^2-n-10}{2n^2-n-1}\right)^{n-1}$ *is increasing for* $n\geqq 3$.

Proof. We have

$$\frac{d}{dn}\left(\frac{2n^2-n-10}{2n^2-n-1}\right)^{n-1}$$

$$=\left(\frac{2n^2-n-10}{2n^2-n-1}\right)^{n-1}\left\{\frac{9(4n-1)}{(2n+1)(2n^2-n-10)}-\log\frac{2n^2-n-1}{2n^2-n-10}\right\},$$

and

$$\lim_{n\to\infty}\left\{\frac{9(4n-1)}{(2n+1)(2n^2-n-10)}-\log\frac{2n^2-n-1}{2n^2-n-10}\right\}=0.$$

Then, we have

$$\frac{d}{dn}\left\{\frac{9(4n-1)}{(2n+1)(2n^2-n-10)}-\log\frac{2n^2-n-1}{2n^2-n-10}\right\}$$

$$=\frac{9(-32n^3+12n^2-61)}{(2n+1)^2(2n^2-n-10)^2}+\frac{9(4n-1)}{(2n^2-n-1)(2n^2-n-10)}$$

$$=-\frac{9(16n^4-40n^3+96n^2+80n-71)}{(n-1)(2n+1)^2(n+2)^2(2n-5)^2},$$

and

$$16n^4-40n^3+96n^2+80n-71>0 \quad \text{for} \quad n\geqq 3.$$

Hence, we see that

$$\frac{9(4n-1)}{(2n+1)(2n^2-n-10)}-\log\frac{2n^2-n-1}{2n^2-n-10}$$

is decreasing with respect to n ($\geqq 3$) and so it must be positive for $n\geqq 3$, which implies the statement of this lemma. Q.E.D.

From Lemmas 5.7~5.9, we obtain immediately the following lemma.

LEMMA 5.10. *The function* $L(n, b)$ *defined by* (5.5) *is increasing with respect to* n *and* b *in the domain of the nb-plane*: $4\leqq n$, $\frac{2n+4}{2n+1}<b<2.3$.

PROPOSITION 2. *When* $9.5\leqq n\leqq 16$, *we have*

$$\frac{2nBH(X)}{n-1}+\frac{XF_2(X)}{(X-1)^2}\{\tilde{\lambda}(X)-\tilde{\lambda}(b_n)\}>0 \qquad \textit{for} \quad 1<X\leqq\frac{2n+4}{2n+1}.$$

Proof. We have

$$\frac{2n+10}{2n+1}<b_n<2.3 \qquad \text{for} \quad n\geqq 4$$

by means of Lemma 4.1, Lemma 4.3 and $b_4<2.3$. Hence, by Lemma 5.6 it is sufficient to prove that $L(n, b_n)>0$ for $9.5\leqq n\leqq 16$. Here, we present the table of approximately calculated values of $b_n: b'_n<b_n<b''_n=b'_n+10^{-4}$ for $9.5\leqq n\leqq 16$ analogous to Table 1.

Table 2

n	b'_n	$\frac{2n+10}{2n+1}$	n	b'_n	$\frac{2n+10}{2n+1}$	n	b'_n	$\frac{2n+10}{2n+1}$
9.5	1.5998	1.4500	11.8	1.4886	1.3659	14.1	1.4119	1.3082
.6	.5940	.4455	.9	.4847	.3629	.2	.4091	.3061
.7	.5882	.4412	12.0	1.4808	1.3600	.3	.4063	.3041
.8	.5826	.4369	.1	.4770	.3571	.4	.4036	.3020
.9	.5770	.4327	.2	.4733	.3543	.5	.4009	.3000
10.0	1.5716	1.4286	.3	.4696	.3516	.6	.3983	.2980
.1	.5663	.4245	.4	.4660	.3488	.7	.3957	.2961
.2	.5610	.2206	.5	.4624	.3462	.8	.3931	.2941
.3	.5559	.4167	.6	.4589	.3435	.9	.3905	.2922
.4	.5508	.4128	.7	.4554	.3409	15.0	1.3880	1.2903
.5	.5459	.4091	.8	.4520	.3383	.1	.3855	.2885
.6	.5410	.4054	.9	.4487	.3358	.2	.3831	.2866
.7	.5362	.4019	13.0	1.4453	1.3333	.3	.3807	.2848
.8	.5315	.3982	.1	.4421	.3309	.4	.3783	.2830
.9	.5269	.3947	.2	.4389	.3285	.5	.3759	.2813
11.0	1.5223	1.3913	.3	.4357	.3261	.6	.3736	.2795
.1	.5179	.3879	.4	.4326	.3237	.7	.3713	.2777
.2	.5135	.3846	.5	.4295	.3214	.8	.3690	.2761
.3	.5091	.3816	.6	.4265	.3191	.9	.3668	.2744
.4	.5049	.3782	7	.4235	.3169	16.0	1.3645	1.2727
.5	.5007	.3750	.8	.4205	.3147	.1	.3623	.2711
.6	.4966	.3719	.9	.4176	.3125			
.7	.4926	3689	14.0	1.4147	1.3103			

On the other hand, calculating the values of $L(n, x)$ for the pair (n, x) as follow, we obtain the result:

n	x	$L(n, x)$	Remarks
9.5	1.5805	$1.14576/10^6$	$1.5805<b'_{9.8}=1.5826$
9.8	1.5525	$6.35827/10^8$	$1.5525<b'_{10.3}=1.5559$
10.3	1.5113	$6.17078/10^7$	$1.5113<b'_{11.2}=1.5135$
11.2	1.4505	$2.74328/10^6$	$1.4505<b'_{12.8}=1.4520$
12.8	1.3713	$2.76191/10^7$	$1.3713=b'_{15.7}=1.3713$
15.7	1.2813	$3.61775/10^6$	$1.2813<b'_{16.0}=1.3645$

This result tells us $L(n, b_n)>0$ for $9.5\leqq n\leqq 16$ by means of Lemma 5.10.

Q.E.D.

Finally, we state also the following proposition, which can be proved by a method analogous to the one of Proposition 2 and whose proof will be found in another coming paper [17] by the same reason as Proposition 1.

PROPOSITION 3. *When* $9.5\leqq n\leqq 16$, *we have*

$$\frac{2nBH(X)}{n-1}+\frac{XF_2(X)}{(X-1)^2}\{\tilde{\lambda}(X)-\tilde{\lambda}(b_n)\}>0 \qquad \textit{for} \quad \frac{2n+4}{2n+1}\leqq X\leqq\frac{2n+10}{2n+1}.$$

By means of Lemma 5.2, Propositions 1~3 and the argument stated in the beginning of this section, we obtain the following

PROPOSITION 4. *When* $9.7\leqq n\leqq 16$, *we have*

$$V(x, b_n)>0 \qquad \textit{for} \quad X_n^{-1}(b_n)\leqq x<1.$$

§6. Evaluation of $W(x, x_1)$ (I).

We know the following facts on $W(x, x_1)$.

LEMMA 3.1 *in* (III). *When* $n\geqq 2$, *we have*

$$W(x, X(x))>0 \quad \textit{and} \quad \lim_{x_1\to n} W(x, x_1)=+\infty \qquad \textit{for} \quad 0<x<1.$$

LEMMA 3.4 *in* (III). *We have*

$$\frac{\partial W(x, x_1)}{\partial x_1}\geqq 0,$$

if and only if

$$\frac{F_0(X)-F_0(x)}{F_1(X)-F_1(x)}\geqq\frac{(n-1)x_1^2(n-x_1)^{n-1}}{n+(n-1)x_1} \qquad \textit{for} \quad 0<x<1,\ X(x)\leqq x_1<n, \tag{6.1}$$

where $X=X(x)$ *and*

$$F_0(x):=\begin{cases}(n-x)^{-n+3/2}(x-1)^{-3}f_0(x) & \text{for } 0\leqq x<n,\ x\neq 1\\ n(2n-1)/6\sqrt{n-1} & \text{for } x=1\end{cases}$$

and

$$F_1(x):=\begin{cases}(n-x)^{-2n+3/2}(1-x)^{-3}f_1(x) & \text{for } 0\leqq x<n,\ x\neq 1\\ n(4n+1)/6(n-1)^{n+1/2} & \text{for } x=1\,.\end{cases}$$

By means of Lemma 3.5 in (III) the function:

$$\frac{(n-1)x^2(n-x)^{n-1}}{n+(n-1)x} \qquad \text{on} \quad 0\leqq x\leqq n$$

is increasing in $0<x<\beta$ and decreasing in $\beta<x<n$, where

$$\beta=\beta_n:=(\sqrt{2n^2-2n+1}-1)/(n-1) \tag{6.2}$$

And β_n is increasing with respect to n, tends to $\sqrt{2}$ as $n\to\infty$ and $1<\beta_n$.

LEMMA 6.1. *For a fixed constant $b(1<b<n)$, if we have*

$$\frac{F_0(X)-F_0(x)}{F_1(X)-F_1(x)}\geqq\frac{(n-1)b}{n+(n-1)b}\cdot\phi(X)\,, \qquad X=X(x)\,, \quad X^{-1}(b)\leqq x<1\,, \tag{6.3}$$

then (6.1) *holds for $X^{-1}(b)\leqq x<1$ and $X(x)\leqq x_1\leqq b$.*

Proof. Case: $\beta<b$. For $X^{-1}(b)\leqq x\leqq X^{-1}(\beta)$, the condition: (6.1) holds for $X(x)\leqq x_1\leqq b$ is equivalent to

$$\frac{F_0(X)-F_0(x)}{F_1(X)-F_1(x)}\geqq\frac{(n-1)X^2(n-X)^{n-1}}{n+(n-1)X}=\frac{(n-1)X}{n+(n-1)X}\cdot\phi(X)$$

and we have

$$\frac{(n-1)X}{n+(n-1)X}\leqq\frac{(n-1)b}{n+(n-1)b}\,.$$

For $X^{-1}(\beta)<x<1$, the condition above is equivalent to

$$\frac{F_0(X)-F_0(x)}{F_1(X)\geqq F_1(x)}\geqq\frac{(n-1)\beta^2(n-\beta)^{n-1}}{n+(n-1)\beta}=\frac{(n-1)\beta}{n+(n-1)\beta}\cdot\phi(\beta)$$

and

$$\frac{(n-1)\beta}{n+(n-1)\beta}\leqq\frac{(n-1)b}{n+(n-1)b}\,, \qquad \phi(\beta)<\phi(X)\,.$$

Hence, in this case we obtain immediately the statement.

Case: $b\leqq\beta$. For $X^{-1}(b)\leqq x<1$, the condition: (6.1) holds for $X(x)\leqq x_1\leqq b$ is equivalent to

$$\frac{F_0(X)-F_0(x)}{F_1(X)-F_1(x)}\geqq\frac{(n-1)b^2(n-b)^{n-1}}{n+(n-1)b}=\frac{(n-1)b}{n+(n-1)b}\cdot\phi(b)$$

and we have $\psi(b)\leqq\psi(X)$. Hence, we obtain easily the statement. Q.E.D.

Observing (6.3), we introduce an auxiliary function as follows:

$$G_b(x):=F_0(x)-\frac{(n-1)b}{n+(n-1)b}\psi(x)F_1(x) \quad \text{for} \quad 0<x<n, \tag{6.4}$$

then (6.3) can be written as

$$G_b(x)\leqq G_b(X(x)) \quad \text{for} \quad X^{-1}(b)\leqq x<1. \tag{6.5}$$

$G_b(x)$ can be written by (1.5) and (1.6) as

$$G_b(x)=\frac{1}{(x-1)^3(n-x)^{n-1/2}}\left\{(n-x)f_0(x)+\frac{(n-1)b}{n+(n-1)b}xf_1(x)\right\}$$

and

$$\{n+(n-1)b\}(n-x)f_0(x)+(n-1)bxf_1(x)$$
$$=\{n+(n-1)b\}(n-x)f_0(x)+(n-1)bx\{f_0(x)-2n(x-1)(B-\psi(x))\}$$
$$=n[\{n+(n-1)b-x\}f_0(x)-2b(n-1)x(x-1)\{B-\psi(x)\}],$$

i.e.

$$G_b(x)=\frac{n}{n+(n-1)b}\left[\frac{n+(n-1)b-x}{(x-1)^3(n-x)^{n-1/2}}\cdot f_0(x)-\frac{2b(n-1)x}{(n-x)^{n-1/2}}\cdot\frac{B-\psi(x)}{(x-1)^2}\right]. \tag{6.6}$$

Now, we shall calculate carefully the derivative of $G_b(x)$. By (2.7) in (II) and (4.9) in (I) we have

$$\frac{d}{dx}\left[\frac{n+(n-1)b-x}{n-x}F_0(x)-\frac{2b(n-1)x}{(n-x)^{n-1/2}}\cdot\frac{B-\psi(x)}{(x-1)^2}\right]$$
$$=\frac{(n-1)b}{(n-x)^2}F_0(x)+\frac{n+(n-1)b-x}{n-x}\cdot\frac{F_2(x)}{2(x-1)^4(n-x)^{n-1/2}}$$
$$-b(n-1)\cdot\frac{2n+(2n-3)x}{(n-x)^{n+1/2}}\cdot\frac{B-x(n-x)^{n-1}}{(x-1)^2}+\frac{2b(n-1)x}{(n-x)^{n-1/2}}$$
$$\cdot\frac{2B-(n-x)^{n-2}\{(n-2)x^2+n\}}{(x-1)^3}$$
$$=\frac{\Phi(x)}{2(x-1)^4(n-x)^{n+1/2}},$$

where

$$\Phi(x):=2b(n-1)(x-1)f_0(x)+\{n+(n-1)b-x\}\{-P_2(x)B+(n-x)^{n-1}P_3(x)\}$$
$$-2b(n-1)(x-1)^2\{2n+(2n-3)x\}\{B-x(n-x)^{n-1}\}$$
$$+4b(n-1)x(n-x)(x-1)\{2B-(n-x)^{n-2}((n-2)x^2+n)\},$$

putting $F_2(x)=-P_2(x)B+(n-x)^{n-1}P_3(x)$. Then, $\Phi(x)$ can be written as

$$\Phi(x) = -L_{b3}(x)B + (n-x)^{n-1}L_{b4}(x)\,, \tag{6.7}$$

where

$$\begin{aligned} L_{b3}(x) := {} & b(n-1)[2(2n+1)x^3-(10n-7)x^2-2(2u^2+5n-1)x+16n^2-8n+1] \\ & -(2n+1)x^3+(6n^2+11n-8)x^2-(4n^3+26n^2-24n+3)x+n(16n^2-16n+3) \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} L_{b4}(x) := {} & b(n-1)[2x^4-(3n-7)x^3+(2n^2-13n+2)x^2+(4n^2-7n+1)x \\ & +n(6n-1)]+(n-1)x^4-(3n^2-8n+8)x^3+(2n^3-11n^2+21n-3)x^2 \\ & +n(4n^2-19n+6)x+3n^2(2n-1)\,. \end{aligned} \tag{6.9}$$

Thus, we obtain the formula:

$$\frac{n+(n-1)b}{n}G_b'(x) = \frac{1}{2(x-1)^4(n-x)^{n+1/2}}[-L_{b3}(x)B+(n-x)^{n-1}L_{b4}(x)]\,. \tag{6.10}$$

LEMMA 6.2. *When $n \geqq 4.2$, $L_{b3}(x)$ is decreasing with respect to x in $0 < x \leqq b$, assuming $b_n \leqq b < 2.25 = 9/4$.*

Proof. We have from (6.8)

$$\begin{aligned} L_{b3}'(x) = {} & 3(2n+1)\{2b(n-1)-1\}x^2-2\{b(n-1)(10n-7)-6n^2-11n+8\}x \\ & -2b(n-1)(2n^2+5n-1)-(4n^3+26n^2-24n+3)\,. \end{aligned}$$

Since the coefficient of x^2 of $L_{b3}'(x)$ is positive and $L_{b3}'(0)<0$, we see that $L_{b3}'(x) < 0$ for $0 < x_2 < b$ if and only if $L_{b3}'(b) \leqq 0$. Now, we have

$$\begin{aligned} -L_{b3}'(b) = {} & -6(2n^2-n-1)b^3+(20n^2-28n+17)b^2+2(2n^3-3n^2-17n+9)b \\ & +4n^3+26n^2-24n+3\,. \end{aligned}$$

Since regarding

$$\frac{1}{2}\frac{\partial}{\partial b}(-L_{b3}'(b)) = -9(2n^2-n-1)b^2+(20n^2-28n+17)b+2n^3-3n^2-17n+9$$

we have

$$\frac{1}{2}\frac{\partial}{\partial b}(-L_{b3}'(b))\,|_{b=0} = 2n^3-3n^2-17n+9>0\,,$$

$$\frac{1}{2}\frac{\partial}{\partial b}(-L_{b3}'(b))\,|_{b=1} = (n-1)(2n^2+n-35)>0\,,$$

we can say $L_{b3}'(b) \leqq 0$ if $-L_{b3}'(b)\,|_{b=2.25} \geqq 0$. Now, we have

$$-L_{b3}'(b)\,|_{b=2.25} = -68.34375(2n^2-n-1)+5.0625(20n^2-28n+17)$$

$$+4.5(2n^3-3n^2-17n+9)+4n^3+26n^2-24n+3$$

$$=13n^3-22.9375n^2-173.90625n+197.90625 :=Z(n),$$

on which we can show :

i) for $n\geqq 4.8$, $Z(n)\geqq 15.51375n+197.90625>0$,

ii) for $4.2\leqq n\leqq 4.8$, $Z(n)\geqq -40.92375n+197.90625>0$.

Hence, we obtain the statement of this lemma. Q. E. D.

Remark. We know that $b'_n<b_n<b''_n=b'_n+10^{-4}$ for $n=4$, 4.1 and 4.2 as follows: $b''_4=2.2680$, $b''_{4.1}=2.2453$, $b''_{4.2}=2.2231$. Regarding the final argument of the proof of Lemma 6.2, we can show that

$$Z(n)\geqq -45.204375n+197.90625>0 \qquad \text{for} \quad 4.15\leqq n\leqq 4.37\,.$$

LEMMA 6.3. *When $n\geqq 4.2$, $L_{b3}(x)>0$ for $0\leqq x<b$, where $b_n\leqq b<2.25$.*

Proof. By means of Lemma 6.2, we obtain from (6.8)

$$L_{b3}(x)\geqq L_{b3}(b)=2(2n^2-n-1)b^4-(10n^2-17n+7)b^3-2(2n^3+3n^2-6n+1)b^2$$

$$-(16n^3-24n^2+9n-1)b-(2n+1)b^3+(6n^2+11n-8)b^2$$

$$-(4n^3-26n^2-24n+3)b+16n^3-16n^2+3n$$

$$=2(2n^2-n-1)b^4-(10n^2-15n+8)b^3-(4n^3-23n+10)b^2$$

$$-(12n^3-50n^2+33n-4)b+16n^3-16n^2+3n :=\Phi_n(b)\,.$$

First, noticing $b_n>1$, we have for $b_n\leqq b\leqq 9/4$

$$\Phi_n(b)>-(6n^2-13n+10)b^3-(4n^3-23n+1)b^2+(12n^3-50n^2+33n-4)b$$

$$+16n^3-16n^2+3n$$

$$\geqq -\left\{(6n^2-13n+10)\cdot\frac{9}{4}+(4n^3-23n+10)\right\}b^2+(12n^3-50n^2+33n-4)b$$

$$+16n^3-16n^2+3n\,,$$

and hence

$$4\Phi_n(b)\geqq -(16n^3+54n^2-209n+130)b^2+4(12n^3-50n^2+33n-4)b$$

$$+4(16n^3-16n^2+3n)\,.$$

Noticing $16n^3+54n^2-209n+130>0$ and $16n^3-16n^2+3n>0$, we shall show that the right hand side of the above inequality is positive at $b=9/4$ when $n\geqq 7$. In fact, we have

$$-81(16n^3+54n^2-209n+130)+144(12n^3-50n^2+33n-4)+64(16n^3-16n^2+3n)$$

$$=1456n^3-12598n^2+21873n-11106\geqq 5031n-11106>0\,.$$

Thus, we see that the statement is true, when $n \geqq 7$.

Next, we consider the case: $4.2 \leqq n \leqq 7$. By Lemma 4.3 and Table 1, we have

$$b_n \geqq b_7' = 1.7949 \quad \text{for} \quad 4 \leqq n \leqq 7 .$$

Since we see easily that $(10n^2-15n+8)/4(2n^2-n-1) < 1.3$, we obtain

$$\Phi_n(b) > \{2(2n^2-n-1)b_n^2-(10n^2-15n+8)b_n-(4n^3-23n+10)\} b^2$$
$$+(12n^3-50n^2+33n-4)b+16n^3-16n^2+3n \quad \text{for} \quad b_n \leqq b \leqq 9/4 .$$

Now, we have

$$2(2n^2-n-1)b_n^2-(10n^2-15n+8)b_n-(4n^3-23n-1)$$
$$\leqq \frac{81}{8}(2n^2-n-1)-\frac{9}{4}(10n^2-15n+8)-4n^3-23n-10$$
$$= -\frac{1}{8}(32n^3+18n^2-373n+305) < 0$$

and hence if the right hand side of the above inequality is positive at $b=9/4$, then we shall obtain $\Phi_n(b) > 0$ for $0 \leqq b \leqq 9/4$. In fact, putting $b=9/4$ and multipling it by 16 we obtain

$$162(2n^2-n-1)b_n^2-81(10n^2-15n+8)b_n-81(4n^3-23n-10)$$
$$+36(12n^3-50n^2+33n-4)+16(16n^3-16n^2+3n)$$
$$=162(2n^2-n-1)b_n^2-81(10n^2-15n+8)b_n+364n^3-2056n^2-3099n-954 := \mu(n) .$$

Since we have

$$162 \cdot (b_7')^2 = 521.9098 \cdots > 521 \quad \text{and} \quad 81 \cdot b_7' = 145.3869 \cdots < 146 ,$$

we have

$$\mu(n) > 521(2n^2-n-1)-146(10n^2-15n+8)+364n^3-2056n^2+3099n-954$$
$$=364n^3-2474n^2+4768n-2643 \geqq 696n-2643 > 0 \quad \text{for} \quad n \geqq 4 .$$

Thus, we see that the statement is also true, when $4.2 \leqq n \leqq 7$. Q. E. D.

Now, from (6.8) and (6.9) we obtain easily

$$L_{b3}(1) = L_{b4}(1) = 12(b+1)(n-1)^3. \tag{6.11}$$

Therefore we have

$$\left[\frac{(n-x)^{n-1}L_{b4}(x)}{L_{b3}(x)}\right]_{x=1} = B . \tag{6.12}$$

Noticing Lemma 6.3 and (6.12), we obtain from (6.8) and (6.9)

$$\frac{d}{dx}\frac{(n-x)^{n-1}L_{b4}(x)}{L_{b3}(x)} = \frac{(n-x)^{n-2}}{(L_{b3}(x))^2}$$

$$\times[\{-(n-1)L_{b4}(x)+(n-x)L'_{b4}(x)\}L_{b3}(x)-L_{b4}(x)\cdot(n-x)L'_{b3}(x)]$$

and

$$-(n-1)L_{b4}(x)+(n-x)L'_{b4}(x)$$
$$=b(n-1)[-2(n+3)x^4+(3n^2+7n-14)x^3-2(n^3-n^2-16n+1)x^2-n(19n-3)x$$
$$-2n^3]-(n-1)(n+3)x^4+(3n^3+2n^2-12n+16)x^3-(2n^4-14n^2+42n-3)x^2$$
$$-3n(n^2-12n+2)x-n^2(2n^2+10n-3),$$
$$(n-x)L'_{b3}(x)=b(n-1)[-6(2n+1)x^3+2(6n^2+13n-7)x^2-2(8n^2-12n+1)x$$
$$-2n(2n^2+5n-1)]+3(2n+1)x^3-(18n^2+25n-16)x^2$$
$$+(16n^3+48n^2-40n+3)x-n(4n^3+26n^2-24n+3).$$

Therefore, arranging terms carefully we can obtain

$$\frac{d}{dx}\frac{(n-x)^{n-1}L_{b4}(x)}{L_{b3}(x)}=\frac{n(n-1)(x-1)^3(n-x)^{n-2}}{(L_{b3}(x))^2}P_{b4}(x),\tag{6.13}$$

where

$$P_{b4}(x)=b^2(n-1)\{-4(2n+1)x^4+6(2n^2-5n+1)x^3-(8n^3-78n^2+57n-5)x^2\tag{6.14}$$
$$-(40n^3-110n^2+23n-1)x+2n(4n^3-36n^2+12n-1)\}$$
$$+2b\{-(n-2)(2n+1)x^4+3(2n^3-9n^2+5n-2)x^3-(4n^4-48n^3+73n^2$$
$$-51n+4)x^2-n(32n^3-100n^2+91n-11)x+n^2(8n^3-44n^2+46n-7)\}$$
$$-(2n+1)x^4-3(4n^2+3)x^3+n(26n^2-11n+30)x^2$$
$$-3n^2(8n^2-6n+11)x+4n^3(2n^2-2n+3).$$

LEMMA 6.4. *When $n\geqq 6$, $P_{b4}(x)$ is decreasing in $0\leqq x\leqq b$, where $b_n\leqq b\leqq 2$.*

Proof. We get from (6.14)

$$P'_{b4}(x)=b^2(n-1)\{-16(2n+1)x^3+18(2n^2-5n+1)x^2\tag{6.15}$$
$$-2(8n^3-78n^2+57n-5)x-(40n^3-110n^2+23n-1)\}$$
$$+2b\{-4(n-2)(2n+1)x^3+9(2n^3-9n^2+5n-2)x^2$$
$$-2(4n^4-48n^3+73n^2-51n+4)x-n(32n^3-100n^2+91n-11)\}$$
$$-4(2n+1)x^3-9(4n^2+3)x^2+2n(26n^2-11n+30)x$$
$$-3n^2(8n^2-6n+11),$$

which we denote simply $-C_3x^3+C_2x^2-C_1x-C_0$, i.e.

$$C_3 := 4(2n+1)\{4b^2(n-1)+2b(n-2)-1\}$$
$$>4(2n+1)\{4(n-1)+2(n-2)-1\}=12(2n+1)(2n-3)>0\,,$$
$$C_2 := 18b^2(2n^3-7n^2+6n-1)+18b(2n^3-9n^2+5n-2)-9(4n^2+3)$$
$$>9\{2(2n^3-7n^2+6n-1)+2(2n^3-9n^2+5n-2)-4n^2-3\}$$
$$=9(8n^3-36n^2+22n-9)>0 \qquad \text{for} \quad n\geqq 4\,,$$
$$C_1 := 2b^2(8n^4-86n^3+135n^2-62n+5)+4b(4n^4-48n^3+73n^2-51n+4)$$
$$-2(26n^3-11n^2+30n)\,,$$
$$C_0 := b^2(40n^4-150n^3+133n^2-24n+1)+2b(32n^4-100n^3+91n^2-11n)$$
$$+3(8n^4-6n^3+11n^2)>0 \qquad \text{for} \quad n\geqq 4\,.$$

i) Case $C_1\geqq 0$. For $0<x\leqq 1$, we have

$$P'_{b4}(x)<C_2-C_0=-\{b^2(40n^4-186n^3+259n^2-132n+19)$$
$$+2b(32n^4-118n^3+172n^2-56n+18)+24n^4-18n^3+69n^2+27\}<0$$
$$\text{for} \quad n\geqq 4\,.$$

Then, for $1<x\leqq b$ we have

$$P'_{b4}(x)<-C_3+C_2b^2-C_1-C_0=18b^4(2n^3-7n^2+6n-1)$$
$$+18b^3(2n^3-9n^2+5n-2)-b^2(56n^4-322n^3+471n^2-164n+22)$$
$$-2b(40n^4-196n^3+245n^2-125n)-24n^4+70n^3-55n^2+68n+4$$
$$<b^2\{18\cdot 4(2n^3-7n^2+6n-1)+18\cdot 2(2n^3-9n^2+5n-2)$$
$$-56n^4+322n^3-471n^2+164n-22\}$$
$$-2b(40n^4-196n^3+245n^2-125n)-24n^4+70n^3-55n^2+68n+4$$
$$=-b^2(56n^4-538n^3+1299n^2-776n+166)-2b(40n^4-196n^3+245n^2-125n)$$
$$-24n^4+70n^3-55n^2+68n+4$$
$$\leqq -[b^2(96n^4-734n^3+1544n^2-901n+166)+24n^4-70n^3+55n^2-68n-4]$$
$$\text{for} \quad n\geqq 4 \quad \text{and} \quad 1<b\leqq 2\,,$$

since

$$40n^3-196n^2+245n-125\geqq 101n-125>0 \qquad \text{for} \quad n\geqq 4\,.$$

Furthermore, we have

$$b^2(96n^4-734n^3+1544n^2-901n+166)+24n^4-70n^3+55n^2-68n-4$$
$$\geqq b^2(144n^2-901n+166)+159n^2-68n-4\,,$$

which is positive for n such that $144n^2-901n+166\geqq 0$ and for n such that $144n^2-901n+166<0$, i.e.

$$n<\frac{901+\sqrt{(901)^2-4\cdot 144\cdot 166}}{2\cdot 144}\doteqdot 6.0669,$$

$$b^2(144n^2-901n+166)+159n^2-68n-4$$

$$\geqq 4(144n^2-901n+166)+159n^2-68n-4$$

$$=735n^2-3672n+660>0 \qquad \text{for} \quad n\geqq 5.$$

Hence, we see that $P'_{b4}(x)<0$ for $1<x\leqq b$, when $n\geqq 5$.

ii) Case $C_1<0$. For $0<x\leqq 1$, we have

$$P'_{b4}(x)<C_2-C_1-C_0=-\{b^2(56n^4-358n^3+529n^2-256n+29)$$

$$+2b(40n^4-214n^3+318n^2-158n+26)+24n^4-70n^3+91n^2-60n+27\}$$

$$<-\{b^2(96n^4-572n^3+847n^2-414n+55)+24n^4-70n^3+91n^2-60n+27\},$$

since

$$40n^4-214n^3+318n^2-158n+26\geqq 102n^2-158n+26>0 \qquad \text{for} \quad n\geqq 4,$$

and $1<b\leqq 2$. Furthermore, we see that

$$4b^2(96n^4-572n^3+847n^2-414n+55)+4(24n^4-70n^3+91n^2-60n+27)$$

$$\geqq b^2(408n^4-2358n^3+3479n^2-1716n+247)$$

$$\geqq b^2(575n^2-1716n+247)>0.$$

Hence, we obtain $P'_{b4}(x)<0$ for $0<x\leqq 1$, when $n\geqq 4$.

Then, for $1<x\leqq b$, we have

$$P'_{b4}(x)<-C_3+C_2b^2-C_1b-C_0=18b^4(2n^3-7n^2+6n-1)$$

$$-2b^3(8n^4-104n^3+216n^2-107n+23)-b^2(56n^4-342n^3+493n^2-244n+28)$$

$$-2b(32n^4-126n^3+110n^2-53n-8)-24n^4+18n^3-33n^2+8n+4$$

$$<2b^3\{18(2n^3-7n^2+6n-1)-8n^4+104n^3-216n^2+107n-23\}$$

$$-b^2\{56n^4-342n^3+493n^2-244n+28+32n^4-126n^3+110n^2-53n-8\}$$

$$-24n^4+18n^3-33n^2+8n+4$$

$$=-2b^3(8n^4-140n^3+342n^2-215n+41)-b^2(88n^4-468n^3+603n^2-297n+20)$$

$$-24n^4+18n^3-33n^2+8n+4.$$

Since we have

$$88n^4-468n^3+603n^2-297n+20\geqq 139n^2-297n+20>0 \qquad \text{for} \quad n\geqq 4,$$

we obtain

$$8P'_{b4}(x) < -b^3\{16(8n^4-140n^3+342n^2-215n+41)$$

$$+4(88n^4-468n^3+603n^2-297n+20)+24n^4-18n^3+33n^2-8n-4\}$$

$$= -b^3(504n^4-4130n^3+7917n^2-4636n+732)$$

$$\leqq -b^3(1281n^2-4636n+732) < 0 \quad \text{for} \quad n \geqq 6\,.$$

Hence, we see that $P'_{b4}(x) < 0$ for $1 < x \leqq b$, when $n \geqq 6$.

Thus, we obtain this lemma, collecting the two cases. Q. E. D.

LEMMA 6.5. *When* $n \geqq \frac{9+\sqrt{77}}{2} \doteqdot 8.8874$, $P_{b4}(1) > 0$ *for* $1 < b \leqq 2$.

Proof. From (6.14) we obtain

$$P_{b4}(1) = 8(n-1)^3(b+1)\{b(n^2-13n+1)+n^2-n+1\}\,. \tag{6.16}$$

Hence, $P_{b4}(1) > 0$ for $n \geqq \frac{13+\sqrt{165}}{2} \doteqdot 12.9226$. Then, for $n < \frac{13+\sqrt{165}}{2}$, $P_{b4}(1) > 0$ if and only if

$$b < \frac{n^2-n+1}{-n^2+13n-1}\,.$$

Therefore, from the condition

$$2 < \frac{n^2-n+1}{-n^2+13n-1}\,,$$

which is equivalent to $n^2-9n+1 > 0$, we obtain $n > \frac{9+\sqrt{77}}{2} \doteqdot 8.8874$. Thus we obtain this lemma. Q. E. D.

§ 7. Evaluation $W(x, x_1)$ (II).

In this section, we shall deal with the fact described in the end of Introduction of the present paper.

By the argument in § 6, especially Lemma 3.4 in (III) and Lemma 6.1, it is sufficient to prove (6.5) in order to attain our purpose. Then, assuming $9 \leqq n \leqq 16$, we see that from Lemma 6.4 and Lemma 6.5

$$P_{b4}(x) > 0 \quad \text{for} \quad 0 \leqq x \leqq 1\,, \tag{7.1}$$

where $b_n \leqq b \leqq 2$, and hence

$$\frac{d}{dx}\frac{(n-x)^{n-1}L_{b4}(x)}{L_{b3}(x)} < 0 \quad \text{for} \quad 0 \leqq x \leqq 1$$

by (6.13), and so

$$G_b'(x)>0 \qquad \text{for} \quad 0<x<1$$

by (6.12). Analogously we see that

i) if $P_{b4}(b)\geqq 0$, then $G_b'(x)>0$ for $1<x<b$;

ii) if $P_{b4}(b)<0$ and $(n-b)^{n-1}L_{b4}(b)/L_{b3}(b)\geqq B$, then $G_b'(x)>0$ for $1<x<b$;

iii) if $P_{b4}(b)<0$ and $(n-b)^{n-1}L_{b4}(b)/L_{b3}(b)<B$, then there exists γ $(1<\gamma<b)$ such that $G_b'(x)>0$ for $1<x<\gamma$ and $G_b'(x)<0$ for $\gamma<x<b$.

Considering these facts about $G_b(x)$, we see that if we can prove

$$G_b(b)\geqq G_b(1)\,, \tag{7.2}$$

then it implies (6.5). This inequality (7.2) is automatically satisfied in the above Cases i) and ii).

Now, from (6.6), (2.3) in (II) and (4.8) in (II) we obtain

$$\frac{n+(n-1)b}{n}G_b(1)=(b+1)F_0(1)-\frac{2b}{(n-1)^{n-3/2}}\lim_{x\to 1}\frac{B-\phi(x)}{(x-1)^2}$$

$$=\frac{(b+1)n(2n-1)}{6\sqrt{n-1}}-\frac{bn}{\sqrt{n-1}}=\frac{n\{(2n-7)b+2n-1\}}{6\sqrt{n-1}}\,.$$

Next, we obtain analogously

$$\frac{n+(n-1)b}{n}G_b(b)=\frac{1}{(n-b)^{n-1/2}(b-1)^3}$$

$$\times[\{(n-2)b+n\}f_0(b)-2(n-1)b^2(b-1)\{B-b(n-b)^{n-1}\}]\,,$$

in which the expression in the brackets can be written by (1.5) as

$$\{n(2n-1)+2(n^2-3n+1)b+nb^2-2(n-1)b^3\}B$$

$$-(n-b)^{n-1}\{n^2+n(n-3)b+(n^2-2n+2)b^2+n(n-1)b^3-2(n-1)b^4\}\,.$$

Therefore the condition (7.2) is equivalent to

$$\frac{(n-1)^{n-1}\{n(2n-1)+2(n^2-3n+1)b+nb^2-2(n-1)b^3\}}{(n-b)^{n-1/2}(b-1)^3}$$

$$\geqq\frac{n^2+n(n-3)b+(n^2-2n+2)b^2+n(n-1)b^3-2(n-1)b^4}{(n-b)^{1/2}(b-1)^3}+\frac{n\{(2n-7)b+2n-1\}}{6(n-1)^{1/2}}$$

or

$$\left(\frac{n-1}{n-b}\right)^{n-1}\geqq\frac{n^2+n(n-3)b+(n^2-2n+2)b^2+n(n-1)b^3-2(n-1)b^4}{n(2n-1)+2(n^2-3n+1)b+nb^2-2(n-1)b^3} \tag{7.3}$$

$$+\frac{1}{6}\sqrt{\frac{n-b}{n-1}}\cdot\frac{n\{(2n-7)b+2n-1\}(b-1)^3}{n(2n-1)+2(n^2-3n+1)b+nb^2-2(n-1)b^3}\,.$$

Now, noticing $b_n<1.6=8/5$ for $9.5\leqq n\leqq 16$ since $b''_{9.5}=1.5999$ (see Table 1) and Lemma 4.3, we assume $9.5\leqq n\leqq 16$ in the following and set $b=1.6$ in (7.3).

Then (7.3) can be written as

$$(7.4)\qquad \left(\frac{5n-5}{5n-8}\right)^{n-1} \geqq \frac{5785n^2-16952n+11392}{5(650n^2-2029n+1424)} + \sqrt{\frac{5n-8}{5n-5}}\cdot\frac{9n(26n-61)}{10(650n^2-2029n+1424)}\,.$$

LEMMA 7.1. *There exists* $\xi_0(9<\xi_0<10)$ *such that* (7.4) *holds for* $n \geqq \xi_0$ *and not for* $6 \leqq n < \xi_0$.

Proof. Since regarding (7.4) we have

$$\left(\frac{5n-5}{5n-8}\right)^{n-1}\Big|_{n=9} = \left(\frac{40}{42}\right)^8 \doteqdot 1.865805\,,$$

$$(\text{the right hand side})|_{n=9} = \frac{327409}{5\cdot 35813} + \sqrt{\frac{37}{40}}\cdot\frac{81\cdot 173}{10\cdot 35813} \doteqdot 1.866069$$

and

$$\left(\frac{5n-5}{5n-8}\right)^{n-1}\Big|_{n=10} = \left(\frac{45}{42}\right)^9 \doteqdot 1.860669\,,$$

$$(\text{the right hand side})|_{n=10} = \frac{420372}{5\cdot 46134} + \sqrt{\frac{42}{45}}\cdot\frac{9\cdot 199}{46134} = 1.859901\,,$$

we see that there exists at least one ξ_0 such that the equality of (7.4) holds for $n=\xi_0$ and $9<\xi_0<10$.

Now, assume that there exist at least two $\xi(>9)$ such that the equality of (7.4) hold for $n=\xi$. Then, we can find a $\xi(>9)$ out of them such that at $n=\xi$

$$(7.5)\qquad \frac{d}{dn}\left(\frac{5n-5}{5n-8}\right)^{n-1} \leqq \frac{d}{dn}(\text{the right hand side of (7.4)})\,.$$

Since we have

$$(7.6)\qquad \frac{d}{dn}\left(\frac{5n-5}{5n-8}\right)^{n-1} = \left(\frac{5n-5}{5n-8}\right)^{n-1}\left\{\log\frac{5n-5}{5n-8} - \frac{3}{5n-8}\right\},$$

and

$$(7.7)\qquad \frac{d}{dn}(\text{the right hand side of (7.4)})$$

$$= -\frac{9(15977n^2-37024n+22784)}{(650n^2-2029n+1424)^2} - \frac{9}{20}\cdot\frac{80340n^4-803972n^3+2521173n^2-3182640n+1389824}{\sqrt{5(n-1)^3(5n-8)}\cdot(650n^2-2029n+1424)^2}\,,$$

hence (7.5) can be replaced by

$$\left\{\frac{5785n^2-16952n+11392}{5(650n^2-2029n+1424)}+\sqrt{\frac{5n-8}{5n-5}}\cdot\frac{9n(26n-61)}{10(650n^2-2029n+1424)}\right\}$$

$$\times\left\{-\log\frac{5n-5}{5n-8}+\frac{3}{5n-8}\right\}\geqq\frac{9(15977n^2-37024n+22784)}{(650n^2-2029n+1424)^2}$$

$$+\frac{9}{20}\cdot\frac{80340n^4-803972n^3+2521173n^2-3182640n+1389824}{\sqrt{5(n-1)^3(5n-8)}\cdot(650n^2-2029n+1424)^2}$$

and multipling the both sides by $10(650n^2-2029n+1424)^2$ we obtain

(7.8)
$$(650n^2-2029n+1424)\left\{2(5785n^2-16952n+11392)\right.$$

$$\left.+\sqrt{\frac{5n-8}{5n-5}}\cdot 9n(26n-61)\right\}\cdot\left\{-\log\frac{5n-5}{5n-8}+\frac{3}{5n-8}\right\}$$

$$\geqq 90(15977n^2-37024n+22784)$$

$$+\frac{9}{2}\cdot\frac{80340n^4-803972n^3+2521173n^2-3182640n+1389824}{\sqrt{5(n-1)^3(5n-8)}}.$$

On the other hand, we have

$$-\log\frac{5n-5}{5n-8}+\frac{3}{5n-8}<\frac{9}{2(5n-8)^2},$$

and hence using this inequality we can obtain from (7.8) the following

$$\frac{(650n^2-2029n+1424)(5785n^2-16952n+11392)}{(5n-8)^2}-10(15977n^2-37024n+22784)$$

$$\geqq\frac{1}{\sqrt{(5n-5)(5n-8)}}\left\{\frac{80340n^4-803972n^3+2521173n^2-3182640n+1389824}{2(n-1)}\right.$$

$$\left.-\frac{9n(26n-61)(650n^2-2029n+1424)}{2(5n-8)}\right\}.$$

Arranging the terms carefully, this inequality can be written as

(7.9)
$$-\frac{1}{(5n-8)^2}\cdot(234000n^4+718965n^3-4497768n^2+5331456n-1640448)$$

$$\geqq\frac{1}{2\cdot\sqrt{5(n-1)^3(5n-8)^3}}(249600n^5-3678844n^4+16758868n^3$$

$$-3385671n^2+31628464n-11118592).$$

The ξ mentioned above satisfies (7.9).

Now, we have

$$234000n^4+718965n^3-4497768n^2+5331456n-1640448$$

$$\geqq 2122092n^2+5331456n-1640448>0\qquad\text{for}\quad n\geqq 4$$

and

$$249600n^5-3678844n^4+16758868n^3-33853671n^2+31628464n-11118592$$
$$\geqq 3366872n^3-33853671n^2+31628464n-11118592$$
$$\geqq 40162057n-11118592>0 \qquad \text{for} \quad n\geqq 9\,.$$

Therefore the left hand side of (7.9) must be negative and the right hand side positive for $n=\xi$. Thus, we have reached a contradiction. Q. E. D

LEMMA 7.2. *Regarding* ξ_0 *in Lemma* 7.1, *we have* $9.2<\xi_0<9.3$.

Proof. We have

$$\left(\frac{5n-5}{5n-8}\right)^{n-1}\Bigg|_{n=9.2}=\left(\frac{41}{38}\right)^{8.2}=1.864671\,,$$

$$(\text{the right hand side of (7.4)})|_{n=9.2}=\frac{345076}{5\times 37773.2}+\sqrt{\frac{38}{41}}\cdot\frac{9\times 9.2\times 178.2}{377732}$$
$$=1.864700$$

and

$$\left(\frac{5n-5}{5n-8}\right)^{n-1}\Bigg|_{n=9.3}=\left(\frac{41.5}{38.5}\right)^{8.3}=1.864126\,,$$

$$(\text{the right hand side of (7.4)})|_{n=9.3}=\frac{354083.05}{5\times 38772.8}+\sqrt{\frac{38.5}{41.5}}\cdot\frac{9\times 9.3\times 180.8}{387728}$$
$$=1.864043\,.$$

These facts and Lemma 7.1 imply $9.2<\xi_0<9.3$. Q. E. D.

By means of the argument in the beginning of this section and Lemmas 7.1 and 7.2 we obtain the follwing

PROPOSITION 5. *When* $9.5\leqq n\leqq 16$, *we have*

$$W(x,\ x_1)>0 \qquad \textit{for} \quad X_n^{-1}(b_n)\leqq x<1 \quad \textit{and} \quad X_n(x)\leqq x_1\leqq b_n\,.$$

Combining Proposition 4 and Proposition 5 with Theorem C in (V), we obtain the main theorem of this paper as follow.

THEOREM C. *The period function* T *as a function of* τ *and* n *is monotone decreasing with respect to* $n\geqq 9.7$ *for any fixed* $\tau\,(0<\tau<1)$.

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