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A CHARACTERIZATION OF THE EXPONENTIAL FUNCTION BY PRODUCT II

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§1. Introduction. In our previous paper [2] we proved the following result.

THEOREM A. *Suppose that $f(z)$ is an entire function of order $q=2p+1$ having only negative zeros. Setting $\phi(z^2)=f(z)f(-z)$, $g(z)=\phi(-z)/\phi(0)$, we assume that $g(z)$ is a canonical product. Further we assume that there is an arbitrarily small $\beta>0$ such that if $|g(r)|\geqq 1$,*

$$\log|g(re^{i\beta})|\leqq(\cos\beta q/2)\log|g(r)|$$

for all sufficiently large r and if $|g(r)|\leqq 1$,

$$\log|g(re^{i\beta})|\geqq(\cos\beta q/2)\log|g(r)|$$

for all sufficiently large r. Then $f(z)=e^{P(z)}$ where $P(z)$ is a polynomial of degree q, or else

$$\lim_{r\to\infty}\frac{\log M(r, f)}{r^q}=+\infty .$$

The purpose of this paper is to improve Theorem A and prove the following.

THEOREM. *Suppose that $f(z)$ is an entire function of order $q=2p+1$ having only negative zeros. Setting $\phi(z^2)=f(z)f(-z)$, $g(z)=\phi(-z)/\phi(0)$, we assume that there is an arbitrarily small $\beta>0$ such that if $|g(r)|\geqq 1$ for all sufficiently large r,*

$$\log|g(re^{i\beta})g(re^{-i\beta})|\leqq 2(\cos\beta q/2)\log|g(r)| \tag{1}$$

for all sufficiently large r and if $|g(r)|\leqq 1$ for all sufficiently large r,

$$\log|g(re^{i\beta})g(re^{-i\beta})|\geqq 2(\cos\beta q/2)\log|g(r)| \tag{2}$$

for all sufficiently large r. Then $f(z)=e^{P(z)}$ where $P(z)$ is a polynomial of degree q, or else

$$\lim_{r\to\infty}\frac{\log M(r, f)}{r^q}=+\infty . \tag{3}$$

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In order to prove our theorem we need the following two lemmas.

LEMMA 1. [2]. *Suppose that* $g(z)=e^{Q(z)}g_1(z)$ *is an entire function of finite order having only negative zeros, where* $Q(z)$ *is a polynomial and* $g_1(z)$ *is a canonical product. Then the sign of* $\log|g(r)|$ *is definite for* $r \geqq r_0$ *where* r_0 *is a positive number, unless*

$$\deg(\operatorname{Re} Q(r))=0 \quad \text{and} \quad g_1(z)=1. \tag{4}$$

LEMMA 2. *Let* $0<t_1<t_2<\infty$. *Let* $B(t)$ *be a nondecreasing convex function of* $\log t$ *on each interval of* $(0, t_1)$, (t_1, t_2), (t_2, ∞) *with* $B(0)=B(0+)=0$ *and* $B(t)=0(t^{\rho})$ $(t\to\infty)$ *for some* $\rho\in(0, 1)$. *Let* $b(re^{i\theta})$ *be the function which is harmonic in the slit plane* $|\theta|<\pi$, *is zero on the positive axis and tends to* $B(r)$ *as* $\theta\to\pi-$ *with the possible exception of* $r=t_1, t_2$. *Then we have*

$$b(r)=\int_0^{\infty}[b_\theta(t)+b_\theta(-t)]Q(r, t)dt \tag{5}$$

where

$$Q(r, t)=\frac{2r\log r/t}{\pi^2(r^2-t^2)}.$$

This is a slight generalization of Proposition 5 in Baernstein [1] and the proof is similar to the one in [1]. Hence we omit the proof of Lemma 2.

§ 2. **Proof of Theorem.** Let $f(z)$ be an entire function satisfying the hypotheses in Theorem. We suppose that (3) is false, i. e.,

$$\liminf_{r\to\infty}\frac{\log M(r, f)}{r^q}<+\infty.$$

Since $\phi(z^2)=f(z)f(-z)$, $g(z)=\phi(-z)/\phi(0)$ and $\log M(r^2, \phi)\leqq 2\log M(r, f)$, there exists a sequence $\{r_n\}=r$ which tends to $+\infty$, such that

$$\frac{\log M(r, g)}{r^{q/2}}=0(1). \tag{6}$$

We see from Lemma 1 that the sign of $\log|g(r)|$ is definite for all sufficiently large r, with the exception of case (4) in which case we have the required function $f(z)=e^{P(z)}$, $\deg P(z)=q$. In the sequel we confine ourselves to the case that the sign of $\log|g(r)|$ is positive for all sufficiently large r, because the remaining case can be dealt with in the same way as in § 4 of [2].

If the sign of $\log|g(r)|$ is positive for all sufficiently large r, then (6) yields

$$\liminf_{r\to\infty}\frac{\log|g(r)|}{r^{q/2}}<+\infty.$$

We set $g(z)=e^{Q(z)}g_1(z)$ where $Q(z)$ is a polynomial and $g_1(z)$ is a canonical product and we denote the genus of $g_1(z)$ by k and the degree of $\operatorname{Re}(Q(r))$ by l.

Case (1). $k \geqq l$. Proceeding as in case (1) of §4 of [2], we have

$$\int_r^s \frac{H_\theta^*(te^{i\beta})-(\cos\beta q/2)H_\theta^*(t)}{t^{1+q/2}}dt \tag{7}$$

$$\geqq C_1\frac{\log|g(r)|}{r^{q/2}}-C_2\frac{\log M_\beta(2s,\ g)+\log M_\beta(\sqrt{2}\,s,\ g)}{s^{q/2}}, \qquad (s<R)$$

where $H^*(z)$ is the harmonic function in $\{z: 0<|z|<R,\ 0<\arg z<\beta\}$, which has the following boundary values: $H^*(r)=0$, $H^*(re^{i\beta})=B^*(r^{1/\gamma})$ (B^* is a nondecreasing convex function of $\log t$ on $(0,\infty)$ with $B(0)=B(0+)=0$ and $\gamma=\beta/\pi$) and C_1, C_2 depend only on β and q and $M_\beta(2s,\ g)=\sup_{|\theta|<\beta}|g(2se^{i\theta})|$. Further we have

$$H_\theta^*(te^{i\beta})\leqq \log|g(te^{i\beta})g(te^{-i\beta})|, \tag{8}$$

$$H_\theta^*(t)\geqq 2\log|g(t)|.$$

Now we consider subcases.

Case (1.1). $A=\limsup_{r\to\infty}\dfrac{\log|g(r)|}{r^{q/2}}=+\infty$.

We can find arbitrarily large values of r and s, with $r<s$, such that the right-hand side of (7) is positive from (6). Hence (8) implies that the inequality

$$\log|g(te^{i\beta})g(te^{-i\beta})|-2(\cos\beta q/2)\log|g(t)|>0$$

holds for some $t>r$ and this contradicts our assumption (1).

Case (1.2). $A=0$. There exists a sufficiently large number r_0 such that $(\log|g(r)|)/r^{q/2}>0$ for $r\geqq r_0$. Thus for each fixed $r(\geqq r_0)$ the right-hand side of (7) is positive for all sufficiently large s, and we have again a contradiction.

Case (1.3). $0<A<+\infty$. We define the function $H(z)$ in $D=\{z: 0<\arg z<\beta\}$ by

$$H(re^{i\theta})=\int_{-\theta}^{\theta}\log|g(re^{i\phi})|d\phi.$$

Since $g(z)=e^{Q(z)}g_1(z)$ we have

$$H(re^{i\theta})=\frac{2}{l}|a_l|r^l\sin l\theta\cos\theta_l+\cdots+2|a_1|r\sin\theta\cos\theta_1$$

$$+2\int_0^\theta\log|g_1(re^{i\phi})|d\phi,$$

where $Q(z)=a_{k'}z^{k'}+\cdots+a_1z$, $\deg(\operatorname{Re}Q(r))=l$ ($\leqq k'$) and $\arg a_j=\theta_j$ ($j=1,\cdots,k'$). Since $g(z)$ has only negative zeros, $H(re^{i\theta})$ is harmonic in D. Further we proved in [2] that $H(re^{i\beta})$ is an increasing convex function of $\log r$ for all sufficiently large r, if β is sufficiently small.

Now we construct the harmonic function $U(re^{i\theta})$ in D which majorizes $H(re^{i\theta})$ in D and has the boundary values $U(r)=0$ and $U(re^{i\beta})=B(r^{1/\gamma})$ where B

is a function satisfying all the hypotheses of the B in Lemma 2 and $\gamma=\beta/\pi$.

Since

$$H(re^{i\beta})=G(re^{i\beta})+c'_j r^j+\cdots+c'_l r^l \qquad (j\geqq 1),$$

where

$$G(re^{i\beta})=2\int_0^\beta \log|g_1(re^{i\phi})|d\phi$$

$$=2r^{k+1}\int_0^\infty\Big(\int_0^\beta \frac{n(x)}{x^{k+1}}\frac{x\cos(k+1)\phi+r\cos k\phi}{x^2+r^2+2rx\cos\phi}d\phi\Big)dx\,,$$

we have

$$H(re^{i\beta})=c_m r^m+c_{m+1}r^{m+1}+\cdots\ (m\geqq 1,\ c_m\neq 0)\,.$$

If $c_m<0$, then $H(re^{i\beta})$ is a decreasing function of r for all sufficiently small r. If $c_m>0$, then

$$\frac{\partial^2 H}{\partial(\log r)^2}=m^2c_m r^m+(m+1)^2c_{m+1}r^{m+1}+\cdots,$$

implies that $H(re^{i\beta})$ is an increasing convex function of $\log r$ for all sufficiently small r.

Thus, firstly, we define the function $B(t)$ by

$$B(t)=\begin{cases}0\,, & \text{if } c_m<0\\ H(t^\gamma e^{i\beta})\,, & \text{if } c_m>0\end{cases}\quad \text{for}\quad 0\leqq t\leqq t_1\,, \tag{9}$$

$$B(t)=at\quad (a>0)\quad \text{for}\quad t_1<t<t_2\,, \tag{10}$$

and

$$B(t)=H(t^\gamma e^{i\beta})\qquad \text{for}\quad t_2\leqq t<+\infty \tag{11}$$

where t_1 is a sufficiently small positive number and t_2 is a sufficiently large positive number, which are defined as follows. Since $B(t)$ satisfies all the hypotheses of the B in Lemma 2 with $\rho=\gamma q/2$, the Poisson integral

$$b(re^{i\theta})=\frac{1}{\pi}\int_0^\infty B(t)\frac{r\sin\theta}{t^2+r^2+2\,\mathrm{tr}\cos\theta}dt \tag{12}$$

satisfies all the hypotheses of the b in Lemma 2. Then we have

$$b_\theta(-r)=\int_0^\infty \log\left|1-\frac{r}{t}\right|dB_1(t)$$

where $B_1(t)=tB'(t)$. For any $\varepsilon>0$ and any $t_2>0$, if t_1 is sufficiently small, then we have

$$\int_0^{t_1}\log\left|1-\frac{r}{t}\right|dB_1(t)<\varepsilon\qquad \text{for}\quad t_1<r<t_2\,.$$

Thus, observing that

$$\int_{t_2}^{\infty} \log\left|1-\frac{r}{t}\right| dB_1(t)<0 \qquad \text{for} \quad t_1<r<t_2,$$

we see that

$$b_\theta(-r)<\varepsilon+\int_{t_1}^{t_2} \log\left|1-\frac{r}{t}\right| dB_1(t) \qquad \text{for} \quad t_1<r<t_2.$$

Hence we have for $r\in(t_1, t_2)$, using (10),

$$b(-r)<\varepsilon+at_1\log t_1-at_2\log t_2+a(r-t_1)\log(r-t_1)+a(t_2-r)\log(t_2-r).$$

Thus we can choose a sufficiently small number t_1 and a sufficiently large number t_2 such that

$$b_\theta(-r)<0 : t_1<r<t_2. \tag{13}$$

Now we define

$$U(z)=b(z^{1/\gamma}) \tag{14}$$

in $D=\{z : 0<\arg z<\beta\}$. Choosing a sufficiently large number a in (10), we can see that if $\beta q/2<\pi$

$$H(z)\leqq U(z) \quad \text{in } D. \tag{15}$$

In fact, H and U are harmonic in D and $H(z)\leqq U(z)$ on the boundary with the possible exception of $z=t_1e^{i\beta}$, $t_2e^{i\beta}$ from (9)~(12) and (14). Further we see that $H(z)$ is $O(|z|^{q/2})$ in D by the definition of H and that $U(z)$ is $O(|z|^{q/2})$ in D by (12) and (14). Therefore we can conclude that $H(z)\leqq U(z)$ inside the angle if $\beta q/2<\pi$.

If (1) holds for all $r>0$, then we claim that the following inequality holds

$$\varphi(r^\gamma)\leqq\int_0^\infty \varphi(t^\gamma)\left(1+\cos\frac{\beta q}{2}\right)Q(r, t)dt \tag{16}$$

where

$$\varphi(t^\gamma)=\begin{cases} U_\theta(t^\gamma) & \text{for} \quad 0\leqq t<t_2 \\ 2\log|g(t^\gamma)| & \text{for} \quad t\geqq t_2 \end{cases} \tag{17}$$

if $c_m<0$ and

$$\varphi(t^\gamma)=\begin{cases} U_\theta(t^\gamma) & \text{for} \quad t_1<t<t_2 \\ 2\log|g(t^\gamma)| & \text{for} \quad 0\leqq t\leqq t_1,\ t\geqq t_2. \end{cases} \tag{18}$$

if $c_m>0$.

From Lemma 2, we have

$$U_\theta(r^\gamma)=\int_0^\infty (U_\theta(t^\gamma)+U_\theta(t^\gamma e^{i\beta}))Q(r, t)dt. \tag{19}$$

At first, we consider the case $c_m<0$. Since $U(z)>0$ in the angle $D=\{z : 0<\arg z$

$<\beta\}$ and $B(t)=0$ for $0\leqq t\leqq t_1$ from (9), we have

$$U_\theta(t^\gamma e^{i\beta})\leqq 0\,,\quad U_\theta(t^\gamma)\geqq 0\qquad (0\leqq t\leqq t_1)\,. \tag{20}$$

Hence we have

$$U_\theta(t^\gamma)-U_\theta(t^\gamma e^{i\beta})\leqq\Big(1+\cos\frac{\beta q}{2}\Big)U_\theta(t^\gamma)\,,\qquad (0\leqq t\leqq t_1)\,. \tag{21}$$

For $t_1<t<t_2$, since $b_\theta(-t)=U_\theta(t^\gamma e^{i\beta})$ we have (20) from (13) and also (21) again. Thus we set in $0\leqq t<t_2$

$$\varphi(t^\gamma)=U_\theta(t^\gamma)\,. \tag{22}$$

Next we consider the case $t\geqq t_2$. From $H(r)=U(r)=0$ and (15), we have $H_\theta(t^\gamma)\leqq U_\theta(t^\gamma)$. Hence, from the definition of H, we have

$$2\log|g(t^\gamma)|\leqq U_\theta(t^\gamma)\,. \tag{23}$$

Now we define two functions $H_1(z)$ and $H_2(z)$ in the angle $D_1=\{z:0<\arg z<\beta/2\}$, which are harmonic and subharmonic respectively, as follows:

$$H_1(re^{i\theta})=U(re^{i(\beta/2+\theta)})-U(re^{i(\beta/2-\theta)})\,,$$

$$H_2(re^{i\theta})=\int_{-\beta/2-\theta}^{-\beta/2+\theta}\log|g(re^{i\phi})|\,d\phi+\int_{\beta/2-\theta}^{\beta/2+\theta}\log|g(re^{i\phi})|\,d\phi\,.$$

Then we have $H_1(r)=H_2(r)=0$ and

$$H_2(re^{i\beta/2})=\int_{-\beta}^{\beta}\log|g(re^{i\phi})|\,d\phi=H(re^{i\beta})$$
$$\leqq U(re^{i\beta})=H_1(re^{i\beta/2})\,.$$

Since H_1 and H_2 are both $O(r^{q/2})$ in D_1 as $r\to\infty$, and since $\beta q/4<\pi$, we can conclude that $H_2(z)\leqq H_1(z)$ inside D_1. Further we have $H_2(re^{i\beta/2})=H_1(re^{i\beta/2})$ for $r\geqq t_2^\gamma$ and hence we obtain

$$\varlimsup_{\theta\to\beta/2}\frac{H_2(re^{i\beta/2})-H_2(re^{i\theta})}{\beta/2-\theta}\geqq(H_1)_\theta(re^{i\beta/2})=U_\theta(re^{i\beta})+U_\theta(r)\,,\qquad (r\geqq t_2^\gamma)\,. \tag{24}$$

From the definition of H_2, we have

$$H_2(re^{i\beta/2})-H_2(re^{i\theta})=\int_{-\beta}^{-\beta/2-\theta}\log|g(re^{i\phi})|\,d\phi$$
$$-\int_{-\beta/2+\theta}^{\beta/2-\theta}\log|g(re^{i\phi})|\,d\phi+\int_{\beta/2+\theta}^{\beta}\log|g(re^{i\phi})|\,d\phi\,,$$

and thus we have

$$\varlimsup_{\theta\to\beta/2}\frac{H_2(re^{i\beta/2})-H_2(re^{i\theta})}{\beta/2-\theta}\leqq\log|g(re^{-i\beta})|+2\log|g(r)|$$
$$+\log|g(re^{i\beta})|\,,\qquad (r\geqq t_2^\gamma)\,.$$

Combining this with (24) and (1) we obtain

$$U_\theta(t^\gamma)+U_\theta(t^\gamma e^{i\beta})\leqq 2\Big(1+\cos\frac{\beta q}{2}\Big)\log|g(t^\gamma)| \qquad \text{for} \quad t>t_2. \tag{25}$$

Therefore setting $\varphi(t^\gamma)=2\log|g(t^\gamma)|$ for $t\geqq t_2$, from (19), (23) and (25), we have (16) for the function $\varphi(t^\gamma)$ defined by (17) in view of (22).

If $c_m>0$, then we can also prove (16) for the function $\varphi(t^\gamma)$ defined by (18).

Proceeding as in §5 of [2] from (16), we arrive at

$$\lim_{r\to\infty}\frac{\log|g(r^\gamma)|}{r^{\gamma q/2}}=A>0.$$

Hence, by Valiron's Tauberian Theorem [3], we have

$$n(r,\,0,\,g)\sim\frac{A}{\pi}r^{q/2},$$

and so

$$n(r,\,0,\,f)\sim\frac{A}{\pi}r^{q}.$$

Therefore we have $\delta(0,\,f)=1$. Proceeding as in the proof of Theorem 2 of [2], we have $A=0$, which is impossible.

Next we suppose that (1) holds for all $r\geqq t_0>0$. Then there exists a positive C such that $h(z)=g(z)/C$ satisfies (1) for all $r>0$. In fact, set

$$\varphi(t)=\log|g(te^{i\beta})g(te^{-i\beta})|-2(\cos\beta q/2)\log|g(t)|\,.$$

$$\max_{0\leq t\leq t_0}\varphi(t)=M(>0)$$

and

$$C=\exp(M/2(1-\cos\beta q/2))\,.$$

Then it is easily seen that $h(z)$ satisfies (1) for all r.

We show an inequality corresponding to (16), using $h(z)$. Setting

$$\tilde{b}(re^{i\theta})=b(re^{i\theta})-2\theta\log C$$

where b is the Poisson integral of (12) constructed by $g(z)$, we can see

$$\tilde{b}_\theta(r)=\int_0^\infty(\tilde{b}_\theta(t)+\tilde{b}_\theta(-t))Q(r,\,t)dt \tag{26}$$

where $Q(r,\,t)=(2r\log r/t)/\pi^2(r^2-t^2)$. In fact, by contour integration

$$\int_0^\infty Q(r,\,t)dt=1/2$$

and so we have (26) from (5).

If we define $\tilde{U}(z)=\tilde{b}(z^{1/\gamma})$ in $D=\{z:0<\arg z<\beta\}$, then we have from (26)

$$\tilde{U}_\theta(r^\gamma)=\int_0^\infty(\tilde{U}_\theta(t^\gamma)+\tilde{U}_\theta(t^\gamma e^{i\beta}))Q(r,\ t)dt\ ,$$

where $\tilde{U}_\theta(r^\gamma e^{i\theta})=U_\theta(r^\gamma e^{i\theta})-2\log C$.

Now we define two functions $\tilde{H}_1(z)$ and $\tilde{H}_2(z)$ in the angle $D_1=\{z:0<\arg z<\beta/2\}$ as follows:

$$\tilde{H}_1(re^{i\theta})=\tilde{U}(re^{i(\beta/2+\theta)})-\tilde{U}(re^{i(\beta/2-\theta)})\ ,$$

$$\tilde{H}_2(re^{i\theta})=\int_{-\beta/2-\theta}^{-\beta/2+\theta}\log|h(re^{i\phi})|d\phi+\int_{\beta/2-\theta}^{\beta/2+\theta}\log|h(re^{i\phi})|d\phi\ .$$

Then we have $\tilde{H}_1(r)=\tilde{H}_2(r)=0$ and

$$\tilde{H}_2(re^{i\beta/2})=H(re^{i\beta})-2\beta\log C\leqq\tilde{U}(re^{i\beta})=\tilde{H}_1(re^{i\beta/2})\ .$$

Hence we have $\tilde{H}_2(z)\leqq\tilde{H}_1(z)$ in D_1. Proceeding as in the previous case, we have the following inequality:

$$\tilde{\varphi}(r^\gamma)\leqq\int_0^\infty\tilde{\varphi}(t^\gamma)\Big(1+\cos\frac{\beta q}{2}\Big)Q(r,\ t)dt$$

where

$$\tilde{\varphi}(t^\gamma)=\begin{cases}\tilde{U}_\theta(t^\gamma) & \text{for}\quad 0\leqq t<t_2\\ 2\log|h(t^\gamma)| & \text{for}\quad t\geqq t_2\ ,\end{cases}$$

if $c_m<0$ and

$$\tilde{\varphi}(t^\gamma)=\begin{cases}\tilde{U}_\theta(t^\gamma) & \text{for}\quad t_1<t<t_2\\ 2\log|h(t^\gamma)| & \text{for}\quad 0\leqq t\leqq t_1,\ t\geqq t_2\ ,\end{cases}$$

if $c_m>0$. Thus we have a contradiction again.

Case (2). $k<l$. Since $g_1(z)$ is a canonical product of $g(z)$, we have

$$|\log|g_1(r)||=r^{k+1}\int_0^\infty\frac{n(x)}{x^{k+1}}\frac{dx}{x+r}$$

$$\leqq r^k\int_0^r\frac{n(x)}{x^{k+1}}dx+r^{k+1}\int_r^\infty\frac{n(x)}{x^{k+1}}dx$$

and so we have $|\log|g_1(r)||=o(\mathrm{Re}\,Q(r))$. Thus in this case we have

$$A=\limsup_{r\to\infty}\frac{\log|g(r)|}{r^{q/2}}=0\ .$$

Hence proceeding as in the proof of case (1.2), we have a contradiction.

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