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DIRECT CALCULATION OF MAXIMUM LIKELIHOOD ESTIMATOR FOR THE BIVARIATE POISSON DISTRIBUTION

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Summary

To estimate the parameter vector λ of bivariate Poisson distribution [1], [2] we would like to calculate maximum likelihood estimator (MLE) $\hat{\lambda}$. This MLE $\hat{\lambda}$ has not a simple expression as $\bar{X}$, S^2, $\cdots$ etc. We only have information about MLE $\hat{\lambda}$ by normal equations and its variation forms [3]. Holgate [4] shows the asymptotic property of MLE $\hat{\lambda}$.

In this paper we would like to show the calculating method of MLE $\hat{\lambda}$. The method will be constructed by direct calculation of likelihood function and by a searching routine of MLE $\hat{\lambda}$ which maximizes the function value. A sequence of random numbers come from a bivariate Poisson distribution $P(\lambda)$ will be shown. The change of the value of likelihood function varying parameter λ in our rule will be calculated and the work of the searching routine will be discussed in detail. In the last part of this paper a numerical interpretation of our routine will be shown.

Section 1. Bivariate Poisson distribution $P(\lambda)$.

If (X, Y) has a bivariate distribution,

$$P(X=k,\ Y=l)=\sum_{\substack{\beta+\delta=k\\ \gamma+\delta=l}} \frac{\lambda_{10}^{\beta}\lambda_{01}^{\gamma}\lambda_{11}^{\delta}}{\beta\,!\,\gamma\,!\,\delta\,!}\,e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}$$

we shall call (X, Y) has a bivariate Poisson distribution $P(\lambda)$, where k, l, β, γ and δ are nonnegative integers and nonnegative λ_{10}, λ_{01} and λ_{11} are called as parameters and λ means a vector of the three parameters $(\lambda_{10}, \lambda_{01}, \lambda_{11})$.

The moment generating function of the distribution is given by

$$g(s_1,\ s_2)=e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})+\lambda_{10}s_1+\lambda_{01}s_2+\lambda_{11}s_1s_2}\,.$$

And the marginal distribution is given by

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$$P(X=k)=p(k\ ;\ \lambda_{10}+\lambda_{11}) \qquad (k=0,\ 1,\ 2,\ \cdots),$$

$$P(Y=l)=p(l\ ;\ \lambda_{01}+\lambda_{11}) \qquad (l=0,\ 1,\ 2,\ \cdots),$$

where $p(*\ ;\ \lambda)$ is an univariate Poisson density. We will get more information about the distribution from the equalities:

$$E(X)=\lambda_{10}+\lambda_{11}, \qquad \mathrm{Var}(X)=\lambda_{10}+\lambda_{11},$$

$$E(Y)=\lambda_{01}+\lambda_{11}, \qquad \mathrm{Var}(Y)=\lambda_{01}+\lambda_{11},$$

$$\mathrm{Cov}(X,\ Y)=\lambda_{11}.$$

If $(X,\ Y)$ has a bivariate Poisson distribution $P(\lambda)$ then we will get the decomposition rule

$$X=X_{10}+X_{11}, \qquad Y=X_{01}+X_{11}$$

where X_{10}, X_{01} and X_{11} are mutually independent univariate Poisson distributions with parameter λ_{10}, λ_{01} and λ_{11} respectively.

But getting a sample $(x,\ y)$ of the distribution, we do not know the decomposed samples x_{10}, x_{01} and x_{11} satisfying the decomposition rule of the last two equalities. This is the main reason why it is difficult to estimate the parameters from the samples of the distribution.

Section 2. MLE of the parameter λ.

Practical estimation of the parameter $\lambda=(\lambda_{10},\ \lambda_{01},\ \lambda_{11})$.

Denote n independent sample variables of a bivariate Poisson distribution $P(\lambda)$ as

$$(X_1,\ Y_1),\ (X_2,\ Y_2),\ \cdots,\ (X_n,\ Y_n)$$

and denote the practical samples as

$$(x_1,\ y_1),\ (x_2,\ y_2),\ \cdots,\ (x_n,\ y_n).$$

The maximum likelihood estimator MLE of $\lambda_{10}+\lambda_{11}$ will be given by $\sum_{i=1}^{n} x_i/n$ and MLE of $\lambda_{01}+\lambda_{11}$ by $\sum_{i=1}^{n} y_i/n$. To estimate the parameters λ_{10}, λ_{01} and λ_{11} individually we need to discuss the method of the estimation in a more delicate way.

2-1. Practical estimation of covariance value λ_{11}.

We shall consider the problem how to estimate λ_{11}.

THEOREM 1. *From n independent samples* $(x_1,\ y_1)$, $(x_2,\ y_2)$, $\cdots$, $(x_n,\ y_n)$, MLE *of* λ_{10}, λ_{01} *and* λ_{11} *is given by next three equalities,*

$$\frac{1}{n}\sum_{i=1}^{n}\frac{p(x_i-1,\ y_i)}{p(x_i,\ y_i)}=1,$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{p(x_i,\ y_i-1)}{p(x_i,\ y_i)}=1$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\frac{p(x_i-1,\ y_i-1)}{p(x_i,\ y_i)}=1\,,$$

where $p(k,\ l)=P(X=k,\ Y=l)$ *with parameters* λ_{10}, λ_{01} *and* λ_{11}.

We usually call the relations the *normal equalities*.

Proof of the theorem. To maximize the likelihood function $\prod_{i=1}^{n}p(x_i,\ y_i)$ with respect to the parameters λ_{10}, λ_{01} and λ_{11}, we have to differentiate the logarithm of likelihood function about the parameters and to put its value zero.

$$\frac{\partial}{\partial\lambda_{10}}\prod_{i=1}^{n}p(x_i,\ y_i)=\sum_{i=1}^{n}\frac{\frac{\partial}{\partial\lambda_{10}}p(x_i,\ y_i)}{p(x_i,\ y_i)}\prod_{i=1}^{n}p(x_i,\ y_i)=0\,,$$

where the differential of the probability density is given by a convenient relation

$$\frac{\partial}{\partial\lambda_{10}}p(x,\ y)=p(x-1,\ y)-p(x,\ y)\,.$$

Because, we have

$$\begin{aligned}\frac{\partial}{\partial\lambda_{10}}p(k,\ l)&=\frac{\partial}{\partial\lambda_{10}}\sum_{\substack{\beta+\delta=k\\ \gamma+\delta=l}}\frac{\lambda_{10}^{\beta}\lambda_{01}^{\gamma}\lambda_{11}^{\delta}}{\beta!\,\gamma!\,\delta!}e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}\\ &=\sum_{\substack{\beta+\delta=k\\ \gamma+\delta=l\\ \beta-1\geqq 0}}\frac{\lambda_{10}^{\beta-1}\lambda_{01}^{\gamma}\lambda_{11}^{\delta}}{(\beta-1)!\,\gamma!\,\delta!}e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}-\sum_{\substack{\beta+\delta=k\\ \gamma+\delta=l}}\frac{\lambda_{10}^{\beta}\lambda_{01}^{\gamma}\lambda_{11}^{\delta}}{\beta!\,\gamma!\,\delta!}e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}\\ &=\sum_{\substack{\beta+\delta=k-1\\ \gamma+\delta=l}}\frac{\lambda_{10}^{\beta}\lambda_{01}^{\gamma}\lambda_{11}^{\delta}}{\beta!\,\gamma!\,\delta!}e^{-\lambda_{10}-\lambda_{01}-\lambda_{11}}-p(k,\ l)\\ &=p(k-1,\ l)-p(k,\ l)\,.\end{aligned}$$

Therefore, we can verify,

$$\begin{aligned}\frac{\partial}{\partial\lambda_{10}}\prod_{i=1}^{n}p(x_i,\ y_i)&=\sum_{i=1}^{n}\frac{\frac{\partial}{\partial\lambda_{10}}p(x_i,\ y_i)}{p(x_i,\ y_i)}\prod_{i=1}^{n}p(x_i,\ y_i)\\ &=\sum_{i=1}^{n}\frac{p(x_i-1,\ y_i)-p(x_i,\ y_i)}{p(x_i,\ y_i)}\prod_{i=1}^{n}p(x_i,\ y_i)\\ &=\left[\sum_{i=1}^{n}\frac{p(x_i-1,\ y_i)}{p(x_i,\ y_i)}-n\right]\prod_{i=1}^{n}p(x_i,\ y_i)\,.\end{aligned}$$

The equivalent condition of the normal equation

$$\frac{\partial}{\partial\lambda_{10}}\prod_{i=1}^{n} p(x_i,\ y_i)=0$$

is given by

$$\sum_{i=1}^{n}\frac{p(x_i-1,\ y_i)}{p(x_i,\ y_i)}=n\ .$$

By using similar calculations about

$$\frac{\partial}{\partial\lambda_{01}}\prod_{i=1}^{n} p(x_i,\ y_i)=0$$

and

$$\frac{\partial}{\partial\lambda_{11}}\prod_{i=1}^{n} p(x_i,\ y_i)=0\ ,$$

we will be given the equivalent conditions

$$\sum_{i=1}^{n}\frac{p(x_i,\ y_i-1)}{p(x_i,\ y_i)}=n$$

and

$$\sum_{i=1}^{n}\frac{p(x_i-1,\ y_i-1)}{p(x_i,\ y_i)}=n$$

respectively. ■

THEOREM 2. *MLE of* λ_{10}, λ_{01} *and* λ_{11} *denoted as* $\hat{\lambda}_{10}$, $\hat{\lambda}_{01}$ *and* $\hat{\lambda}_{11}$ *satisfy* $\hat{\lambda}_{10}+\hat{\lambda}_{11}=\bar{x}$ *and* $\hat{\lambda}_{01}+\hat{\lambda}_{11}=\bar{y}$.

Proof of the theorem. From the first normal equation

$$\sum_{i=1}^{n}\frac{p(x_i-1,\ y_i)}{p(x_i,\ y_i)}=n$$

and a relation:

$$kp(k,\ l)=\lambda_{10}p(k-1,\ l)+\lambda_{11}p(k-1,\ l-1)\ ,$$

we have

$$\sum_{i=1}^{n}\frac{p(x_i-1,\ y_i)-p(x_i,\ y_i)}{p(x_i,\ y_i)}=0$$

and

$$\lambda_{10}p(x_i-1,\ y_i)=x_i p(x_i,\ y_i)-\lambda_{11}p(x_i-1,\ y_i-1)\ .$$

Then, the first normal equation is expressed as following:

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{\lambda_{10}p(x_i-1,\ y_i)-\lambda_{10}p(x_i,\ y_i)}{p(x_i,\ y_i)}\\
&=\sum_{i=1}^{n}\frac{x_i p(x_i,\ y_i)-\lambda_{11}p(x_i-1,\ y_i-1)-\lambda_{10}p(x_i,\ y_i)}{p(x_i,\ y_i)}\\
&=\sum_{i=1}^{n}\left[(x_i-\lambda_{10})-\lambda_{11}\frac{p(x_i-1,\ y_i-1)}{p(x_i,\ y_i)}\right]\\
&=\sum_{i=1}^{n}x_i-n\lambda_{10}-\lambda_{11}\sum_{i=1}^{n}\frac{p(x_i-1,\ y_i-1)}{p(x_i,\ y_i)}=0\ ,
\end{aligned}$$

and by the third normal equation, we have

$$\sum_{i=1}^{n} x_i - n(\lambda_{10}+\lambda_{11}) = 0$$

and a concluding equation

$$\lambda_{10}+\lambda_{11} = \sum_{i=1}^{n} x_i/n = \bar{x} .$$

Similarly, we have another concluding equation from the second and the third normal equations

$$\lambda_{01}+\lambda_{11} = \sum_{i=1}^{n} y_i/n = \bar{y} .$$

These two simple relations come only from the normal equations and λ_{10}, λ_{01} and λ_{11} must be symbolized by MLE $\hat{\lambda}_{10}$, $\hat{\lambda}_{01}$ and $\hat{\lambda}_{11}$. ■

Further calculation about $\hat{\lambda}_{10}$, $\hat{\lambda}_{01}$ and $\hat{\lambda}_{11}$ is very difficult and we have to calculate the individual estimator by numerical method. In the next section we shall treat the practical calculating method showing how to get MLE $\hat{\lambda}_{10}$, $\hat{\lambda}_{01}$ and $\hat{\lambda}_{11}$.

2-2. Practical calculation of MLE $\hat{\lambda}=(\hat{\lambda}_{10}, \hat{\lambda}_{01}, \hat{\lambda}_{11})$.

THEOREM 3. *MLE $\hat{\lambda}_{10}$, $\hat{\lambda}_{01}$ and $\hat{\lambda}_{11}$ satisfy the relations*

$$\hat{\lambda}_{10}+\hat{\lambda}_{11}=\bar{x} , \quad \hat{\lambda}_{01}+\hat{\lambda}_{11}=\bar{y} \quad and \quad 0 \leqq \hat{\lambda}_{11} \leqq \min(\bar{x}, \bar{y})$$

which maximize the logarithm of likelihood function $\log \prod_{i=1}^{n} p(x_i, y_i)$.

Proof of the theorem. This theorem is a representation of the last theorem for our calculation of MLE. Denote $m=\min(\bar{x}, \bar{y})$ for simplicity of notation, λ_{10}, λ_{01} and λ_{11} are nonnegative parameters, so that we have $0 \leqq \hat{\lambda}_{11} \leqq m$. To get $\hat{\lambda}_{11}$ we have to move λ_{11} in the interval $[0, m]$ which maximize the logarithm of likelihood function *llf*. ■

To calculate MLE $\hat{\lambda}_{10}$, $\hat{\lambda}_{01}$ and $\hat{\lambda}_{11}$ numerically, we have to compare the value of *llf* on the scanning space of the three parameters. This theorem states that we may find MLE which maximizes the value of *llf* in one dimensional parameter space, that is, our scanning space of the parameters reduce to one dimensional subspace;

$$\lambda_{10}=\bar{x}-\lambda_{11} , \quad \lambda_{01}=\bar{y}-\lambda_{11} \quad \text{and} \quad \lambda_{11} \in [0, m] .$$

At the begining of computation, we compare the value *llf* in the rule $\lambda_{11}=0.0, 1.0, 2.0, \cdots \leqq m=(\bar{x}, \bar{y})$ and $\lambda_{10}=\bar{x}-\lambda_{11}$, $\lambda_{01}=\bar{y}-\lambda_{11}$, that is, λ_{11} moves on nonnegative integers from 0 to the integer lower than m. We will find one λ_{11} which maximizes *llf* or two λ_{11} which maximize the function. Usually we get only one λ_{11} which maximizes *llf* and occasionally we look for two λ_{11} which maximize the function. In the first case we have to look for λ_{11} in the interval of both sides of λ_{11} which maximizes the function and we have adjusted the scanning step in the reduced parameter space to one quater of preceding scanning step 1.0.

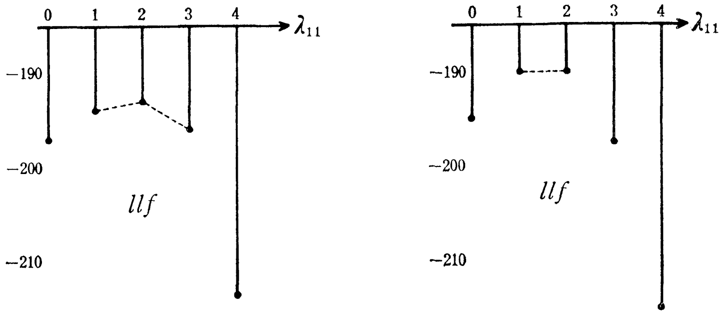


Fig. 1

2-3. Automatic reduction rule of λ_{11} scanning space.

Let us discuss in detail the reduction rule. We put the scanning space of λ_{11} as D_0 initially,

$$D_0=\{0,\ 1,\ 2,\ \cdots,\ m_0\},$$

where m_0 equals to the maximum integer lower than $m=\min(\bar{x},\ \bar{y})$.

(case 1) $\lambda_{11}=0$ maximizes *llf* in the scanning space D_0.

(case 2) One of $\lambda_{11}=k$ in $1,\ 2,\ \cdots,\ m_0-1$ maximizes *llf* in the scanning space D_0.

(case 3) Two of $\lambda_{11}=k,\ k+1$ in $0,\ 1,\ 2,\ \cdots,\ m_0$ maximize *llf* in the scanning space D_0.

(case 4) $\lambda_{11}=m_0$ maximizes *llf* in the scanning space D_0.

Scanning $\lambda_{11}\in D_0$ as to maximize *llf*, we have four cases (case 1), $\cdots$, (case 4). In every case, to compute MLE $\hat{\lambda}_{11}$ more detail, we have to reduce the scanning space and the scanning step. We have used the secondary scanning step as one quater of the preceding one. Then we have a new scanning space D_1 as following:

(case 1) $D_1=\left\{0,\ \frac{1}{4},\ \frac{2}{4},\ \frac{3}{4},\ 1\right\}$

(case 2) $D_1=\left\{k-1,\ k-1+\frac{1}{4},\ k-1+\frac{2}{4},\ k-1+\frac{3}{4},\ k,\ k+\frac{1}{4},\ k+\frac{2}{4},\ k+\frac{3}{4},\ k+1\right\}$

(case 3) $D_1=\left\{k,\ k+\frac{1}{4},\ k+\frac{2}{4},\ k+\frac{3}{4},\ k+1\right\}$

(case 4) $D_1=\left\{m_0-1,\ m_0-1+\frac{1}{4},\ m_0-1+\frac{2}{4},\ m_0-1+\frac{3}{4},\ m_0,\ m_0+\frac{1}{4},\ \cdots,\ m_0+\frac{t}{4}\right\}$

where t is the maximum integer in 0, 1, 2, 3 such that

$$m_0+\frac{t}{4}\leqq m=\min(\bar{x}, \bar{y}).$$

In the primary step, we compare the value of llf by λ_{11} scanning only the integral values on $[0, m]$, as to find λ_{11} which maximizes llf. Secondary step, if we find one λ_{11} maximizing llf as (case 2), we can reduce the scanning interval $[0, m]$ to $[k-1, k+1]$ where k is the value of λ_{11} maximizing llf. If we find one $\lambda_{11}=0$ maximizing llf as (case 1), we can reduce the scanning interval to $[0, 1]$ and if we find one $\lambda_{11}=m_0$ maximizing llf as (case 4), we can reduce the scanning interval to $[m_0-1, m]$. If we find two $\lambda_{11}=k$, $k+1$ $(k=0, 1, \cdots, m_0-1)$ maximizing llf we can reduce the scanning interval to $[k, k+1]$ as (case 3). Our scanning of λ_{11} is made in the reduced interval and the scanning step is adjusted to a quater step of preceding scanning step, where the scanning step of $\hat{\lambda}_{11}$ in the reduced interval may change under another reduction rule and the total computing time will change. Then we can reduce our scanning space D_0 to D_1 as denoted above.

Under this inductive routine, if we set 0.001 as the exactness of the calculation of MLE $\hat{\lambda}_{11}$, then we will obtain MLE $\hat{\lambda}_{11}$ involving the exactness after high resolution computing method.

Section 3. A sequence of random numbers from $P(\lambda)$ and computation of MLE $\hat{\lambda}_{11}$ by computer.

In this section a result of computer simulation will be demonstrated, one is a change of llf under our reduced linear space, and the other is a computing process of finding MLE $\hat{\lambda}_{11}$.

3-1. Simulation of a sequence of bivariate Poisson distribution $P(\lambda)$.

Given parameters λ_{10}, λ_{01} and λ_{11}, X_{10}, X_{01} and X_{11} are independent univariate Poisson distributions then (X, Y) from a bivariate Poisson distribution is expressed by

$$X=X_{10}+X_{11} \quad \text{and} \quad Y=X_{01}+X_{11}.$$

To make a sequence of random numbers of the distribution $P(\lambda)$, we should make three series of independent univariate Poisson random variables X_{10}, X_{01} and X_{11}.

Following figure is our flow-chart of making bivariate Poisson random variables.

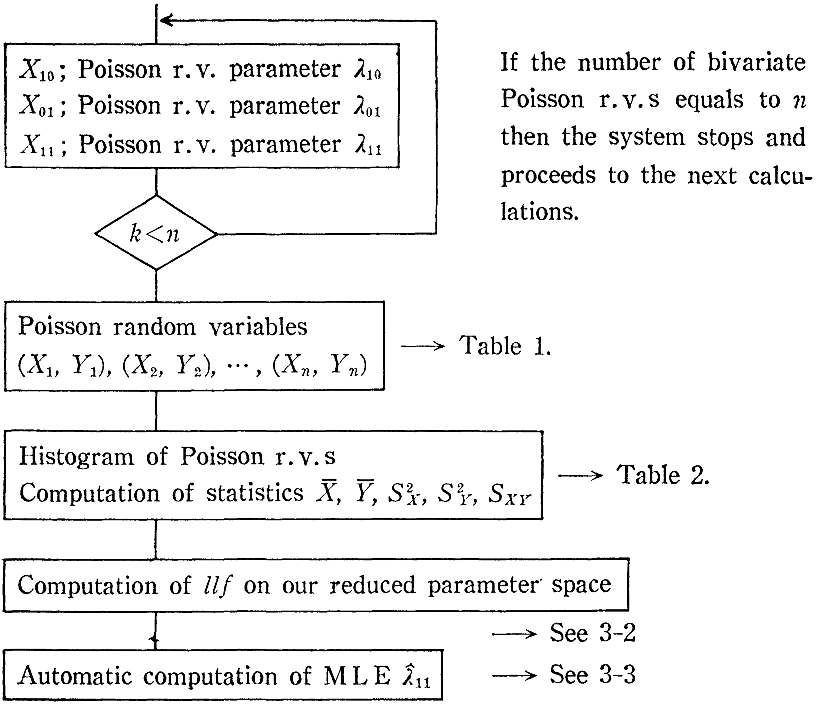


Fig. 2

Poisson random variables with parameters $\lambda_{10}=3.0$, $\lambda_{01}=3.0$ and $\lambda_{11}=2.0$

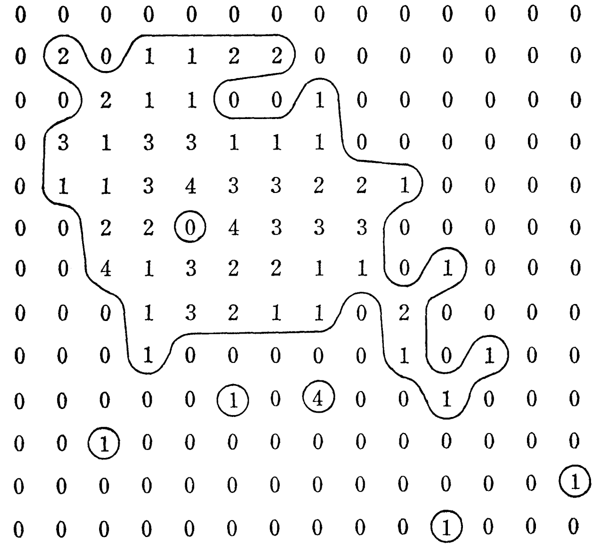
(6, 5)	(9, 5)	(2, 4)	(7, 3)	(6, 4)	(6, 4)	(9, 7)	(9, 7)	(1, 3)	(4, 6)
(5, 7)	(3, 7)	(3, 4)	(3, 3)	(4, 5)	(4, 4)	(5, 7)	(5, 5)	(1, 6)	(6, 10)
(4, 3)	(6, 2)	(5, 6)	(5, 6)	(6, 2)	(6, 6)	(3, 5)	(3, 2)	(5, 7)	(1, 4)
(3, 3)	(7, 5)	(6, 5)	(6, 4)	(2, 3)	(7, 5)	(6, 3)	(3, 3)	(6, 2)	(4, 6)
(1, 5)	(9, 10)	(5, 6)	(4, 5)	(10, 2)	(4, 2)	(7, 4)	(5, 8)	(4, 4)	(1, 1)
(5, 2)	(7, 9)	(4, 7)	(4, 4)	(4, 8)	(7, 4)	(5, 5)	(5, 5)	(4, 6)	(11, 13)
(5, 3)	(6, 6)	(3, 6)	(8, 11)	(7, 9)	(5, 8)	(4, 3)	(5, 5)	(2, 7)	(6, 7)
(3, 1)	(6, 8)	(4, 5)	(12, 10)	(2, 2)	(8, 3)	(8, 9)	(6, 2)	(9, 7)	(3, 4)
(5, 2)	(4, 9)	(5, 8)	(4, 7)	(9, 7)	(5, 3)	(4, 1)	(7, 7)	(3, 1)	(4, 4)
(1, 5)	(1, 6)	(3, 1)	(4, 8)	(7, 6)	(3, 4)	(7, 4)	(2, 2)	(1, 1)	(4, 3)

Table 1

Histogram of the Poisson random variables

$l=0, 1, 2, \cdots$

$k=0, 1, 2, \vdots$



0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	2	0	1	1	2	2	0	0	0	0	0	0	0
0	0	2	1	1	0	0	1	0	0	0	0	0	0
0	3	1	3	3	1	1	1	0	0	0	0	0	0
0	1	1	3	4	3	3	2	2	1	0	0	0	0
0	0	2	2	⓪	4	3	3	3	0	0	0	0	0
0	0	4	1	3	2	2	1	1	0	1	0	0	0
0	0	0	1	3	2	1	1	0	2	0	0	0	0
0	0	0	1	0	0	0	0	0	1	0	1	0	0
0	0	0	0	0	①	0	④	0	0	1	0	0	0
0	0	①	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	①
0	0	0	0	0	0	0	0	0	0	①	0	0	0

Statistics based on the Poisson random variables.

$$\bar{X}=4.93, \qquad S_X^2=5.3251$$
$$\bar{Y}=5.03, \qquad S_Y^2=6.1891$$
$$S_{XY}=2.5021$$

where $S_{XY}=\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})/n$.

Table 2

3-2. The change of *llf* under our scanning rule.

Our reduced one dimensional parameter space is expressed as

$$\lambda_{10}=\bar{X}-\lambda_{11}, \quad \lambda_{01}=\bar{Y}-\lambda_{11} \quad \text{and} \quad \lambda_{11}\in[0, m]$$

where $m=\min(\bar{x}, \bar{y})$. From $\bar{x}=4.93$, $\bar{y}=5.03$, we get $m=4.93$ so that our reduced parameter space is expressed;

$$\lambda_{10}=4.93-\lambda_{11}, \quad \lambda_{01}=4.93-\lambda_{11} \quad \text{and} \quad \lambda_{11}\in[0, 4.93].$$

We tried to calculate *llf* on the space under a scanning step 0.1 of the third parameter λ_{11}. Next table and graph explain a relation of variable λ_{11} and *llf*. And we will find MLE of λ_{11} close to $\lambda_{11}=1.9$.

λ_{11}	llf	λ_{11}	llf
0.0	−197.25	1.5	−193.2464
0.5	−195.38	1.6	−193.1536
1.0	−194.06	1.7	−193.0868
1.5	−193.25	1.8	−193.0479
2.0	−193.06	1.9	−193.0389
2.5	−193.79	2.0	−193.0625
3.0	−196.07	2.1	−193.1216
3.5	−201.36	2.2	−193.2196
4.0	−213.58	2.3	−193.3607
4.5	−247.47	2.4	−193.5494
		2.5	−193.7915

Table 3

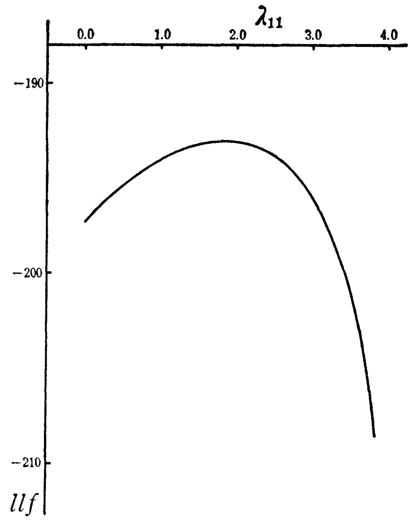


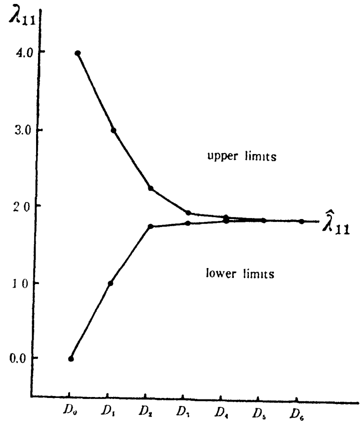
Fig. 3

3-3. Automatic computing process of MLE $\hat{\lambda}_{11}$.

If we would like to know the function of our automatic computing process of MLE $\hat{\lambda}_{11}$, we can easily pull out the scanning spaces D_0, D_1, $\cdots$ and llf values on each spaces. Following table and graph explain the function.

	Lower limit	Upper limit	Scan
D_0	0	4	1
D_1	1	3	1/4
D_2	1+3/4	1+5/4	1/16
D_3	1+3/4+1/16	1+3/4+3/16	1/64
D_4	1+3/4+1/16+3/64	1+3/4+1/16+5/64	1/256
D_5	1+3/4+1/16+3/64+4/256	1+3/4+1/16+3/64+6/256	1/1024
D_6	1+3/4+1/16+3/64+4/256 +3/1024	1+3/4+1/16+3/64+4/256 +5/1024	

Table 4



Graph of upper and lower limits of reduced scanning spaces D_0, D_1, $\cdots$.

Fig. 4

After our automatic computing process of MLE, we get MLE $\hat{\lambda}_{11}=1.879$ and from the assertion of theorem 3, we get MLE $\hat{\lambda}_{10}=4.93-1.879=3.051$ and MLE $\hat{\lambda}_{01}=5.03-1.879=3.151$. These are the maximum likelihood estimators of the bivariate Poisson random variables simulated by computer. We have another estimator for λ_{11},

$$S_{XY}=\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})/n=\sum_{i=1}^{n}X_iY_i/n-\bar{X}\bar{Y}=2.5021.$$

The parameter used to the simulation was $\lambda_{10}=3.0$, $\lambda_{01}=3.0$ and $\lambda_{11}=2.0$. And our MLE is expressed as $\hat{\lambda}_{10}=3.051$, $\hat{\lambda}_{01}=3.151$ and $\hat{\lambda}_{11}=1.879$. Another estimation for λ_{11} is given by $S_{XY}=2.5021$.

Remark. The aim of the last section is to answer the questions: how to make the bivariate Poisson random variables (by simulation) and how to calculate MLE of the parameter $\lambda=(\lambda_{10}, \lambda_{01}, \lambda_{11})$. But the next question to answer would be how to check the fitness of the bivariate Poisson distribution for given bivariate data (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) as given in table 1.

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