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# FIELDS OF TOTALLY ISOTROPIC SUBSPACES AND ALMOST COMPLEX STRUCTURES

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**Abstract**

Our aim is to prove global existence of a differentiable field of totally isotropic planes over the sphere $S_4$, to obtain from this result a counterexample nullifying a conjecture about some manifolds or fibre bundles.

Probably according an assertion given in rather conjectural way in [2], some people, papers or books consider the following statement as an obviousness:

*If the complexified tangent bundle $T_C(M)$ of a $2r$-dimensional real $C^\infty$ manifold $M$ is a Whitney sum:*

(1) $$T_C(M)=\eta\oplus\eta'$$

*where $\eta$, $\eta'$ are r-complex subbundles, such that for any $x$ belonging to $M$, $\eta'_x=\bar{\eta}_x$, then $M$ owns an almost complex structure. More generally, if a real vector fiber bundle $\xi$, with base $M$ and rank $r$ is such that the complexified bundle $\xi_C$ is a Whitney sum:*

*$\xi_C=\xi_1\oplus\xi'_1$, $\xi_1$, $\xi'_1$, rank $r$ complexe subbundles, with $\xi'_1(x)=\bar{\xi}_1(x)$, $\forall x\in M$, then $\xi$ owns an r-complex structure.*

This paper intends to etablish that statement is *erroneous*, building a such decomposition over $M=S_4$: indeed it is well known that $S_4$ doesn't admit any almost complex structure. One can define the sphere $S_4$ by angular parameters $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$, according the following equations:

(2) $$\begin{cases} x_1=\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4, & \\ x_2=\sin\theta_1\cos\theta_2\cos\theta_3\cos\theta_4, & -\pi\leqq\theta_1\leqq\pi, \\ x_3=\sin\theta_2\cos\theta_3\cos\theta_4, & \\ x_4=\sin\theta_3\cos\theta_4, & -\dfrac{\pi}{2}\leqq\theta_2,\ \theta_3,\ \theta_4\leqq\dfrac{\pi}{2}, \\ x_5=\sin\theta_4. & \end{cases}$$

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With inequalities in the strict meaning, these equations define a diffeomorphism from a hypercubic set in $\boldsymbol{R}^4$ to an open dense set in $S_4$.

LEMMA 1. *There exists over the sphere $S_3$, defined by:*

$$x_2^2+x_3^2+x_4^2+x_5^2=1,$$

*a global field of real orthonormed frames.*

It's a classical result: $T(S_3)$ is a trivializable bundle. If the $e_i$, $i=1, 2, 3, 4, 5$ constitute a direct orthonormed frame in $\boldsymbol{R}^5$, it is sufficient to choose:

$$\text{(3)} \qquad \begin{cases} U=-x_3e_2+x_2e_3-x_5e_4+x_4e_5, \\ V=-x_4e_2+x_5e_3+x_2e_4-x_3e_5, \\ W=-x_5e_2-x_4e_3+x_3e_4+x_2e_5. \end{cases}$$

LEMMA 2. *There exists over the sphere $S_2$, defined by:*

$$x_3^2+x_4^2+x_5^2=1,$$

*a field of isotropic directions in the complexified tangent bundle $T_C(S_2)$.*

Let $S_2$ be defined by:

$$\text{(4)} \qquad x_3=\cos\theta_3\cos\theta_4,\ x_4=\sin\theta_3\cos\theta_4,\ x_5=\sin\theta_4,$$

$$-\pi\leqq\theta_3\leqq\pi, \qquad -\frac{\pi}{2}\leqq\theta_4\leqq\frac{\pi}{2}.$$

Let $\frac{1}{2}(\mathring{S}_1)_0$ be the half open circle corresponding to $\theta_3=0$, $-\frac{\pi}{2}<\theta_4<\frac{\pi}{2}$, and $\frac{1}{2}(\mathring{S}_1)_{\theta_3}$ the intersection of $S_2$ with the half plane $\theta_3$, for some $\theta_3$.

Over $\frac{1}{2}(\mathring{S}_1)_0$, in $T_C(S_2)$, we consider a field of isotropic vectors, everywhere different from 0.

$$u_0=-e_3\sin\theta_4+e_5\cos\theta_4+ie_4;$$

from which by a $\theta_3$-rotation around $e_5$ we deduce:

$$\text{(5)} \qquad u_{\theta_3}=-e_3(\sin\theta_4\cos\theta_3+i\sin\theta_3)-e_4(\sin\theta_4\sin\theta_3-i\cos\theta_3)+e_5\cos\theta_4.$$

a field of isotropic vectors, everywhere different from 0, over $\frac{1}{2}(\mathring{S}_1)_{\theta_3}$, in $T_C(S_2)$. Varying $\theta_3$, between $(-\pi)$ and $(+\pi)$ appears a field of isotropic vectors over $S_2$. Indeed, it is sufficient to verificate that $\theta_4$ tending to $\pm\frac{\pi}{2}$(we obtain thus $\pm e_5$), gives an unique isotropic direction, to verify:

$$(\exp i\theta_3)(-e_3+ie_4), \qquad \text{if} \quad \theta_4 \to \frac{\pi}{2}.$$

Elsewhere there is no difficulty.

LEMMA 3. *$S_3$ is a principal fibre bundle, with base $S_2$ and fibre $S_1$.*

This is a classical result pointed out by Hopf (good reference in Dieudonné, Cours d'Analyse, Tome 3, p. 71-72). $S_3$ is defined by the equations:

$$(6) \qquad \begin{cases} x_2=\cos\theta_2\cos\theta_3\cos\theta_4, & \\ x_3=\sin\theta_2\cos\theta_3\cos\theta_4, & -\pi\leqq\theta_2\leqq\pi, \\ x_4=\sin\theta_3\cos\theta_4, & \\ x_5=\sin\theta_4 & -\frac{\pi}{2}\leqq\theta_3,\ \theta_4\leqq\frac{\pi}{2}. \end{cases}$$

$\mathring{S}_3$ is the open set corresponding to:

$$-\pi<\theta_2<\pi,\ -\frac{\pi}{2}<\theta_3,\ \theta_4<\frac{\pi}{2},$$

diffeomorphic with a cubic set in $\boldsymbol{R}^3$ and if we choose:

$$-\frac{\pi}{2}<\theta_2,\ \theta_3,\ \theta_4<\frac{\pi}{2},$$

we define $\frac{1}{2}(\mathring{S}_3)$ also diffeomorphic with a cubic set. According the quoted book, the bundle projection $p$ over $S_2$:

$$x'=p(x)=p(x_2,\ x_3,\ x_4,\ x_5)$$

is stated by:

$$x_3'=2\,\mathrm{Im}\,(u\bar{v}),\ x_4'=2\,\mathrm{Re}\,(u\bar{v}),\ x_5'=|u|^2-|v|^2$$

with: $u=x_2+ix_5,\ v=x_3+ix_4$.

If $x\in\frac{1}{2}S_2$, the hemisphere corresponding to $-\frac{\pi}{2}\leqq\theta_3\leqq\frac{\pi}{2}$ in (4)—also defined by $\theta_2=\frac{\pi}{2}$ in (6)—$x'$ is the point of $\frac{1}{2}(S_2)$ with coordinates:

$$(7) \qquad \begin{cases} x_3'=\cos\theta_3\cos\left(2\theta_4-\frac{\pi}{2}\right), \\ x_4'=\sin\theta_3\cos\left(2\theta_4-\frac{\pi}{2}\right), \\ x_5'=\sin\left(2\theta_4-\frac{\pi}{2}\right); \end{cases}$$

whereas, for the symmetric hemisphere $\frac{1}{2}(S_2')\left(\theta_2=-\frac{\pi}{2}\text{ in (6)}\right)$ we obtain.

$$\text{(8)}\qquad \begin{cases} x_3'=-\cos\theta_3\cos\left(2\theta_4-\frac{\pi}{2}\right), \\ x_4'=\sin\theta_3\cos\left(2\theta_4-\frac{\pi}{2}\right), \\ x_5'=\sin\left(2\theta_4-\frac{\pi}{2}\right). \end{cases}$$

Writing abusively $x=(\theta_2, \theta_3, \theta_4)$, when $x$ belongs to $\mathring{S}_3$, so that $\left(\theta_2, \theta_3, \frac{\theta_4}{2}+\frac{\pi}{4}\right)$ also belong to $\mathring{S}_3$, we define $p': \mathring{S}_3\to S_2$ according:

$$x^0=p\left(\theta_2, \theta_3, \frac{\theta_4}{2}+\frac{\pi}{4}\right)=p'(\theta_2, \theta_3, \theta_4)=p'(x)$$

and we see immediately:

$$x=p'(x), \qquad \text{if} \quad x\in S_2\cap\mathring{S}_3.$$

*Demonstration.*

*First step.* Let $A$, $A_1$ be two fields of vectors over $\frac{1}{2}(\mathring{S}_3)$ in $T_C(S_4)$, with:

$$\begin{cases} A=x_3^0U+x_4^0V+x_5^0W+ie_1, \\ A_1=\alpha_3U+\alpha_4V+\alpha_5W, \end{cases}$$

we have put $x^0=p'(x)$, $x\in\frac{1}{2}(\mathring{S}_3)$, $U$, $V$, $W$ respectively for $U(x)$, $V(x)$, $W(x)$, and $\alpha_3$, $\alpha_4$, $\alpha_5$ for $\alpha_3(x^0)$, $\alpha_4(x_0)$, $\alpha_5(x^0)$, which are the components of the isotropic direction introduced by lemma 2, and:

$$(\alpha_3)^2(x^0)+(\alpha_4)^2(x_0)+(\alpha_5)^2(x_0)=0,$$

$$\alpha_3(x_0)x_3^0+\alpha_4(x_0)x_4^0+\alpha_5(x_0)x_5^0=0.$$

We observe that definition is tributary of an arbitrary complex coefficient.

One can easily verify that the pair $(A, A_1)$ defines over $\frac{1}{2}(\mathring{S}_3)$ a field of totally isotropic planes.

*Second step*: Now consider the rotation $\theta_1-\frac{\pi}{2}$, $-\pi\leqq\theta_1\leqq\pi$, in the plane $(e_1, e_2)$; $\frac{1}{2}(\mathring{S}_3)$ generates an open set $\mathring{S}_4$ of $S_4$, dense in $S_4$, $A$ and $A_1$ give $\hat{A}$ and $\hat{A}_1$ respectively:

$$\hat{A}=-[(x_3^0x_3+x_4^0x_4+x_5^0x_5)\cos\theta_1+i\sin\theta_1]e_1$$

$$-[(x_3^0x_3+x_4^0x_4+x_5^0x_5)\sin\theta_1-i\cos\theta_1]e_2$$
$$+(x_3^0x_2+x_4^0x_5-x_5^0x_4)e_3+(-x_3^0x_5+x_4^0x_2+x_5^0x_3)e_4$$
$$+(x_3^0x_4-x_4^0x_3+x_5^0x_2)e_5,$$

$$\hat{A}_1=-[(\alpha_3x_3+\alpha_4x_4+\alpha_5x_5)\cos\theta_1]e_1-[(\alpha_3x_3+\alpha_4x_4+\alpha_5x_5)\sin\theta_1]e_2$$
$$+(\alpha_3x_2+\alpha_4x_5-\alpha_5x_4)e_3+(-\alpha_3x_5+\alpha_4x_2+\alpha_5x_3)e_4$$
$$+(\alpha_3x_4-\alpha_4x_3+\alpha_5x_2)e_5.$$

Thus, $\hat{A}$ and $\hat{A}_1$ define over $\mathring{S}_4$, in $T_C(\mathring{S}_4)$, a totally isotropic differentiable field of planes.

Concerning $S_4$, we must verify that any ambiguity appears when $\theta_2$, $\theta_3$, $\theta_4$ tend either to $\pm\frac{\pi}{2}$, so that we can complete the definition of $\hat{A}$ and $\hat{A}_1$ by mean of limits. Indeed, we verify, in each case that:

$\hat{A}$ determines over $S_2$ the direction:

$$(-\cos\theta_1+i\sin\theta_1)(e_1+ie_2).$$

and $\hat{A}_1$, the direction:

$$\alpha_3(x^0)e_3+\alpha_4(x^0)e_4+\alpha_5(x^0)e_5.$$

This ends the proof, because $(\hat{A}, \hat{A}_1)$ defines a subbundle $\eta$, with rank 2, such that:

$$T_x^C(S_4)=\eta_x\oplus\bar{\eta}_x,$$

the metric being positive definite, but the vector bundle $\eta$, doesn't admit a direct Whitney factor $\eta'$ such that cocycles of $\eta$ and $\eta'$ are conjugate (only $\eta_x=\bar{\eta}_x$, $\forall x\in M$), otherwise $S_4$ would own an almost complex structure, that is impossible. However classicaly, $S_4$ admits a spinor structure (in the strict meaning) and it's possible to prove that statement (1) caracterises spinor structure, in the large meaning. We demonstrated that in [1, p. 61].

## REFERENCES


[1] A. CRUMEYROLLE, Spin fibrations and generalised twistors, Proceedings of Symposia in Pure Mathematics, Stanford 1973. A. M. S. Providence R. I. 1975. Vol. XXVII. Part 1, p. 53.

[2] C. EHRESMANN, Sur les variétés presque complexes, Proceed. of the international congress of math. Cambridge U. S. A. 1950, vol. II, p. 414.



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