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ON THE BOUNDARY OBSTRUCTIONS TO THE JACOBIAN PROBLEM

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§ 1. Introduction

Let K be an algebraically closed field of characteristic zero and let $f(X, Y)$ and $g(X, Y)$ be polynomials with K-coefficients which satisfy the Jacobian condition:

$$J(f, g)=f_X g_Y - f_Y g_X = 1 \tag{1.1}$$

where f_X, f_Y etc. are respective partial derivatives. The so-called Jacobian conjecture is the following.

(J. C. I) "If (1.1) is satisfied, X and Y are polynomials of f and g".

Typical examples are given by elementary transformations which are defined by finite compositions of the following transformations.

(i) $(f, g)=(aX+bY+e, cX+dY+e')$ where $ad-bc\neq 0$ or

(ii) $(f, g)=(X, Y+h(X))$ where $h(X)$ is an arbitrary polynomial.

By the theorem of Jung [J], (J. C. I) is equivalent to

(J. C. II) "If (1.1) is satisfied, (f, g) is an elementary transformation".

Let m=degree (g) and let g_m be the m-th homogeneous part of g. Among the various results about (J. C. I), the following is due to Abyankar [Ab]:
$g_m=0$ has at most 2 points in $P^1(K)$ if (1.1) is satisfied by f and g.
It is easy to prove that (J. C. II) is equivalent to (J. C. III) (See § 4.):

(J. C. III) "If $g_m=0$ consists of two points, there is no polynomial f such that $J(f, g)=1$".

In this paper, we study the necessary conditions ("boundary obstruction") of the boundaries of the Newton polygon $N(g)$ for the existence of f such that $J(f, g)=1$. Unfortunately there exist polynomials which have no obstructions on the boundary. Our main results are in § 6 (Theorem (6.3) etc.).

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§ 2. Jacobian problem for weighted homogeneous rational functions I.

Parts of the results of this and the next section are obtained by Abyankar [Ab] but our approach is made from a different point of view.

DEFINITION (2.1). A polynomial $f(X, Y)$ is called a weighted homogeneous polynomial of type (a, b; d) if $f(t^a X, t^b Y)=t^d f(X, Y)$ for any $t\in K$. Here a and b are integers such that (i) a and b are coprime if $ab\neq 0$ and (ii) $(a, b)=(1, 0)$ or $(0, 1)$ if $ab=0$. (a, b) are called the weights and d is called the degree of f with respect to the weights (a, b). We denote it by $d=\deg_{(a,b)}f$.

EXAMPLE (2.2) $X^2(Y+1)$ is a weighted homogeneous polynomial of type $(1, 0; 2)$. $X^2(XY+1)$ is of type $(1, -1; 2)$.

DEFINITION (2.3). A rational function $F(X, Y)=f(X, Y)/g(X, Y)$ is called a weighted homogeneous rational function of type $(a, b; d)$ if f and g have the same weights (a, b) and $d=\deg_{(a,b)}f-\deg_{(a,b)}g$. From the equation $F(t^a X, t^b Y)=t^d F(X, Y)$, we obtain the Euler equation:

$$dF(X, Y)=aXF_X(X, Y)+bYF_Y(X, Y)\,. \tag{2.4}$$

PROPOSITION (2.5). *Let $F(X, Y)\neq 0$ be a weighted homogeneous rational function of type $(a, b; d)$. Then $F(X, Y)$ can be uniquely factored as $cX^pY^q\prod_{i=1}^{m}(X^b+c_iY^a)^{n_i}$ where $c, c_1, \cdots, c_m$ are non-zero and $c_i\neq c_j$ for $i\neq j$.*

Proof. We may prove the assertion for a weighted homogeneous polynomial f. If $ab=0$, the assertion is nothing but the unique factorization property of a polynomial of one variable. Assume that $ab\neq 0$. We can write $F(X, Y)=X^pY^qf_1(X^b, Y^a)$ for some homogeneous polynomial $f_1(X, Y)$. Thus the assertion is reduced to the homogeneous case which is well known.

DEFINITION (2.6). For a given $F(X, Y)$ as above, we define $p=\mathrm{val}_X F$, $q=\mathrm{val}_Y F$ and $n_i=\mathrm{val}_{\sigma_i}F$ where $\sigma_i=X^b+c_iY^a$ for $i=1, \cdots, m$.

PROPOSITION (2.7). ((17.4), [Ab]). *Let $F(X, Y)$ and $G(X, Y)$ be non-zero weighted homogeneous rational functions with the same weights (a, b). Let $d_1=\deg_{(a,b)}F$ and $d_2=\deg_{(a,b)}G$. Assume that $J(F, G)=0$. Then there exists a constant c such that $F^{d_2}=cG^{d_1}$.*

Proof. From the assumption $F_XG_Y-F_YG_X=0$, we get $d_1FG_Y-d_2F_YG=(aXF_X+bYF_Y)G_Y-F_Y(aXG_X+bYG_Y)=0$. Similarly we get $d_2F_XG-d_1FG_X=0$. Thus taking the differential of F^{d_2}/G^{d_1}, we get

$$d(F^{d_2}/G^{d_1})=\left\{\frac{d_2F_X}{F}-\frac{d_1G_X}{G}\right\}\frac{F^{d_2}}{G^{d_1}}dX+\left\{\frac{d_2F_Y}{F}-\frac{d_1G_Y}{G}\right\}\frac{F^{d_2}}{G^{d_1}}dY=0\,.$$

This completes the proof.

The following lemma plays a key role in the following sections.

LEMMA (2.8). *Let $F(X, Y)$ be a weighted homogeneous rational function of type $(a, b; d)$. Let σ be one of the divisors X, Y and X^b+cY^a for some $c\neq 0$. Assume that $d\neq 0$ and* $\mathrm{val}_\sigma F=0$. *Then we have* $\mathrm{val}_\sigma J(\sigma, F)=0$.

Proof. First observe that $J(\sigma, F)$ is a weighted homogeneous rational function of type $(a, b; d'-a-b)$ where $d'=d+\deg_{(a,b)}\sigma$. Let $F(X, Y)=X^pY^q\prod_{i=1}^m(X^b+c_iY^a)^{n_i}$. By the assumption, $d=pa+qb+\sum_{i=1}^m n_iab\neq 0$. We put $\sigma_i=X^b+c_iY^a$ for brevity's sake.

Case 1. Suppose $\sigma=X$. Then $\mathrm{val}_\sigma F=0$ implies $p=0$. Let F_1, F_2 and G be rational functions. The following property of the Jacobian is used throughout this paper.

$$J(G, F_1F_2)=J(G, F_1)F_2+J(G, F_2)F_1\,. \tag{2.8.1}$$

As $J(X, Y)=1$ and $J(X, \sigma_i)=c_iaY^{a-1}$, we have

$$J(X, F)=qY^{q-1}\prod_{i=1}^m\sigma_i^{n_i}+\sum_{i=1}^m n_ic_iaY^{q+a-1}\sigma_i^{n_i-1}\quad(\prod_{j\neq i}\sigma_j^{n_j})\,.$$

Substituting $X=0$, we have

$$J(X, F)_{|X=0}=\alpha Y^{na+q-1} \tag{2.8.2}$$

where $n=\sum_{i=1}^m n_i$ and $\alpha=(q+na)\prod_i c_i^{n_i}$. As $d=qb+nab=(q+na)b$ is not zero, α is not zero. Thus (2.8.2) implies $\mathrm{val}_X J(X, F)=0$. (If b is a negative integer, the sustitution $X=0$ should be replaced by $X'=0$ where $X'=X^{-1}$.)

The case that $\sigma=Y$ is treated in the exact same way.

Case 2. Suppose that $\sigma=X^b+cY^a$ where $c\neq 0$, $c\neq c_i$ for $i=1, \cdots, m$. As $J(\sigma, \sigma_i)=(c_i-c)abX^{b-1}Y^{a-1}$, we have

$$J(\sigma, F)=-pacX^{p-1}Y^{q+a-1}\prod_{i=1}^m\sigma_i^{n_i}+qbX^{p+b-1}Y^{q-1}\prod_{i=1}^m\sigma_i^{n_i}$$
$$+\sum_{i=1}^m n_i(c_i-c)abX^{p+b-1}Y^{q+a-1}\sigma_i^{n_i-1}\prod_{j\neq i}\sigma_j^{n_j}\,.$$

Restricting $J(\sigma, F)$ to $X^b+cY^a=0$, we obtain

$$J(\sigma, F)_{|\sigma=0}=d\prod_{i=1}^m(c_i-c)^{n_i}X^{p+b-1}Y^{na+q-1}\neq 0$$

by the assumption. This implies $\mathrm{val}_\sigma J(\sigma, F)=0$, completing the proof.

THEOREM (2.9). *Let $h(X, Y)$ be a weighted homogeneous polynomial of type $(a, b; d)$ and suppose that $a>0$, $b>0$. Then there exists a weighted homogeneous rational function $\varphi(X, Y)$ such that $J(\varphi, h)=1$ if and only if either*

(i) $h=\sigma_1^p\sigma_2^q$ *where σ_1 and σ_2 are linear forms and $p\geqq 0$, $q\geqq 0$, $p\neq q$ or*

(ii) $h=cX^p(Y+c'X^b)^q$ *where $c, c'\neq 0$ and $a=1$ and p and q are as in* (i), *or*

(iii) $h=cY^p(X+c'Y^a)^q$ *where $c, c'\neq 0$, $b=1$ and p and q are as in* (i). (Compare with (18.9)—(18.12), [Ab].)

Proof. Let $h=c\sigma_1^{\alpha_1}\cdots\sigma_k^{\alpha_k}$ be the factorization of h as in Proposition (2.5). We assume that $\sigma_1=X$, $\sigma_2=Y$ and $\sigma_i=X^b+c_iY^a$ for $i\geqq 3$ and α_1 and α_2 are non-negative and α_i ($i\geqq 3$) is positive. Suppose that φ is a weighted homogeneous rational function such that $J(\varphi, h)=1$.

ASSERTION 1. (i) $\mathrm{val}_{\sigma_i}\varphi\geqq 1-\alpha_i$ *if* $\alpha_i\neq 0$,

$\geqq 0$ *if* $\alpha_i=0$,

(ii) $\mathrm{val}_{\sigma}\varphi\geqq 0$ *for any* $\sigma\neq\sigma_1, \cdots, \sigma_k$.

Proof. The second part of (i) and (ii) can be treated in the same way. Let σ be a divisor such that $\mathrm{val}_\sigma h=0$. Let $\varphi=\sigma^\alpha\varphi_1$ and $\mathrm{val}_\sigma\varphi_1=0$. Suppose $\alpha=\mathrm{val}_\sigma\varphi<0$. Then we have

$$(*)\qquad 1=J(\varphi, h)=\alpha\sigma^{\alpha-1}\varphi_1 J(\sigma, h)+\sigma^\alpha J(\varphi_1, h).$$

$\mathrm{val}_\sigma J(\sigma, h)=0$ by Lemma (2.8) and $\mathrm{val}_\sigma J(\varphi_1, h)\geqq 0$ because a and b are positive. Taking val_σ of (*), we get a contradiction $0=\alpha-1<0$. Now take σ_i such that $\alpha_i>0$ and let $\beta_i=\mathrm{val}_{\sigma_i}\varphi$ and $\psi=\varphi^{\alpha_i}h^{-\beta_i}$.

Case 1. Suppose that $\deg_{(a,b)}\psi=\alpha_i\deg_{(a,b)}\varphi-\beta_i\deg_{(a,b)}h=0$. By the assumption $J(\varphi, h)=1$, we have

$$\deg_{(a,b)}\varphi+\deg_{(a,b)}h=a+b>0.$$

Therefore $\dfrac{\beta_i}{\alpha_i}=\deg_{(a,b)}\varphi/\deg_{(a,b)}h>-1$ and this implies $\beta_i>-\alpha_i$ which is the assertion.

Case 2. Suppose that $\deg_{(a,b)}\psi\neq 0$. Then using (2.8) and the equality

$$J(\psi, h)=\alpha_i\sigma_i^{\alpha_i-1}J(\psi, \sigma_i)\prod_{j\neq i}\sigma_j^{\alpha_j}+\sigma_i^{\alpha_i}J(\psi, \prod_{j\neq i}\sigma_j^{\alpha_j}),$$

we get $\mathrm{val}_{\sigma_i}J(\psi, h)=\alpha_i-1$. On the other hand, we can write $J(\psi, h)=J(\varphi^{\alpha_i}, h)h^{-\beta_i}=\alpha_i\varphi^{\alpha_i-1}h^{-\beta_i}$. Therefore $\mathrm{val}_{\sigma_i}J(\psi, h)=(\alpha_i-1)\beta_i-\beta_i\alpha_i=-\beta_i$. Combining the two equalities, we get $\beta_i=1-\alpha_i$. This completes the proof of Assertion 1.

By Assertion 1 and the equality $a+b=\deg_{(a,b)}\varphi+\deg_{(a,b)}h$, we get the following inequality.

$$(2.9.1)\qquad \begin{aligned} a+b&\geqq(\alpha_1+\beta_1)a+(\alpha_2+\beta_2)b+\sum_{i\geqq 3}(\alpha_i+\beta_i)ab\\ &\geqq(\alpha_1+\beta_1)a+(\alpha_2+\beta_2)b+(k-2)ab.\end{aligned}$$

Note that $\alpha_i+\beta_i\geqq 1$ if $\alpha_i\neq 0$ and $\alpha_i+\beta_i\geqq 0$ if $\alpha_i=0$.

First case: $k=2$. Then $h=cX^{\alpha_1}Y^{\alpha_2}$. Suppose that $\alpha_1\neq\alpha_2$. Then $\varphi=\frac{1}{c(\alpha_2-\alpha_1)}X^{1-\alpha_1}Y^{1-\alpha_2}$ clearly satisfies $J(\varphi, h)=1$. Suppose that $\alpha_1=\alpha_2>0$. Then by the above inequalities, φ must be written as $c'X^{1-\alpha_1}Y^{1-\alpha_2}$ for some c'. However this is absurd because $J(\varphi, h)=0$.

Second case: $k=3$. As $0<a+b-ab=-(a-1)(b-1)+1$, we must have $a=1$ or $b=1$. ($a=b=2$ is not allowed.) Assume that $b=1$. As $a+1\geqq(\alpha_1+\beta_1)a+a+(\alpha_2+\beta_2)$, we must have $\alpha_1=0$ or $\alpha_2=0$ and $a=1$. (i) Suppose that $\alpha_1=0$. Then $h=cY^{\alpha_2}(X+c_3Y^a)^{\alpha_3}$. If $\alpha_2\neq\alpha_3$, $\frac{1}{(\alpha_2-\alpha_3)c}Y^{1-\alpha_2}(X+c_3Y^a)^{1-\alpha_3}$ is the desired solution. If $\alpha_2=\alpha_3>0$, by the inequality (2.9.1), φ must be $c'Y^{1-\alpha_2}(X+c_3Y^a)^{1-\alpha_3}$ which gives the contradiction $J(\varphi, h)=0$. (ii) Suppose that $\alpha_2=0$ and $a=1$. By the same discussion as in the case $k=2$, $\alpha_1\neq\alpha_3$ is the necessary and sufficient condition for the existence of φ.

The case that $k=3$ and $a=1$ can be discussed in a similar way so that h is either $cX^{\alpha_1}(X^b+c_3Y)^{\alpha_3}(\alpha_1\neq\alpha_3)$ or $cY^{\alpha_2}(X+c_3Y)^{\alpha_3}(\alpha_2\neq\alpha_3)$.

Third case: $k=4$. As $a+b\geqq 2ab$, we get $a=b=1$. By (2.9.1), h must be $c\sigma_3^{\alpha_3}\sigma_4^{\alpha_4}$. As σ_3 and σ_4 are linear forms, we obtain, by the same discussion as in case 1, that $\sigma_3\neq\sigma_4$ is the necessary and sufficient condition. The case that $k>4$ is clearly impossible by (2.9.1). This completes the proof of Theorem (2.9).

§ 3. Newton polygon and the Jacobian problem.

Let $f(X, Y)=\sum a_{\nu,\mu}X^\nu Y^\mu$ be a polynomial. We define the Newton polygon $N(f)$ by the convex hull of points (ν, μ) for which $a_{\nu,\mu}\neq 0$. This is a compact polyhedron in R^2. For a face Δ (Δ may be a vertex) of the boundary $\partial N(f)$, let $f_\Delta(X, Y)$ be the partial sum $\sum_{(\nu,\mu)\in\Delta}a_{\nu,\mu}X^\nu Y^\mu$. There are integers a, b and d such that a and b are coprime and $f_\Delta(X, Y)$ is a weighted homogeneous polynomial of type $(a, b; d)$. If dim $\Delta=1$ and Δ and the origin are not colinear, a, b and d are unique if we assume $d>0$.

DEFINITION (3.1). We call (a, b) the weights of Δ. Let $\Gamma_\infty(f)$ be the union of Δ's which have positive weights. See Figure A.

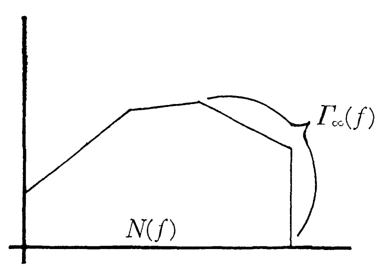


Figure A.

Let $f(X, Y)$ and $g(X, Y)$ be polynomials which satisfy the Jacobian condition (1.1). Let a and b be coprime integers. Let $f=f_{-n'}+f_{-n'+1}+\cdots+f_n$ and $g=g_{-m'}+g_{-m'+1}+\cdots+g_m$ be the gradations of f and g respectively. Namely $f_i(X, Y)=\sum_{a\nu+b\mu=i} a_{\nu\mu}X^\nu Y^\mu$. Note that we can write f_n as f_Δ for some $\Delta\in\partial N(f)$.

We consider the equation $J(f, g)=1$. As $J(f_i, g_j)$ is a weighted homogeneous polynomial of degree $i+j-a-b$, $J(f, g)$ has the gradation $J(f, g)_k=\sum_{i+j=k+a+b} J(f_i, g_j)$. In particular, we have

PROPOSITION (3.2). $J(f_n, g_m)=0$ *if* $n+m\neq a+b$.

Write $g_m=h^e$ *so that* e *is a positive integer and* h *is a square-free weighted homogeneous polynomial of degree* r. $(er=m)$.

PROPOSITION (3.3). *For any* $N>0$, *there exists a rational function* $\hat{g}(X, Y)$ *such that* $\hat{g}$ *is a finite sum of weighted homogeneous rational functions so that* $\deg_{(a,b)}(g-\hat{g}^e)<-N$.

Proof. Let $\hat{g}=\hat{g}_r+\hat{g}_{r-1}+\cdots+\hat{g}_{-M}$, where $\hat{g}_j$ is defined inductively by

$$(3.3.1)\qquad \hat{g}_r=h \quad\text{and}\quad \sum_{i_1+\cdots+i_e=j}\hat{g}_{i_1}\hat{g}_{i_2}\cdots\hat{g}_{i_e}=g_j \quad\text{for}\quad j<m.$$

For example, $\hat{g}_{r-1}=g_{m-1}/eh^{e-1}$, $\hat{g}_{r-2}=\left\{g_{m-2}-\binom{e}{2}h^{e-2}\hat{g}_{r-1}^2\right\}/eh^{e-1}$. By definition, $(\hat{g}^e)_j=g_j$ for $j\geqq(e-1)r-M$. Thus the assertion is immediate if we take $M>0$ large enough.

LEMMA (3.4). *If* $m>0$, *there exists a weighted homogeneous rational function* φ *of degree* $(a+b-m)$ *such that* $J(\varphi, g_m)=1$.

Proof. Take $N>0$ large enough and let $\hat{g}$ be as in Proposition (3.3). By Proposition (3.2) and Proposition (2.7), we can write $f_n=c_qh^q$ for some $c_q\neq0$. $(qr=n)$. Then $\deg_{(a,b)}(f-c_q\hat{g}^q)<n$ and we have

$$(3.4.1)\qquad J(f-c_q\hat{g}^q, g)_i=J(f, g)_i \quad\text{for}\quad i\geqq0.$$

To see this, let $R=g-\hat{g}^e$. Then $\deg_{(a,b)}R<-N$ and we have

$$\begin{aligned}J(\hat{g}^q, g)_i&=J(\hat{g}^q, \hat{g}^e+R)_i\\&=J(\hat{g}^q, R)_i=0 \quad\text{for}\quad i\geqq0,\end{aligned}$$

because $qr+\deg_{(a,b)}R-(a+b)<0$. Let s be the minimal integer such that $sr>a+b-m$. We repeat the same argument for $f-c_q\hat{g}^q$ and g using (3.4.1).

I. Assume that $s\geqq0$. By the inductive argument, we find constants c_q, $c_{q-1}, \cdots, c_s$ so that

(3.4.2) $$\deg_{(a,b)}\Bigl(f-\sum_{j=s}^{q} c_j\hat{g}^j\Bigr)<sr \quad\text{and}\quad J\Bigl(f-\sum_{j=s}^{q} c_j\hat{g}^j,\ g\Bigr)_i=J(f,\ g)_i$$

for $i\geqq 0$. Let φ be the maximal gradation part of $f-\sum_{j=s}^{q} c_j\hat{g}^j$. By (3.4.2), we have $\deg_{(a,b)}\varphi=a+b-m$ and $J(\varphi,\ g_m)=J(f,\ g)_0=1$.

II. Assume that $s<0$. We can find constants $c_q, \cdots, c_0$ so that

(3.4.3) $$\deg_{(a,b)}\Bigl(f-\sum_{j=0}^{q} c_j\hat{g}^j\Bigr)<0 \quad\text{and}\quad J\Bigl(f-\sum_{j=0}^{q} c_j\hat{g}^j,\ g\Bigr)_i=J(f,\ g)_i$$

for $i\geqq 0$. Let $\check{g}$ be the sum of weighted homogeneous factors of degree greater than $-N$ in the formal sum

$$\hat{g}^{-1}=\hat{g}_r^{-1}(1+\hat{g}_{r-1}/\hat{g}_r+\cdots+\hat{g}_{-M}/\hat{g}_r)^{-1}$$
$$=h^{-1}\sum_{j=0}^{\infty}(-1)^j k^j \quad\text{where}\quad k=\sum_{j=-M}^{r-1}\hat{g}_j/\hat{g}_r\,.$$

We can write $\hat{g}\check{g}=1+S$ and $\deg_{(a,b)}S\leqq r-N$. Now we consider

(3.4.4) $$J(\check{g}^i,\ g)=J(\check{g}^i,\ \hat{g}^e+R)\,.$$

It is easy to see that $\deg_{(a,b)}J(\check{g}^i,\ R)<0$ for $i\geqq 0$. We consider

$$J(\check{g}^i,\ \hat{g}^e)\hat{g}^i=J(\check{g}^i\hat{g}^i,\ \hat{g}^e)=J((1+S)^i,\ \hat{g}^e)=J((1+S)^i-1,\ \hat{g}^e)\,.$$

If $0<i\leqq -s$ and M and N are large enough, we see that $\deg_{(a,b)}J(\check{g}^i,\ \hat{g}^e)$ is negative. Thus we have

(3.4.5) $$\deg_{(a,b)}J(\check{g}^i,\ g)<0 \qquad\text{for}\quad 0<i\leqq -s\,.$$

The rest of the argument is exactly parallel to that of I. Suppose that we have chosen constants $c_q, \cdots, c_i$ such that $0\geqq i>s$ and $\deg_{(a,b)}f^{(i)}<ir$ and $J(f^{(i)},\ g)_k=J(f,\ g)_k$ for $k\geqq 0$ where $f^{(i)}=f-\sum_{j=0}^{q} c_j\hat{g}^j-\sum_{k=i}^{-1} c_k\check{g}^{-k}$. By Proposition (2.7), we can find a constant c_{i-1} such that $\deg_{(a,b)}(f^{(i)}-c_{i-1}\check{g}^{-i+1})<(i-1)r$. Let $f^{(i-1)}=f^{(i)}-c_{i-1}\check{g}^{-i+1}$. Then by (3.4.5) we have

(3.4.6) $$J(f^{(i-1)},\ g)_k=J(f,\ g)_k \qquad\text{for}\quad k\geqq 0\,.$$

We stop the argument at $f^{(s)}$ and let φ be $(f^{(s)})_{a+b-m}$. Then φ is the desired function.

COROLLARY (3.5). *Let f and g be as in Lemma* (3.4). *For any face or vertex Δ of $\Gamma_\infty(g)$, $g_\Delta(X, Y)$ is one of* (i), (ii) *and* (iii) *of Theorem* (2.9).

Proof. Take positive integers a, b and d so that $g_\Delta(X, Y)$ is a weighted homogeneous polynomial of type $(a, b; d)$. The assertion is immediate from

Lemma (3.4) and Theorem (2.9). (Thus $\Gamma_\infty(g)$ has at most one 1-dimensional simplex.)

§ 4. Equivalence of (J. C. II) and (J. C. III).

We prove the equivalence of (J.C. II) and (J.C. III) in § 1. Let (f, g) be a polynomial pair which satisfies the Jacobian condition (1.1). Assume that (f, g) is an elementary transformation. By an inductive argument on the number of compositions of transformations of type (i) and (ii) in § 1 and by Propositions (2.7) and (3.2), we can see easily that $g_m=0$ has a unique solution in $P^1(K)$. Thus (J.C. II) implies (J.C. III). Now assume that (J.C. III) is verified. We prove (J.C. II) by the induction on m=degree g.

Case 1. $m=1$. Then we may assume that $g(X, Y)=X$. Then (1.1) can be solved directly so that we get $f(X, Y)=-Y+h(X)$ for some polynomial $h(X)$. This is clearly an elementary transformation.

Case 2. $m>1$. We may assume that $g_m(X, Y)=X^m$ by a linear change of coordinates if necessary. We consider the Newton polygon $N(g)$ and we take the face Δ of $\partial N(g)$ which has the point $(m, 0)$ at the end. By Corollary (3.5), we can write $g_\Delta(X, Y)$ as $c_1X^p(Y+c_2X^b)^q$ for some $b\geqq 2$, $p\geqq 0$ and $q>0$ where c_1 and c_2 are non-zero. We change the coordinates by $X'=X$ and $Y'=Y+c_2X^b$. Then it is easy to see that the degree of $g(X', Y')$ is strictly less than m. Thus $(f(X', Y'), g(X', Y'))$ is an elementary transformation by the induction's hypothesis. Therefore $(f(X, Y), g(X, Y))$ is an elementary transformation.

§ 5. Jacobian problem for weighted homogeneous rational functions II.

Let $h(X, Y)$ be a weighted homogeneous polynomial of type $(a, b; d)$ and we assume that h is not a monomial and $ab\leqq 0$. In this section we study the necessary condition for the existence of a weighted homogeneous rational function $\varphi(X, Y)$ such that $J(\varphi, h)=1$. The Newton polygon $N(h)$ is a line segment $\overline{PQ}$ for some $P=(\alpha, \beta)$ and $Q=(\alpha', \beta')$ where $\alpha\leqq\alpha'$, $\beta\leqq\beta'$ and $\alpha+\beta<\alpha'+\beta'$.

DEFINITION (5.1). We call P and Q the left and right end of $N(h)$ respectively.

(I) Assume that $(a, b)=(1, -1)$. Then $N(h)$ is parallel to the line $Y-X=0$. This case is exceptional by the following property. (*) $\deg_{(1,-1)}J(F, G)=\deg_{(1,-1)}F+\deg_{(1,-1)}G$ for any weighted homogeneous rational functions F and G. Let $h(X, Y)=X^\alpha Y^\beta \prod_{i=1}^{k}(XY+c_i)^{\nu_i}(c_i\neq 0)$. By the assumption, $k\geqq 1$ and $d=\alpha-\beta$. If d is not zero, $\varphi=-XY/dh$ is a desired function. Suppose that $d=0$. By the above property, φ must be written as $(XY)^r\prod_{j=1}^{t}(XY+d_j)^{\mu_j}$. However this gives a contradiction $J(\varphi, h)=0$. Thus we obtain

THEOREM (5.2.). *If* $(a, b)=(1, -1)$, *the necessary and sufficient condition for the existence of* φ *is* $d\neq 0$.

(II) Assume that $a+b\neq 0$ and $(a+b)d>0$. Changing the signs of a, b and d if necessary, we may assume that $a+b$ and d are negative. We may also assume that $a\geqq 0>b$, taking the coordinates $X'=Y$ and $Y'=X$ if necessary. Let $h(X, Y)=X^{\alpha}Y^{\beta}\prod_{i=1}^{k}(X^{-b}Y^{a}+c_i)^{\nu_i}=X^{\tilde{\alpha}}Y^{\beta}\prod_{i=1}^{k}(Y^{a}+c_iX^{b})^{\nu_i}$ where $\{c_i\}$ are non-zero and mutually distinct and $\tilde{\alpha}=\alpha-\Sigma\nu_i b$. Suppose that there exists a weighted homogeneous rational function φ such that $J(\varphi, h)=1$. Let $\varphi(X, Y)=c_0X^{\gamma}Y^{\delta}\prod_{j=1}^{k+t}(X^{-b}Y^{a}+c_j)^{\mu_j}$.

ASSERTION (5.3). (i) $\nu_i+\mu_i\geqq 1$ *for* $i\leqq k$ *and* (ii) μ_j *is non-negative for* $j=k+1, \cdots, k+t$.

Proof. The proof is parallel to that of Assertion 1 in the proof of Theorem (2.9). First, the assertion (ii) is immediate from Lemma (2.8) and the following equality: $1=J(\varphi, h)=J(\varphi_j, h)\xi_j^{\mu_j}+J(\xi_j, h)\mu_j\xi_j^{\mu_j-1}\varphi_j$ where $\varphi_j=\varphi/\xi_j^{\mu_j}$ and $\xi_j=Y^{a}+c_jX^{b}$. To prove (i), let $\psi_i=\varphi^{\nu_i}h^{-\mu_i}$. Assume that $\deg_{(a,b)}\psi_i=\nu_i\deg_{(a,b)}\varphi-\mu_i d=0$. Combining this with the equality $\deg_{(a,b)}\varphi=-d+a+b$, we obtain

$$-\mu_i/\nu_i=-\deg_{(a,b)}\varphi/d=1-(a+b)/d<1\,.$$

Thus we get $-\mu_i<\nu_i$ i.e. $\nu_i+\mu_i\geqq 1$. Assume that $\deg_{(a,b)}\psi_i\neq 0$. We consider two expressions of $J(\psi_i, h)$. First it is equal to $\nu_i\varphi^{\nu_i-1}h^{-\mu_i}$ as $J(\varphi, h)=1$ and $J(h, h)=0$. Secondly we can write $J(\psi_i, h)$ as $J(\psi_i, h/\xi_i^{\nu_i})\xi_i^{\nu_i}+J(\psi_i, \xi_i)\nu_i\xi_i^{\nu_i-1}h/\xi_i^{\nu_i}$. Comparing the val_{ξ_i}'s of both expressions, we get by Lemma (2.8)

$$(\nu_i-1)\mu_i-\mu_i\nu_i=\nu_i-1\,.$$

Namely we get $\nu_i+\mu_i=1$, completing the proof of the assertion.

ASSERTION (5.4). *Let* $Q=(\alpha', \beta')$ *be the right end of* $N(h)$ *and assume that* $\alpha'<\beta'$. *Then there exists a weighted homogeneous rational function* φ *such that* $J(\varphi, h)=1$ *if and only if* $a=\alpha=0$ *and* $k=1$.

Proof. Let $\psi=\varphi h$. Then by Assertion (5.3), ψ is a Laurent polynomial (i.e. $\psi\in K[X, Y, X^{-1}, Y^{-1}]$) such that

(5.4.1) $$\psi=c_0X^{\alpha+\gamma}Y^{\beta+\delta}\prod_{j=1}^{k+t}(X^{-b}Y^{a}+c_j)^{\nu_j+\mu_j}\quad\text{and}$$

(5.4.2) $$J(\psi, h)=h\,.$$

Let $P'=(\varepsilon, \delta)$ and $Q'=(\varepsilon', \delta')$ be the left and right ends of $N(\psi)$ respectively. Let L be the line which contains (1.1) and which is parallel to the segment $\overline{PQ}$.

As $\deg_{(a,b)}\psi = a+b$, L contains P' and Q'. Let P'' and Q'' be the intersections of L and $\overline{OP}$ and $\overline{OQ}$ respectively. See Figure B.

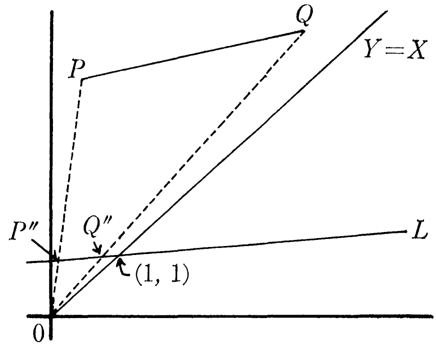


Figure B.

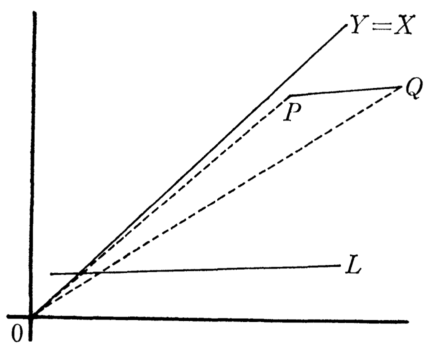
As $J(X^{\varepsilon'}Y^{\delta'}, X^{\alpha'}Y^{\beta'}) = (\varepsilon'\beta' - \delta'\alpha')X^{\varepsilon'+\alpha'-1}Y^{\delta'+\beta'-1}$, $(\varepsilon'+\alpha'-1, \delta'+\beta'-1)$ is the right end of $N(J(\psi, h))$ if $\varepsilon'\beta' - \alpha'\delta' \neq 0$. By (5.4.2), this implies that $Q'=(1.1)$. Thus we get that $Q'=(1.1)$ or $Q'=Q''$ and Q'' is an integral point in the latter case. By the same discussion, $P'=(1, 1)$ or P''. In the latter case, P'' must be an integral point. In our case Q'' is not an integral point because $0<\alpha'<\beta'$. (See Figure B.) Thus $Q'=(1, 1)$. P'' is an integral point if and only if $\overline{PQ}$ is parallel to the X-axis and P is on the Y-axis. Namely $a=\alpha=0$. As ψ is not a monomial by (5.4.1) and Assertion (5.3), the case that $P'=Q'=(1, 1)$ is impossible. Thus $P'=P''=(1, 0)$ and $a=\alpha=0$. By (5.4.1), k must be one. Thus $h = Y^{\beta}(X+c_1)^{\nu_1}$ and $\varphi = c_0 Y^{1-\beta}(X+c_1)^{1-\nu_1}$, where $c_0 = 1/(\nu_1 - \beta)$, is the desired solution.

Now we consider the case that $\alpha' \geqq \beta'$.

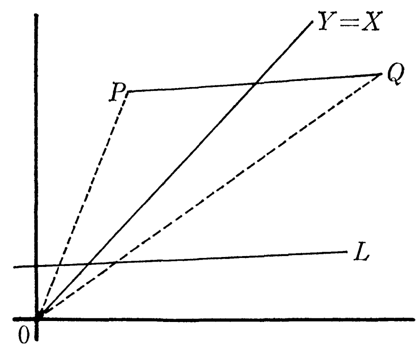
ASSERTION (5.5). *$\alpha \neq \beta$ and $\alpha' \neq \beta'$.*

Proof. Assume that $\alpha=\beta$. Then by the above discussion, $P'=P''=(1, 1)$. This is impossible because $J(\psi, h)$ cannot contain the non zero term $cX^{\alpha}Y^{\beta}$ as $J(XY, X^{\alpha}Y^{\beta})=0$. The case that $\alpha'=\beta'$ is impossible by the same argument.

We have two possible configurations.



C. I.



C. II.

Case C. I. $\alpha>\beta$. As P'' is on the right side of (1, 1), we must have $P'=(1, 1)$ to have a non-zero term $cX^{\alpha}Y^{\beta}$ in $J(\psi, h)$. If Q'' is not an integral point, we have that $Q'=(1.1)$ which is impossible by (5.4.1). Thus it is necessary that Q'' is an integral point. (The sufficient condition is difficult to describe.)

Case C. II. First note that P'' is integral if and only if $a=\alpha=0$. Remember that ψ is not a monomial. Thus if $(a, \alpha)\neq(0, 0)$, Q'' must be an integral point. We do not try to clarify the sufficient condition. (This is an algebraic condition on $\{c_i\}$ where $h(X, Y)=X^{\alpha}Y^{\beta}\prod_{i=1}^{k}(X^{b}+c_{i}Y^{a})^{\nu_i}$ if $k, \alpha, \beta, \nu_1, \cdots, \nu_k$ are fixed.)

As a conclusion, we have:

THEOREM (5.5). *Assume that $a\geqq 0>b$, $a+b<0$ and $d<0$. The following are necessary conditions for the existence of a weighted homogeneous rational function φ such that $J(\varphi, h)=1$.*

(i) *The end points P, Q of $N(h)$ are not on the line $Y-X=0$.*

(ii) *If $(a, \alpha)\neq(0, 0)$, Q'' must be an integral point. Namely there exists a positive integer s such that $(1+sa)\alpha'=(1-sb)\beta'$. In particular, Q must satisfy $\alpha'>\beta'$.*

Remark (5.6). The above conditions are not sufficient for the existence of φ. We give some examples.

(A-I) Assume that $Q''=(1+a, 1-b)$ i.e. $(1+a)\alpha'=(1-b)\beta'$. By (5.4.1), we must have $k=1$ and $h(X, Y)=X^{\alpha}Y^{\beta}(X^{-b}Y^{a}+c_1)^{\nu_1}$. In this case we can solve ψ as $c_0XY(X^{-b}Y^{a}+c_1)$, $c_0=1/(\beta-\alpha)c_1$ and $\varphi=\psi/h$.

(A-II) Assume that $(1+2a)\alpha'=(1-2b)\beta'$ and $k=2$. By (5.4.1), ψ must be $c_0XY(X^{-b}Y^{a}+c_1)(X^{-b}Y^{a}+c_2)$. By an easy calculation, ψ is a solution if and only if c_1 and c_2 satisfy the following equation:

$$(\nu_1c_1+\nu_2c_2)\begin{vmatrix}1-2b, & 1+2a\\ \alpha'+b, & \beta'-a\end{vmatrix}+(c_1+c_2)\begin{vmatrix}1-b, & 1+a\\ \alpha', & \beta'\end{vmatrix}=0\,.$$

(B) Assume that $(a+b)d<0$. We may assume that $a\geqq 0>b$, $a+b<0$ and $d>0$. This case is more difficult. The main reason is that Assertion (5.3) is not true in general. For example, let $h(X, Y)=X(X^{3}Y^{2}+c_1)^{2}$ and let $\varphi=\frac{-Y}{3}(X^{3}Y^{2}+c_1)^{-3}(X^{3}Y^{2}+3c_1)$. It is easy to see that $J(\varphi, h)=1$.

§ 6. Boundary obstructions and further remarks.

Let (f, g) be a pair of polynomials which satisfy the Jacobian condition. We assume that (f, g) is not an elementary transformation. Then by finite changes of coordinates of type (i) and (ii) in § 1 if necessary, we may assume that $g_m(X, Y)=X^{p}Y^{q}$ $(p>q>0$ and $m=p+q)$ where $m=$degree (g). Then the Newton polygon $N(g)$ is included in the rectangle $OPQR$ in Figure D by Corollary (3.5).

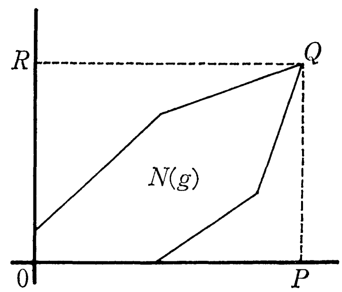


Figure D.

We may also assume that $0 \in N(g)$ and $0 \in N(f)$ by adding constants if necessary.

LEMMA (6.1). *Let f and g be as above. Then the polygons $N(f)$ and $N(g)$ are similar.*

Proof. Let Δ be a 1-dimensional simplex of the boundary $N(g)$ which is not colinear with the origin. Let $(a, b; d)$ be the weights of Δ. Let $f_{\Delta'}(X, Y)$ be the maximal gradation part of f with respect to (a, b) where Δ' is a face of $N(f)$. Assume that $\deg_{(a,b)} f_{\Delta'} + d = a + b$. Then we have $J(f_{\Delta'}, g_\Delta) = 1$. As $f_{\Delta'}$ and g_Δ are polynomials, we may assume that, for example, $(1, 0) \in \Delta$ and $(0, 1) \in \Delta'$. Let S and S' be the right ends of Δ and Δ' respectively. See Figure E.

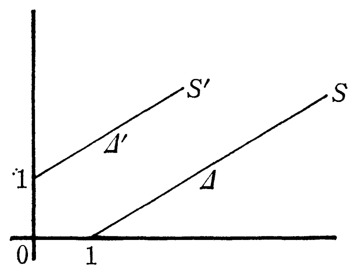


Figure E.

As S and S' and the origin are not colinear, $J(f_{\Delta'}, g_\Delta)$ contains a non-zero term $cX^{\alpha+\alpha'-1}Y^{\beta+\beta'-1}$ which is absurd. Thus we get $\deg_{(a,b)} f_{\Delta'} + d \neq a + b$. By (3.2), $f_{\Delta'}^{d} / g_{\Delta}^{d'}$ is a constant where $d' = \deg_{(a,b)} f_{\Delta'}$. Let S and T (respectively S' and T') be the ends of Δ (respectively ends of Δ'). Then the triangle OST is similar to the triangle $OS'T'$ and $|\Delta| / |\Delta'| = \overline{ST} / \overline{S'T'} = \overline{OS} / \overline{OS'} = \overline{OT} / \overline{OT'}$. As the faces Δ's of $N(g)$ as above are connected, the assertion is immediate.

COROLLARY (6.2). *Let f and g be as above. $N(f)$ and $N(g)$ contain the points* $(1, 0)$ *and* $(0, 1)$. (Otherwise $J(f, g)$ cannot be 1.)

Let Δ_1 and Δ_2 be the two particular faces of $\partial N(g)$ which have $Q=(p, q)$ as the right ends. See Figure F.

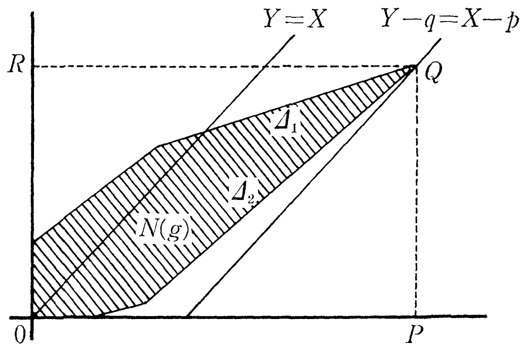


Figure F.

Assume that $\Delta_1=QR$ and let $g_{\Delta_1}(X, Y)=Y^q\prod_{i=1}^{t}(X+c_i)^{\nu_i}$. Then we take the new coordinates $X'=X+c_1$ and $Y'=Y$. Then R is not contained in the Newton polygon of $g(X', Y')$. Note that the under part of $\partial N(g)$ i.e. $\{(x, y)\in\partial N(g); py\leqq qx\}$ remains unchanged. By the same device, we may assume that $P\notin\partial N(g)$. Now the results of § 5 can be read as

THEOREM (6.5). (*Boundary obstructions*). *For any simplex Δ of $\partial N(g)$ which is not colinear with the origin, there exists a weighted homogeneous rational function φ such that $J(\varphi, g)=1$. Let (a_i, b_i) be the weights of Δ_i $(i=1, 2)$. In particular, we have the following.*

(i) *There exists a positive integer s such that $(1+s|a_1|)p=(1+s|b_1|)q$.*

(ii) *$a_2>0>b_2$ and $a_2+b_2\leqq 0$. (See Figure F.)*

Remark (6.4). Let (f, g) be a pair of polynomials which satisfy the Jacobian condition (1.1). Then f and g do not have any critical points in K^2 as functions from K^2 to K. However the converse is not true in general. For example, let $g_1(X, Y)=X+\sum_{i=1}^{k}c_i(X^aY^b)^i$ and assume that $c_k\neq 0$, $a>1$ and $b>0$. It is easy to see that g_1 has no critical point. As $N(g_1)$ does not contain (0.1), there is no polynomial f such that $J(f, g)=1$. A similar example is given by $g_2(X, Y)=X+c_2X^2+\cdots+c_mX^m+aX^nY$ where $n>m$ and $a\neq 0$.

Remark (6.5). Let $g(X, Y)$ be a polynomial with $g_m(X, Y)=X^pY^q$ where m=degree (g) and $m=p+q$ and $p, q\geqq 1$. Let Δ be a 1-dimensional simplex of $\partial N(g)$ such that Δ is not colinear with the origin. Write $g_\Delta(X, Y)=h_\Delta(X, Y)^{e(\Delta)}$ where $h_\Delta(X, Y)$ is a weighted homogeneous, square free polynomial and $e(\Delta)$ is a positive integer. Let $e(g)$ be the greatest common divisor of such $e(\Delta)$'s. The following is related to "Segre's Lemma" ([B-C-W]) and it might be well known to specialists.

LEMMA (6.5.1). *Assume that there exists a polynomial* $f(X, Y)$ *such that* $J(f, g)=1$. *Then* $e(g)>1$.

Proof. For a 1-dimensional simplex Δ of $\partial N(g)$ which is not colinear with the origin, let Δ' be the simplex of $\partial N(f)$ which corresponds to Δ by Lemma (6.1). Let m=degree (g) and n=degree (f). Assume that $e(g)=1$.

ASSERTION. *m divides n.*

Proof. Let $\frac{n}{m}=\frac{n_1}{m_1}$ where n_1 and m_1 are coprime. Let $d(\Delta)$ be the degree of $h_\Delta(X, Y)$ with respect to the weights of Δ and let d and d' be the respective degrees of $g_\Delta(X, Y)$ and $f_{\Delta'}(X, Y)$. By the proposition (2.7) and (3.2), $f_{\Delta'}(X, Y)=ch_\Delta(X, Y)^{k(\Delta)}$ where $k(\Delta)$ is defined by $d'=k(\Delta)d(\Delta)$. By Lemma (6.1), this implies that $d'=d\frac{n_1}{m_1}=d(\Delta)e(\Delta)n_1/m_1$ is a multiple of $d(\Delta)$. As $e(g)=1$ by the assumption, this is possible only if $m_1=1$. This completes the proof of the assertion.

The rest of the argument is well known. Let $f_1(X, Y)=f(X, Y)-cg(X, Y)^{n_1}$. Then degree $(f_1)<$degree (f) and $J(f_1, g)=1$. By the inductive argument, we come to the situation that $J(f_s, g)=1$ and degree $(f_s)<$degree (g) which is impossible.

Remark (6.6). The final remark is a bit unfortunate for us: There exist polynomials without any obstructions from the boundary.

EXAMPLE (6.6.1). Let $g(X, Y)=Y^n(X^2Y+1)^{3n}+X^{2n}(XY+1)^{4n}-X^{6n}Y^{4n}$. Then g has no obstruction on the boundary, i.e. there exists a weighted homogeneous rational function φ_Δ such that $J(g_\Delta, \varphi_\Delta)=1$ for any simplex Δ of $\partial N(g)$ which is not colinear with the origin. However there does not exist any polynomial $f(X, Y)$ such that $J(g, f)=1$ because g has many critical points. We finish this paper with the following question.

Question: Is there any polynomial $g(X, Y)$ such that (i) $g_m(X, Y)=X^pY^q$ where m=degree g and $p+q=m$ and $p, q>0$ and (ii) g has no obstruction on the boundary and (iii) g has no critical point?.

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