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KODAI MATH. J.
6 (1983), 397—407

ON THE GROWTH OF MEROMORPHIC FUNCTIONS OF ORDER LESS THAN 1/2, II

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1. Let $f(z)$ be meromorphic in the plane. We denote the order and lower order of $f(z)$ by ρ and μ, respectively. And we set

$$m^*(r, f)=\min_{|z|=r}|f(z)| .$$

A nonconstant meromorphic function $f(z)$ of finite order ρ is further classified as having maximal, mean, or minimal type according as

$$\limsup_{r\to\infty} T(r, f)/r^{\rho}$$

is infinite, positive, or zero, respectively.

Ostrowskii [5] and Edrei [3] proved

THEOREM A. *Let $f(z)$ be meromorphic of order ρ $(0\leqq\rho<1/2)$. Suppose there is a $\delta\in(0, 1]$ such that*

$$\cos\pi\rho-1+\delta>0 \tag{1}$$

and

$$N(r, \infty, f)\leqq(1-\delta)T(r, f)+O(\log r) \quad (r\to\infty). \tag{2}$$

Then, given $\varepsilon>0$,

$$\log m^*(r, f)>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta-\varepsilon)T(r, f) \tag{3}$$

on an unbounded sequence of r.

From this, we deduce the following result immediately.

COROLLARY 1. *Let $f(z)$ be meromorphic of order ρ $(0\leqq\rho<1/2)$. Suppose $\delta(\infty, f)>1-\cos\pi\rho$. Then, given $\varepsilon>0$,*

$$\log m^*(r, f)>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta(\infty, f)-\varepsilon)T(r, f)$$

Received January 26, 1983

on a sequence of $r \to \infty$.

As is easily shown, we may restate Theorem A in the following manner.

THEOREM A′. *Let* $f(z)$ *be meromorphic of order* ρ $(0 \leqq \rho < 1/2)$, *and suppose there is a* $\delta \in (0, 1]$ *satisfying* (1) *and* (2). *Then there exists a positive continuous function* $h(r)$ $(r \geqq 0)$ *tending to zero as* $r \to \infty$ *such that*

$$\log m^*(r, f) > \frac{\pi\rho}{\sin \pi\rho}(\cos \pi\rho - 1 + \delta)(1 - h(r))T(r, f) \tag{4}$$

for certain arbitrarily large values of r.

At this stage we introduce some notations. Let S_1 be the set consisting of all functions $h(r)$ $(r \geqq 0)$ which are positive, continuous and tend to zero as $r \to \infty$. The set S_2 is defined to consist of all slowly varying functions which belong to S_1. A function $h(r) \in S_2$ is further classified as $h(r) \in S_3$ or $h(r) \in S_4$ according as the integral

$$\int_1^\infty \frac{h(t)}{t}\, dt$$

is finite or not.

In our previous paper [6], we studied the estimate (4).

THEOREM B. ([4, Theorems 1 and 2]) *Let* $f(z)$ *be meromorphic of order* ρ $(0 < \rho < 1/2)$, *and suppose there is a* $\delta \in (0, 1]$ *satisfying* (1) *and* (2).

(I) *If* $f(z)$ *is of mean type, and if* $h(r)$ *belongs to* S_4, *then the estimate* (4) *holds on an unbounded sequence of* r.

(II) *If* $f(z)$ *is of minimal type, then*

$$\log m^*(r, f) > \frac{\pi\rho}{\sin \pi\rho}(\cos \pi\rho - 1 + \delta)T(r, f) - O(\log r) \tag{5}$$

on a sequence of $r \to \infty$. *In particular, if the term* $O(\log r)$ *vanishes in* (2), *so does in* (5).

As corollaries of Theorem B (II) we have the following two results.

COROLLARY 2. *Let* $f(z)$ *be a meromorphic function of order* ρ $(0 < \rho < 1/2)$ *and minimal type. Suppose there is a* $\delta \in (0, 1]$ *satisfying* (1) *and* (2). *Then the estimate* (4) *with* $h(r) = r^{-\lambda} (0 < \lambda < \rho)$ *holds for certain arbitrarily large values of* r.

COROLLARY 3. *Let the assumptions on* $f(z)$ *and* δ *of Corollary* 2 *be unchanged. If* $h(r)$ *belongs to* S_3, *then the estimate* (4) *holds on an unbounded sequence of* r.

Proof of Corollary 2. Assume first that the lower order μ of $f(z)$ is less than ρ. Then as in the proof of Theorem B (II), we deduce that

$$\log m^*(r, f) > \frac{\pi\mu'}{\sin \pi\mu'}(\cos \pi\mu' - 1 + \delta)T(r, f) - O(\log r)$$

for certain arbitrarily large values of r, where $\mu' > \mu$. Since the function $(x/\sin x)(\cos x - 1 + \delta)$ decreases strictly as $x \in [0, \pi/2)$ increases, we obtain

$$\log m^*(r, f) > \frac{\pi\rho}{\sin \pi\rho}(\cos \pi\rho - 1 + \delta)T(r, f)$$

on a sequence of $r \to \infty$. Hence the conclusion of Corollary 2 is trivially true in this case. Assume now that $\mu = \rho$. This condition implies that, for any $\varepsilon > 0$, $T(r, f) \geqq r^{\rho-\varepsilon} (r \geqq r_0(\varepsilon))$. Hence $O(\log r)/T(r, f) = o(r^{-\lambda}) (r \to \infty)$, where $\lambda \in (0, \rho)$. Combining this with the estimate (5), we have the desired result.

Proof of Corollary 3. If $h(r)$ belongs to S_2, then $r^{\varepsilon} h(r) \to \infty (r \to \infty)$ for each $\varepsilon > 0$. This result is due to Karamata [2]. Thus from Corollary 2 we obtain the conclusion.

2. As supplements of Theorem B, we showed

THEOREM C. ([6, § 5 and Remark]) *Let $h(r) \in S_3$ ($h(r) \in S_4$) be given. Let ρ and δ be numbers with $0 < \rho < 1/2$, $1 - \cos \pi\rho < \delta \leqq 1$. Then there exists a meromorphic function $f(z)$ satisfying the following conditions* (i)—(iv) ((i)′, (ii), (iii) and (iv)′).

(i) *$f(z)$ is of order ρ and mean type.*

(i)′ *$f(z)$ is of order ρ and minimal type.*

(ii) $\delta(\infty, f) = \delta$.

(iii) $N(r, \infty, f) \leqq (1-\delta)T(r, f) + O(\log r) \quad (r \to \infty)$.

(iv) $\log m^*(r, f) \leqq \frac{\pi\rho}{\sin \pi\rho}(\cos \pi\rho - 1 + \delta)(1 - h(r))T(r, f)$ *for all sufficiently large r.*

(v) $\log m^*(r, f) \leqq \frac{\pi\rho}{\sin \pi\rho}(\cos \pi\rho - 1 + \delta)(1 + h(r))T(r, f)$ *for all sufficiently large r.*

In this section we prove the following result.

THEOREM 1. *Let $h(r) \in S_1$ be given. Let ρ and δ be numbers with $0 < \rho < 1/2$, $1 - \cos \pi\rho < \delta \leqq 1$. Then there is a meromorphic function $f(z)$ having all the following properties.*

(i) *$f(z)$ is of order ρ and maximal type.*

(ii) $\delta(\infty, f) = \delta$.

(iii) $N(r, \infty, f) \leqq (1-\delta)T(r, f) + O(\log r) \quad (r \to \infty)$.

(iv) $\log m^*(r, f) \leqq \frac{\pi\rho}{\sin \pi\rho}(\cos \pi\rho - 1 + \delta)(1 - h(r))T(r, f)$ *for all sufficiently large r.*

LEMMA 1. *Given $h(r)\in S_1$, there is a function $h_1(r)\in S_2$ such that $h_1(r)\geqq h(r)$* $(r\geqq 0)$.

Proof. Put $M=\max_{r\geqq 0} h(r)$. We give a positive sequence $\{r_n\}_1^\infty$ such that

$$r_{n+1}/r_n\geqq n+1 \quad (n=1, 2, 3, \cdots)$$

and

$$h(r)\leqq M/2^n \quad (r\geqq r_n;\ n=1, 2, \cdots).$$

Now we define $h_1(r)$ as follows:

$$h_1(r)=\begin{cases} M \quad (0\leqq r\leqq r_1), \\ \dfrac{M(\log r_{n+1}-\log r_n)}{2^{n-1}(\log r+\log r_{n+1}-\log r_n^2)} \quad (r_n\leqq r\leqq r_{n+1};\ n=1, 2, \cdots) \end{cases}$$

Clearly $h_1(r)\in S_1$ and $h_1(r)\geqq h(r)$ $(r\geqq 0)$. It remains to prove that $h_1(r)\in S_2$. For this purpose, it is sufficient to show that for every fixed $\lambda>1$

$$\lim_{r\to\infty}\frac{h_1(\lambda r)}{h_1(r)}=1. \tag{6}$$

Assume first that $r_n\leqq r\leqq r_{n+1}/\lambda$. Then

$$\begin{aligned} 1>\frac{h_1(\lambda r)}{h_1(r)} &= \frac{\log r+\log r_{n+1}-2\log r_n}{\log\lambda+\log r+\log r_{n+1}-2\log r_n} \\ &\geqq \frac{\log r_n+\log r_{n+1}-2\log r_n}{\log\lambda+\log r_n+\log r_{n+1}-2\log r_n} \\ &= \frac{\log(r_{n+1}/r_n)}{\log\lambda+\log(r_{n+1}/r_n)}\geqq\frac{\log(n+1)}{\log\lambda+\log(n+1)}\to 1 \quad (n\to\infty) \end{aligned} \tag{7}$$

Assume next that $r_{n+1}/\lambda<r\leqq r_{n+1}$. Then

$$\begin{aligned} 1>\frac{h_1(\lambda r)}{h_1(r)} &= \frac{1}{2}\,\frac{\log r_{n+2}-\log r_{n+1}}{\log\lambda+\log r+\log r_{n+2}-2\log r_{n+1}}\,\frac{\log r+\log r_{n+1}-2\log r_n}{\log r_{n+1}-\log r_n} \\ &\geqq \frac{1}{2}\,\frac{\log r_{n+2}-\log r_{n+1}}{\log\lambda+\log r_{n+2}-\log r_{n+1}}\,\frac{2\log r_{n+1}-2\log r_n-\log\lambda}{\log r_{n+1}-\log r_n} \\ &= \frac{\log(r_{n+2}/r_{n+1})}{\log\lambda+\log(r_{n+2}/r_{n+1})}\,\frac{\log(r_{n+1}/r_n)-(\log r)/2}{\log(r_{n+1}/r_n)}\to 1 \quad (n\to\infty). \end{aligned}$$

Combining (7) and (8), we obtain (6).

LEMMA 2. *Let $h(r)\in S_4$ be given. Let ρ and δ be numbers with $0<\rho<1/2$, $1-\cos\pi\rho<\delta\leqq 1$. Then there is a meromorphic function $f(z)$ of order ρ and maximal type satisfying the following conditions.*

(i) $\delta(\infty, f)=\delta$.

(ii) $N(r, \infty, f)\leqq(1-\delta)T(r, f)+O(\log r)\quad(r\to\infty)$.

(iii) $\log m^*(r, f)<\dfrac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h(r))T(r, f)$ *for all sufficiently large values of* r.

For the construction of such a function $f(z)$, see § 5 in [6].

Proof of Theorem 1. Let $h(r)\in S_1$ be given. If $h(r)\in S_3$, our conclusion is an immediate consequence of Lemma 2. Assume that $h(r)\notin S_3$. By Lemma 1 there is a function $h_1(r)\in S_2$ such that $h_1(r)\geqq h(r)\,(r\geqq 0)$. If $h_1(r)\in S_4$, our conclusion follows from Lemma 2. Assume that $h_1(r)\in S_3$. In this case, we take a function $h_2(r)\in S_4$ arbitrarily, and consider the function $h_1(r)+h_2(r)$. We easily see that $h_1(r)+h_2(r)\in S_4$. Hence from Lemma 2 our conclusion follows. This completes the proof of Theorem 1.

3. The purpose of this section is to give a result similar to Theorem 1 in the case of $\rho=0$.

THEOREM 2. *Let* $h(r)$ *be positive and continuous for* $r>0$ *and, for* $0<r<1$,

$$h(r)=\frac{1-r}{\log(1/r)}. \tag{3.1}$$

Define $\phi(r)\,(r>0)$ *by*

$$\phi(r)=\int_0^r\frac{\phi_1(t)}{t}\,dt, \tag{3.2}$$

where

$$\phi_1(r)=\exp\left\{\int_1^r\frac{h(t)\,|\log t|}{t}\,dt\right\}. \tag{3.3}$$

Suppose

$$h(r)\log r\to 0\quad(r\to\infty), \tag{3.4}$$

$$\frac{\log r}{h(r)\phi_1(r)}\to 0\quad(r\to\infty), \tag{3.5}$$

and, for each $\lambda>0$,

$$\frac{h(\lambda r)}{h(r)}\to 1\quad(r\to\infty). \tag{3.6}$$

Then, if $\delta\in(0, 1]$, *there is a meromorphic function* $f(z)$ *of order zero satisfying the following conditions* (i)—(iii):

(i) $T(r, f)=O(\phi(r))\quad(r\to\infty)$.

(ii) $N(r, \infty, f)\leqq(1-\delta)T(r, f)+O(\log r)\quad(r\to\infty)$.

(iii) $\log m^*(r, f)\leqq(\delta-h(r))T(r, f)$ *for all sufficiently large values of* r.

Remark. An example of a function $h(r)$ satisfying our conditions is

$$h(r)=(\log r)^{-1-\alpha} \quad (0<\alpha<1)$$

for large values of r.

LEMMA 3. (cf. [2, Example 1]). *Define* $\psi_1(r)$ *by* (3.3) *and let* $\psi_2(r)=r\psi_1'(r)$. *If we define* $g(z)$ *by*

$$\log g(z)=\int_0^\infty \log\left(1+\frac{z}{t}\right)d[\psi_1(t)],$$

then, for $\varepsilon>0$

$$\frac{\log m^*(r, g)}{\log M(r, g)}<1-(1-2\varepsilon)\frac{\pi^2}{2}\frac{\psi_2(r)}{\psi(r)} \quad (r\geqq r_0(\varepsilon)).$$

Proof. By (3.3) and (3.4)

$$\frac{\psi_1(\lambda r)}{\psi_1(r)}\to 1 \quad (r\to\infty) \tag{3.7}$$

for each $\lambda>0$. Since $\psi_2(r)=r\psi_1'(r)=h(r)\log r\cdot\psi_1(r)\,(r\geqq 1)$, $\psi_2(r)$ is positive and continuous for $r>1$, and for each $\lambda>0$ we deduce from (3.7) that

$$\frac{\psi_2(\lambda r)}{\psi_2(r)}\to 1 \quad (r\to\infty).$$

Further, by (3.5), $\psi_2(r)\to\infty\,(r\to\infty)$. Thus we may apply the argument in [2, Example 1] to our $\psi(r)$, and have the desired result.

LEMMA 4. *Put*

$$J_1(r)=\int_0^r \frac{\psi_1(t)-\psi_1(r)}{t+r}\,dt$$

and

$$J_2(r)=r\int_r^\infty \frac{\psi_1(t)-\psi_1(r)}{t(t+r)}\,dt.$$

Then

$$J_1(r)\sim-\left\{\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2}\right\}\psi_2(r) \quad (r\to\infty), \tag{3.8}$$

and

$$J_2(r)\sim\left\{\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2}\right\}\psi_2(r) \quad (r\to\infty). \tag{3.9}$$

Proof.

$$J_1(r)=\frac{1}{r}\int_0^r \frac{\phi_1(t)-\phi_1(r)}{1+(t/r)}\,dt=\frac{1}{r}\int_0^r \sum_{n=0}^{\infty}(-1)^n\left(\frac{t}{r}\right)^n \{\phi_1(t)-\phi_1(r)\}\,dt$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{1}{r^{n+1}}\int_0^r t^n\{\phi_1(t)-\phi_1(r)\}\,dt$$

$$=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{r^{n+1}}\int_0^r \frac{t^n}{n+1}\phi_2(t)\,dt=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{r^{n+1}}\int_0^1 \frac{(ru)^n}{n+1}\phi_2(ru)r\,du$$

$$=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n+1}\int_0^1 u^n\phi_2(ru)\,du=\int_0^1\phi_2(ru)\left\{\sum_{n=0}^{\infty}\frac{(-1)^{n+1}u^n}{n+1}\right\}du\,.$$

The inversions in the order of integration and summation are legitimate because

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1 u^n\phi_2(ru)\,du<\infty\,.$$

Now, we put

$$p(u)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}u^n}{n+1}\ (0<u\leqq 1),\ p(u)=0\quad (u>1)\,.$$

Then for $\alpha\in(0, 1)$,

$$\int_0^1 u^{-\alpha}|p(u)|\,du\leqq\int_0^1 u^{-\alpha}\sum_{n=0}^{\infty}\frac{u^n}{n+1}\,du=\sum_{n=0}^{\infty}\frac{1}{n+1}\int_0^1 u^{n-\alpha}du$$

$$=\sum_{n=0}^{\infty}\frac{1}{(n+1)(n-\alpha+1)}<\infty\,.$$

Hence by a result of Aljančić, Bojanić and Tomić [1]

$$\int_0^1\phi_2(ru)p(u)du\sim\phi_2(r)\int_0^1 p(u)du\quad (r\to\infty)\,.$$

Thus

$$J_1(r)\sim-\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}\right\}\phi_2(r)\,.$$

The proof of (3.9) is quite similar to the one of (3.8), so we omit the proof.

Proof of Theorem 2. Let α and β be numbers such that $1\geqq\alpha>\beta\geqq 0$ and $\delta=1-\beta/\alpha$. Let $P(z)=\prod(1+z/a_n)$, $Q(z)=\prod(1-z/b_n)\,(a_n, b_n>0)$ be canonical products satisfying $n(r, 0, P)=[\alpha\phi_1(r)]$, $n(r, 0, Q)=[\beta\phi_1(r)]$, respectively. Then we shall show that $f(z)\equiv P(z)/Q(z)$ satisfies all the conditions (i)—(iii).

We first assert that

$$\log M(r, P)-N(r, 0, P)\leqq 2\alpha\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}+o(1)\right\}\phi_2(r)+\log 2\quad (r\to\infty)\,. \tag{3.10}$$

Clearly

$$\log M(r, P)-N(r, 0, P)=r\int_0^\infty \frac{[\alpha\phi_1(t)]}{t(t+r)}\,dt-\int_0^r \frac{[\alpha\phi_1(t)]}{t}\,dt \tag{3.11}$$

$$=-\int_0^r \frac{[\alpha\phi_1(t)]}{t+r}\,dt+\int_r^\infty \frac{r[\alpha\phi_1(t)]}{t(t+r)}\,dt\equiv I_1(z)+I_2(z), \text{ say}.$$

By (3.8)

$$-I_1(z)\geqq\int_0^r \frac{\alpha\phi_1(t)}{t+r}\,dt-\log 2=\alpha\{\phi_1(r)\log 2+J_1(r)\}-\log 2 \tag{3.12}$$

$$\geqq\alpha\left\{\phi_1(r)\log 2+\left(\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2}+o(1)\right)\phi_2(r)\right\}-\log 2\,.$$

Similarly, by (3.9)

$$I_2(r)<\alpha\int_r^\infty \frac{r\phi_1(t)}{t(t+r)}\,dt<\alpha\{\phi_1(r)\log 2+J_2(r)\} \tag{3.13}$$

$$<\alpha\left\{\phi_1(r)\log 2+\left(\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^2}+o(1)\right)\phi_2(r)\right\}.$$

Substituting (3.12) and (3.13) into (3.11), we obtain (3.10).

Now,

$$T(r, f)\leqq T(r, P)+T(r, Q^{-1})\leqq m(r, P)+m(r, Q)+O(1)$$

$$\leqq\log M(r, P)+\log M(r, Q)+O(1)<2\log M(r, P)+O(1)\,.$$

Hence it follows from (3.10) that

$$T(r, f)<2\{N(r, 0, P)+O(\phi_2(r))\}<2\alpha\phi(r)+O(\phi_2(r))$$

$$=O(\phi(r))\quad (r\to\infty)\,.$$

Next,

$$N(r, \infty, f)=\int_1^r \frac{[\beta\phi_1(t)]}{t}\,dt<\frac{\beta}{\alpha}\int_1^r \frac{\alpha\phi_1(t)}{t}\,dt$$

$$<\frac{\beta}{\alpha}\{N(r, 0, f)+\log r\}=(1-\delta)N(r, 0, f)+O(\log r)$$

$$<(1-\delta)T(r, f)+O(\log r)\,.$$

It remains to show (iii). Using Lemma 3, we deduce that

$$\log m^*(r, P)<\left\{1-(1-2\varepsilon)\frac{\pi^2}{2}\frac{\phi_2(r)}{\phi(r)}\right\}\log M(r, P)\quad (r\geqq r_0(\varepsilon))\,.$$

Further,

$$\log M(r, Q)=r\int_1^\infty \frac{[\beta\phi_1(t)]}{t(t+r)}dt \geqq \frac{\beta}{\alpha} r\int_1^\infty \frac{[\alpha\phi_1(t)]}{t(t+r)}dt-\log(r+1)$$

$$=\frac{\beta}{\alpha}\log M(r, P)-\log(r+1).$$

Hence

$$\log m^*(r, f)=\log m^*(r, P)-\log M(r, Q)$$

$$\text{(3.14)} \qquad <\left\{\delta-(1-2\varepsilon)\frac{\pi^2}{2}\frac{\phi_2(r)}{\phi(r)}\right\}\log M(r, P)+\log(r+1) \quad (r\geqq r_0(\varepsilon)).$$

It follows from (3.10) and (3.14) that

$$\text{(3.15)} \qquad \log m^*(r, f)<\left\{\delta-(1-2\varepsilon)\frac{\pi^2}{2}\frac{\phi_2(r)}{\phi(r)}\right\}\{N(r, 0, P)+\log 2+2\alpha(C+o(1))\phi_2(r)\}$$

$$+\log(r+1) \quad (r\to\infty), \text{ where } C\equiv\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}\in(0, 1).$$

Here we note that

$$N(r, 0, P)>\int_1^r \frac{\alpha\phi_1(t)-1}{t}dt=\alpha\phi(r)-\alpha-\log r.$$

Hence, we deduce from (3.15) that

$$\log m^*(r, f)<\left\{\delta-(1-2\varepsilon)\frac{\pi^2}{2}\frac{\phi_2(r)}{\phi(r)}\right\}\left\{1+\frac{\log 2+2\alpha(C+o(1))\phi_2(r)}{\alpha\phi(r)-\alpha-\log r}\right\}N(r, 0, P)$$

$$+\log(r+1)<\left\{\delta-(1-2\varepsilon)\frac{\pi^2}{2}\frac{\phi_2(r)}{\phi(r)}\right\}\left\{1+3C\frac{\phi_2(r)}{\phi(r)}\right\}N(r, 0, P)+\log(r+1)$$

$$\text{(3.16)} \qquad <\left\{\delta-\left[(1-2\varepsilon)\frac{\pi^2}{2}-3C\right]\frac{\phi_2(r)}{\phi(r)}\right\}N(r, 0, P)+\log(r+1)$$

$$<\left\{\delta-1.1\frac{\phi_2(r)}{\phi(r)}\right\}N(r, 0, P)+\log(r+1) \quad (\varepsilon<4/49)$$

$$<\left\{\delta-1.1\frac{\phi_2(r)}{\phi(r)}+\frac{\log(r+1)}{\alpha\phi(r)-\alpha-\log r}\right\}T(r, f) \quad (r\to\infty).$$

It is a easy consequence of (3.1)—(3.3) that

$$\text{(3.17)} \qquad \log 3\cdot\phi_1(r/3)<\phi(r)<\phi_1(r)\log r \quad (r>1),$$

so that by (3.3) or (3.5)

$$\frac{\phi_2(r)}{\phi(r)}>h(r), \qquad \frac{\log(r+1)}{\alpha\phi(r)-\alpha-\log r}=o(h(r)).$$

Thus, from (3.16) we have

$$\log m^*(r, f) < \{\delta - h(r)\} T(r, f) \quad (r \to \infty).$$

This completes the proof of Theorem 2.

As minor variations of the proof of Lemma 1, we have the following

LEMMA 5. *Let* $h_1(r)$ *be positive and continuous for* $r>0$. *Suppose* $h_1(r)\log r \to 0\ (r \to \infty)$. *Tnen there is a function* $h(r)$ *satisfying all the assumptions of Theorem 2 such that* $h(r) \geqq h_1(r)$ *for large values of* r.

Proof. Put $M = \max_{r \geqq r_0} h_1(r) \log r$, where $r_0(>1)$ is an arbitrarily fixed number. We give a positive sequence $\{r_n\}_1^\infty$ such that

$$r_{n+1}/r_n \geqq n+2 \quad (n=0, 1, 2, \cdots),$$

$$h_1(r) \log r \leqq M/2^n \quad (r \geqq r_n;\ n=0, 1, 2, \cdots),$$

$$\frac{(\log r)^2}{r^{M/2^{n-1}}} < \frac{M}{2^{2n-1}} \quad (r \geqq r_n;\ n=1, 2, 3, \cdots).$$

Define $h(r)$ as follows:

$$h(r) \log r = \begin{cases} 0 & (1<r<r_0), \\ M & (r_0 \leqq r \leqq r_1), \\ \dfrac{M(\log r_{n+1} - \log r_n)}{2^{n-1}(\log r + \log r_{n+1} - \log r_n^2)} & (r_n \leqq r \leqq r_{n+1};\ n=1, 2, \cdots) \end{cases}$$

Clearly, $h(r) \geqq h_1(r)\ (r \geqq r_0)$ and $h(r) \log r \to 0\ (r \to \infty)$. By the proof of Lemma 1

$$\frac{h(\lambda r) \log \lambda r}{h(r) \log r} \longrightarrow 1 \quad (r \to \infty),$$

so that

$$\frac{h(\lambda r)}{h(r)} \longrightarrow 1 \quad (r \to \infty).$$

To prove $(\log r)/(h(r)\psi_1(r)) \to 0\ (r \to \infty)$ (where $\psi_1(r)$ is defined by (3.3).), assume that $r_n \leqq r \leqq r_{n+1}$. Then

$$\frac{\log r}{h(r)\psi_1(r)} < \frac{(\log r)^2}{h(r)(\log r) r^{h(r) \log r}} < \frac{(\log r)^2}{(M/2^{n-1}) r^{M \cdot 2^{1-n}}} < \frac{1}{2^n} \to 0 \quad (n \to \infty).$$

This completes the proof.

Combining Theorem 2 with Lemma 5, we deduce the following

THEOREM 3. *Let* $h(r)$ *be positive and continuous for* $r \geqq r_0$. *Suppose* $h(r) \log r \to 0\ (r \to \infty)$. *Then, if* $\delta \in (0, 1]$, *there is a transcendental meromorphic function* $f(z)$ *of order zero satisfying*

$$N(r, \infty, f)<(1-\delta)T(r, f)+O(\log r) \quad (r \to \infty),$$

and

$$\log m^*(r, f)<(\delta-h(r))T(r, f) \quad \textit{for all sufficiently large values of } r.$$

References

[1] Aljančić, S., Bojanić, R. and Tomić, M., Sur la valeur asymptotique d'une classe des intégrales definies, Publ. Inst. Math. Acad. Serbe Sci. **7** (1954), 81-94.

[2] Barry, P. D., The minimum modulus of small integral and subharmonic functions, Proc. London Math. Soc. (3) **12** (1962), 445-495.

[3] Edrei, A., A local form of the Phragmén-lindelöf indicator, Mathematika, **17** (1970), 149-172.

[4] Karamata, J., Sur un mode de croissance régulière des fonctions, Mathematica (Cluj) **4** (1930), 38-53.

[5] Ostrowskii, I. V., Deficiences of meromorphic functions of order less than one, Dokl. Acad. Nauk. SSSR, **150** (1963), 32-35.

[6] Ueda, H., On the growth of meromorphic functions of order less than 1/2, Kodai Math. J., vol. **6**, No. 2 (1983), 135-146.

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