

ON THE SELF-HOMOTOPY EQUIVALENCES OF THE WEDGE OF CERTAIN COMPLEXES

BY KOHHEI YAMAGUCHI

§ 1. Introduction.

The set of homotopy classes of self-homotopy equivalences of a based space X , which is denoted by $\text{Eq}(X)$, is a group with the multiplication defined by the composition of maps. This group $\text{Eq}(X)$ has now been studied by many authors since the paper of W.D. Barcus and M.G. Barratt ([2]) appeared in 1958. However, generally speaking, we have not yet obtained an effective method for calculating it except classical ones, and the structure of it also has not been clarified sufficiently.

In this paper, we study the group $\text{Eq}(K)$ for a CW -complex K such as

$$(1.1) \quad K = \bigvee_{h=1}^r K_h, \quad K_h = S^n \bigcup_{\alpha_h} e^{n+k+1} \quad (n \geq 3, k \geq 1)$$

under the condition that the attaching class $\alpha_h \in \pi_{n+k}(S^n)$ is a double suspension, $\alpha_h = E^2 \alpha_h''$, and both α_h and $\alpha_h' = E \alpha_h''$ have the same order m_h , for $1 \leq h \leq r$. However, the case $r=1$ has already been treated by S. Oka in [6], so we will consider the case $r \geq 2$.

For each based topological space X and Y , we denote by $[X, Y]$ the homotopy set of all based maps from X to Y , and also by $[\beta, \gamma]$ Whitehead product of β and γ for $\beta \in \pi_t(X)$ and $\gamma \in \pi_s(X)$. Furthermore, we denote by $Z\{\nu\}$ the infinite cyclic group generated by ν . For example, $\pi_m(S^n) = Z\{\iota_m\}$.

In our case, it is well-known that the homotopy set $[K, K]$ becomes a (non-commutative) ring if $n \geq k+3$, and we are mainly concerned with studying the ring structure of it because the group $\text{Eq}(K)$ is the group consisting of all invertible elements of the ring $[K, K]$.

On the other hand, from the view-point of applications, we consider the homotopy type classification of highly-connected manifolds, in particular, the connected sums of sphere bundles over a sphere.

In fact, since such a manifold M has a cell-decomposition with the form $K \bigcup_{\beta} e^{2n+k+1}$ for $\beta \in \pi_{2n+k}(K)$ up to homotopy, it is very important to investigate the group $\text{Eq}(K)$ for this purpose.

Then the results of this paper are as follows:

Received January 19, 1983

First, the additive group structure of $[K, K]$ is given by

THEOREM A.

$$[K, K] = \bigoplus_{h,s=1}^r Z \{ \bar{\sigma}_{hs}, \bar{\lambda}_{hs} \} \oplus G \cong 2r^2 Z \oplus G,$$

where

$$G = \bigoplus_{h=1}^r \pi_h^* \bar{G}_h \cong \bigoplus_{h=1}^r \bar{G}_h$$

and

$$\bar{G}_h = p_{h!}^* \pi_{n+k+1} (\bigvee S^n) / p_h^* (\bigoplus_{s \neq h} j_{s!} \alpha_{h*} \tau_{n+k+1} (S^{n+k})).$$

In particular, we have

COROLLARY B. *If $n \geq k+3$, then there exists an isomorphism*

$$[K, K] \cong \text{Mat}(r, Z) \oplus \text{Mat}(r, Z) \oplus \text{Mat}(r, G_{hs}).$$

Secondly, we investigate the multiplicative structure of $[K, K]$ defined by the composition of maps, and we have

THEOREM C. *Let Θ and Θ' be any elements of $[K, K]$ of the form (See Theorem A):*

$$(4.20) \quad \begin{aligned} \Theta &= \sum_{h,s=1}^r (a_{hs} \bar{\sigma}_{hs} + b_{hs} \bar{\lambda}_{hs}) + g, \\ \Theta' &= \sum_{h,s=1}^r (a'_{hs} \bar{\sigma}_{hs} + b'_{hs} \bar{\lambda}_{hs}) + g', \\ g &= \sum_{h=1}^r g_h, \quad g' = \sum_{h=1}^r g'_h \in G = \bigoplus_{h=1}^r \pi_h^* \bar{G}_h, \\ g_h &= \pi_h^* (p_h^* i_* (\hat{g}_h)), \quad g'_h = \pi_h^* (p_h^* i_* (\hat{g}'_h)) \in \pi_h^* \bar{G}_h, \\ a_{hs}, b_{hs}, a'_{hs}, b'_{hs} &\in Z \quad \text{for } 1 \leq h, s \leq r. \end{aligned}$$

Then the multiplication of $[K, K]$ is given by the formula:

$$(4.21) \quad \Theta \circ \Theta' = \sum_{h,s=1}^r (A_{hs} \bar{\sigma}_{hs} + B_{hs} \bar{\lambda}_{hs}) + \sum_{h=1}^r \pi_h^* (\Gamma_h)$$

where

$$(4.22) \quad \begin{aligned} A_{hs} &= \sum_{q=1}^r (a'_{hq} a_{qs} t_s q t_{qs} / t_{sh}), \\ B_{hs} &= \sum_{q=1}^r \{ a'_{hq} a_{qs} (t_s h t_{nq} t_{qs} - t_{hs} t_s q t_{qn}) / (m_s t_{sq}) \\ &\quad + a'_{hq} b_{qs} t_{nq} + (b'_{hq} a_{qs} m_q t_{qs} / m_s) + b'_{hq} b_{qs} m_q \}, \end{aligned}$$

and

$$\begin{aligned}
 (4.23) \quad \Gamma_h &= \sum_{q=1}^r (a'_{hq}t_{hq} + b'_{hq}m_q) p_{h^*}^* i_{s^*}(\hat{g}_q) \\
 &+ \sum_{q,s=1}^r \{a'_{hq}a_{qs}j_{s^*}(\gamma_{hs}^q) + b'_{hq}a_{qs}j_{s^*}(\beta_{hs}^q)\} \\
 &+ p_{h^*}^* (\sum_{q,s=1}^r a_{qs}t_{sq}j_{s^*}(\pi'_h)) \circ \hat{g}''_h.
 \end{aligned}$$

In the general situation, the above multiplication does not necessarily satisfy the left distributive law

$$(\alpha + \beta) \circ \gamma = \alpha \circ \gamma + \beta \circ \gamma \quad (\alpha, \beta, \gamma \in [K, K]).$$

However, if $n \geq k + 3$, this one holds. So the homotopy set $[K, K]$ becomes a (non-commutative) ring.

Its ring structure is described by the following :

THEOREM D. *Let $n \geq k + 3$, and for each element Θ and $\Theta' \in [K, K]$ we can put*

$$(5.5) \quad \Theta = (\theta_{hs}) \in \text{Mat}(r, [K_h, K_s])$$

and

$$\Theta' = (\theta'_{hs}) \in \text{Mat}(r, [K_h, K_s]),$$

where

$$\begin{aligned}
 (5.6) \quad \theta_{hs} &= a_{hs}\sigma_{hs} + b_{hs}\lambda_{hs} + g_{hs} \in [K_h, K_s], \\
 \theta'_{hs} &= a'_{hs}\sigma_{hs} + b'_{hs}\lambda_{hs} + g'_{hs} \in [K_h, K_s], \\
 g_{hs} &= p_{h^*}^* i_{s^*}(\hat{g}_{hs}) \in G_{hs}, \\
 g'_{hs} &= p_{h^*}^* i_{s^*}(\hat{g}'_{hs}) \in G_{hs}, \\
 a_{hs}, a'_{hs}, b_{hs}, b'_{hs} &\in Z \quad \text{for } 1 \leq h, s \leq r.
 \end{aligned}$$

Then the multiplicative structure of $[K, K]$ is given by the formula:

$$(5.7) \quad \Theta \circ \Theta' = (\theta''_{hs}) \in [K, K] \cong \text{Mat}(r, [K_h, K_s]),$$

where

$$(5.8) \quad \theta''_{hs} = A_{hs}\sigma_{hs} + B_{hs}\lambda_{hs} + \Gamma_{hs} \in [K_h, K_s],$$

$$(5.9) \quad A_{hs} \text{ and } B_{hs} \text{ are defined by (4.22),}$$

and

$$(5.10) \quad \Gamma_{hs} = \sum_{q=1}^r \{ (a'_{hq}t_{hq} + b'_{hq}m_q) p_{h's}^* (\hat{g}_{qs}) \\ + a_{qs}t_{sq} p_{h's}^* (\hat{g}'_{hq}) + b'_{hq}a_{qs}\beta_{hs}^q + a'_{hq}a_{qs}r_{hs}^q \} \\ \text{for } 1 \leq h, s \leq r.$$

Now we suppose that $\alpha_h = \alpha_1$ for $1 \leq h \leq r$. Then the above result can be simplified as follows :

THEOREM E. *Let $n \geq k+3$, and we suppose that*

$$\alpha_h = \alpha_1 \quad \text{for } 1 \leq h \leq r.$$

Then

$$[K, K] \cong M(r; m_1, G_1) \quad \text{as a ring,}$$

where

$$\text{the integer } m_1 \text{ is the order of } \alpha_1 \text{ and} \\ G_1 = \pi_{n+k+1}(S^n) / \alpha_1 \cdot \pi_{n+k+1}(S^{n+k}) = \pi_{n+k+1}(S^n) / E\alpha_1^* \pi_{n+1}(S^n).$$

We denote by $\text{Inv}(M)$ the group of all multiplicative invertible elements of a ring M .

Then the above theorem implies

COROLLARY F. *Under the same assumptions as above, we have an isomorphism*

$$\text{Eq}(K) \cong \text{Inv}(M(r; m_1, G_1)).$$

For example, we can determine the group $\text{Eq}(\bigvee^r (S^n \vee S^{n+m}))$ for $n \geq m+2$, and $\text{Eq}(\bigvee^r E^m CP^2)$ for $m \geq 2$.

Finally, as an application we study the homotopy classification of connected sums of sphere bundles over a sphere which admit cross-sections.

THEOREM G (H. Ishimoto, [3]). *Let $n \geq k+3$ and $k \geq 1$. Let X_h and X'_h be n -sphere bundles over a $(n+k+1)$ -sphere which admit cross-sections, for $1 \leq h \leq r$.*

Then the connected sums $\#_{h=1}^r X_h$ and $\#_{h=1}^r X'_h$ are of the same homotopy type if and only if there exists an unimodular $(r \times r)$ -matrix $A \in \text{GL}(r, Z)$ such that

$$(6.8) \quad \begin{pmatrix} \lambda(X'_1) \\ \vdots \\ \lambda(X'_r) \end{pmatrix} = A \cdot \begin{pmatrix} \lambda(X_1) \\ \vdots \\ \lambda(X_r) \end{pmatrix},$$

where the abelian group $J\pi_{n+k}(SO(n))/P\pi_{n+k+1}(S^n)$ is considered as a left Z -module.

The above result has been already established by H. Ishimoto ([3]) and he also considered the more general cases. However, his method is quite different from ours, and he used essentially the technique of the handlebody theory to prove it. On the contrary, we use simply the elementary homotopy theory, so our technique is applicable to the case of the connected sum of spherical fibrations over a sphere with a cross-section which do not always have the homotopy types of closed smooth manifolds, and we will treat it in the subsequent paper.

This paper is organized as follows: In §2 we construct several homotopy classes of maps needed to determine the structure of $[K_h, K_s]$ and $[K, K]$, and in §3 we determine the additive structure of them. Similarly, in §4 and §5, we determine the multiplicative structure of them. In §6 we classify the homotopy type of connected sums of sphere bundles over a sphere which admit cross-sections.

In the final of this section, the author would like to take this opportunity to thank Professor S. Sasao for his sincere many valuable suggestions and encouragements.

§2. The construction of elements of $[K_h, K_s]$.

For each $1 \leq h \leq r$, let K_h denote a CW complex

$$(2.1) \quad K_h = S^n \bigcup_{\alpha_h} e^{n+k+1}, \quad n \geq 3, k \geq 1,$$

such that the attaching class $\alpha_h \in \pi_{n+k}(S^n)$ satisfying the following condition:

$$(2.2) \quad \text{There is an element } \alpha'_h \in \pi_{n+k-2}(S^{n-2}) \text{ such that } \alpha_h \text{ is a double suspension of } \alpha'_h, \alpha_h = E^2 \alpha'_h, \text{ and both } \alpha'_h = E \alpha'_h \text{ and } \alpha_h \text{ have the same order } m_h.$$

Then it is easy to see that the integer m_h is finite and $\alpha_h = 0$ for $n=3$.

Now let us define two complexes K'_h and K''_h as follows:

$$(2.3) \quad K'_h = S^{n-1} \bigcup_{\alpha'_h} e^{n+k} \quad \text{and} \quad K''_h = S^{n-2} \bigcup_{\alpha'_h} e^{n+k-1}.$$

Then for each $1 \leq h \leq r$, there is a cofibre sequence

$$(2.4) \quad \begin{array}{ccccccc} S^{n+k-2} & \xrightarrow{\alpha''_h} & S^{n-2} & \xrightarrow{i''_h} & K''_h & \xrightarrow{p''_h} & S^{n+k-1} \xrightarrow{\alpha'_h} S^{n-1} \\ & & & & \xrightarrow{i'_h} & \xrightarrow{p'_h} & \\ & & & & K'_h & \xrightarrow{p'_h} & S^{n+k} \xrightarrow{\alpha_h} S^n \xrightarrow{i_h} K_h \xrightarrow{p_h} S^{n+k+1}. \end{array}$$

Since $\alpha_h = E \alpha'_h$ and $\alpha'_h = E \alpha'_h$, we may canonically identify

$$(2.5) \quad K_h = EK'_h \quad \text{and} \quad K'_h = EK''_h.$$

Furthermore the following relations are clear.

$$(2.6) \quad \begin{aligned} \iota_h &= E i'_h, & i'_h &= E i''_h, \\ p_h &= -E p'_h, & p'_h &= -E p''_h, \\ \iota d_h &= E \iota d'_h, & \iota d'_h &= E \iota d''_h, \end{aligned}$$

where $\iota d_h, \iota d'_h$ and $\iota d''_h$ denote the homotopy classes of the identity maps $\iota d_{K_h}, \iota d_{K'_h}$ and $\iota d_{K''_h}$, respectively.

Here, for our purpose, we quote the well-known

LEMMA 2.7 (H. Toda, [12]).

(a) *There is an element $\xi'_h \in \pi_{n+k}(K'_h)$ such that $p'_h * (\xi'_h) = m_h \iota_{n+k}$ and the element ξ'_h is determined up to $i'_h * \pi_{n+k}(S^{n-1})$. The element ξ'_h is called the coextension map of $m_h \iota_{n+k-1}$.*

(b) *There is an element $\beta'_h \in [K'_h, S^{n-1}]$ such that $\beta'_h \circ i'_h = m_h \iota_{n-1}$, and the element β'_h is determined up to $p'_h * \pi_{n+k}(S^{n-1})$. The element β'_h is called the extension map of $m_h \iota_{n-1}$.*

For each $1 \leq h \leq r$, we define two elements ξ_h and β_h by

$$(2.8) \quad \xi_h = -E \xi'_h \in \pi_{n+k+1}(K_h),$$

and

$$\beta_h = E \beta'_h \in [K_h, S^n].$$

Then it follows from Lemma 2.7 and (2.8) that we have

LEMMA 2.9. *The above elements satisfy the following relations:*

- (a) $p_h \circ \xi_h = m_h \iota_{n+k+1}$.
- (b) $\beta_h \circ i_h = m_h \iota_h$.

Similarly, we define two elements λ'_{hs} and μ'_{hs} for $1 \leq h, s \leq r$ by

$$(2.10) \quad \lambda'_{hs} = \xi'_s \circ p'_h, \quad \text{and} \quad \mu'_{hs} = i'_s \circ \beta'_h \quad \text{in} \quad [K'_h, K'_s].$$

In particular, for $h=s$, we put

$$(2.11) \quad \lambda'_h = \lambda'_{hh}, \quad \text{and} \quad \mu'_h = \mu'_{hh}.$$

Furthermore, we define two elements λ_{hs} and μ_{hs} for $1 \leq h, s \leq r$ by

$$(2.12) \quad \lambda_{hs} = E \lambda'_{hs}, \quad \text{and} \quad \mu_{hs} = E \mu'_{hs} \quad \text{in} \quad [K_h, K_s].$$

For $h=s$, we put

$$(2.13) \quad \lambda_h = \lambda_{hh}, \quad \text{and} \quad \mu_h = \mu_{hh}.$$

Then we have easily

LEMMA 2.14.

- (a) $\lambda_{hs} = \xi'_s \circ p_h$.
- (b) $\mu_{hs} = i_s \circ \beta_h$.

Let us consider the relation between the attaching classes of $K'_h, \alpha'_h \in \pi_{n+k-1}(S^{n-1})$, for $1 \leq h \leq r$. For each $\beta \in \pi_{n+k-1}(S^{n-1})$, we denote by $\langle \beta \rangle$ the cyclic group generated by β . Then there is an element $\gamma \in \pi_{n+k-1}(S^{n-1})$ satisfying $\langle \alpha'_h \rangle \cap \langle \alpha'_s \rangle = \langle \gamma \rangle$. Thus, there is a uniquely determined pair of positive integers (t_{hs}, t_{sh}) such that, if $\alpha'_h \neq \alpha'_s$, then

$$(2.15) \quad \begin{aligned} & \text{(i) } t_{hs}\alpha'_s = t_{sh}\alpha'_h = \gamma, \text{ and} \\ & \text{(ii) } 1 \leq t_{hs} \leq m_s, \quad 1 \leq t_{sh} \leq m_h. \end{aligned}$$

In particular, if $\alpha'_h = \alpha'_s$, we put

$$(2.16) \quad t_{hs} = t_{sh} = 1.$$

LEMMA 2.17. *There is an element $\sigma'_{hs} \in [K'_h, K'_s]$ satisfying the conditions $\sigma'_{hs} \circ i'_h = t_{sh}i'_s$ and $p'_s \circ \sigma'_{hs} = t_{hs}p'_h$. In particular, if $\alpha'_h = \alpha'_s$, then we can choose $\sigma'_{hs} = id'_h$.*

Proof. Since $[K'_h, S^{n+k}] \cong \pi_{n+k}(S^{n+k})$, the proof can be easily obtained. Q. E. D.

Now we define the element $\sigma_{hs} \in [K_h, K_s]$ by

$$(2.18) \quad \sigma_{hs} = E\sigma'_{hs}.$$

Then the following is obvious.

$$(2.19) \quad \sigma_{hs} \circ \iota_h = t_{sh}\iota_s, \text{ and } p_s \circ \sigma_{hs} = t_{hs}p_h.$$

Since $K'_h = EK''_h$, the homotopy set $[K'_h, K'_s]$ becomes a group with the track addition.

Here we note that the integer $m_s t_{sh}$ is divisible by m_h and we obtain

PROPOSITION 2.20. *For each tripple of elements $(\sigma'_{hs}, \xi'_{hs}, \beta'_{hs})$, the following relation holds in $[K'_h, K'_s]$:*

$$t_{hs}\lambda'_{hs} + (m_s t_{sh} / m_h)\mu'_{hs} \equiv m_s \sigma'_{hs} \pmod{G'_{hs}},$$

where $G'_{hs} = p'_h * i'_{s*} \pi_{n+k}(S^{n-1})$.

In particular, if $\alpha'_h = \alpha'_s$, then we can choose $\sigma'_{hs} = id'_h$, so we have

COROLLARY 2.21 (S. Oka, [6]). *For each coextension ξ'_h (resp. extension β'_h), there exists an extension β'_h (resp. a coextension ξ'_h) such that*

$$\lambda'_h + \mu'_h = m_h \lambda d'_h \quad \text{in } [K'_h, K'_h].$$

Proof of Proposition 2.20. We put $x = t_{hs}$, $y = m_s t_{sh} / m_h$, and $z = m_s$. First, consider the exact sequence

$$\pi_{n+k}(K'_s) \xrightarrow{p'_h{}^*} [K'_h, K'_s] \xrightarrow{i'_h{}^*} \pi_{n-1}(K'_s).$$

Since

$$\begin{aligned} i'_h{}^*(x\lambda'_{hs} + y\mu'_{hs} - z\sigma'_{hs}) &= x(\xi'_s \circ p'_h \circ i'_h) + y(i'_s \circ \beta'_h \circ i'_h) - z(\sigma'_{hs} \circ i'_h) \\ &= (ym_h)i'_s - (zt_{sh})i'_s \\ &= 0, \end{aligned}$$

there is an element $\gamma \in \pi_{n+k}(K'_s)$ satisfying

$$x\lambda'_{hs} + y\mu'_{hs} - z\sigma'_{hs} = p'_h{}^*(\gamma).$$

Secondly, consider the isomorphism

$$p'_h{}^* : \pi_{n+k}(S^{n+k}) \xrightarrow{\cong} [K'_h, S^{n+k}].$$

Since

$$\begin{aligned} p'_h{}^*(p'_s \circ \gamma) &= p'_s{}^*(p'_h{}^*(\gamma)) \\ &= p'_s{}^*(x\lambda'_{hs} + y\mu'_{hs} - z\sigma'_{hs}) \\ &= x(p'_s \circ \xi'_s \circ p'_h) + y(p'_s \circ i'_s \circ \beta'_h) - z(p'_s \circ \sigma'_{hs}) \\ &= (xm_s)p'_h - (zt_{hs})p'_h \\ &= 0, \end{aligned}$$

we have $p'_s(\gamma) = 0$. If we consider the exact sequence

$$\pi_{n+k}(S^{n-1}) \xrightarrow{i'_s{}^*} \pi_{n+k}(K'_s) \xrightarrow{p'_s{}^*} \pi_{n+k}(S^{n+k}),$$

then we obtain

$$x\lambda'_{hs} + y\mu'_{hs} - z\sigma'_{hs} = p'_h{}^*(\gamma) \equiv 0 \pmod{G'_{hs}}.$$

Thus the proof is completed.

Q. E. D.

Now, by applying the suspension functor homomorphism, Proposition 2.20 can be transformed into

PROPOSITION 2.22. *For each triple of elements $(\sigma_{hs}, \xi_h, \beta_h)$, the following relation holds in $[K_h, K_s]$:*

$$t_{hs}\lambda_{hs} + (m_s t_{sh} m_h)\mu_{hs} \equiv m_s \sigma_{hs} \pmod{G_{hs}},$$

where the group G_{hs} is defined in (2.26).

COROLLARY 2.23 (S. Oka, [6]). *For each extension β_n (resp. coextension ξ_n), there is a coextension ξ_h (resp. an extension β_h) such that*

$$\lambda_h + \mu_h = m_h \iota d_h \quad \text{in } [K_h, K_h].$$

Since $K_h = E^2 K'_h$, the track addition defines an abelian group structure on the homotopy set $[K_h, K_s]$ for $1 \leq h, s \leq r$. Now we determine the additive structure of $[K_h, K_s]$.

First, we need the following two lemmas.

LEMMA 2.24.

$$\pi_{n+k+1}(K_s) = Z \{ \xi_s \} \oplus \iota_{s*} \pi_{n+k+1}(S^n).$$

Proof. Let $pr: (K_s, S^n) \rightarrow (S^{n+k+1}, s_0)$ be a natural projection map. Then it follows from the homotopy excision theorem that the induced homomorphism

$$pr_*: \pi_{n+k+1}(K_s, S^n) \xrightarrow{\cong} \pi_{n+k+1}(S^{n+k+1})$$

is an isomorphism. Hence we have the exact sequence

$$\begin{array}{ccccccc} \pi_{n+k+1}(S^n) & \xrightarrow{i_s^*} & \pi_{n+k+1}(K_s) & \xrightarrow{p_{s*}} & \pi_{n+k+1}(S^{n+k+1}) & \xrightarrow{\partial'} & \pi_{n+k}(S^n). \\ & & & & \uparrow \cong pr_* & \nearrow \partial & \\ & & & & \pi_{n+k+1}(K_s, S^n) & & \end{array}$$

Let $\bar{\alpha}_s: (D^{n+k+1}, S^{n+k}) \rightarrow (K_s, S^n)$ be the characteristic map of the $(n+k+1)$ -cell e^{n+k+1} in K_s . Then it is clear that $\pi_{n+k+1}(K_s, S^n) = Z \{ \bar{\alpha}_s \}$, and $pr_*(\bar{\alpha}_s) = \pm \iota_{n+k+1}$. Since $\partial(\bar{\alpha}_s) = \alpha_s$ and the order of α_s is m_s , we obtain $\text{Im}(p_{s*}) = \{ m_s \iota_{n+k+1} \} \cong Z$. Hence it follows (2.9) that we have

$$\pi_{n+k+1}(K_s) = Z \{ \xi_s \} \oplus \iota_{s*} \pi_{n+k+1}(S^n). \quad \text{Q. E. D.}$$

LEMMA 2.25. *Let $E\alpha_h^*$ denote the induced homomorphism*

$$E\alpha_h^*: \pi_{n+1}(K_s) \longrightarrow \pi_{n+k+1}(K_s).$$

Then $\text{Im}(E\alpha_h^) = i_{s*} \alpha_h \pi_{n+k+1}(S^{n+k}) = \{ i_s \circ \alpha_h \circ \eta_{n+k} \}$, where $\pi_{m+1}(S^m) = \{ \eta_m \} = Z/2Z = Z_2$ for $m \geq 3$.*

Proof. If $n=3$, then $\alpha_h=0$ and the statement is trivial. Otherwise, the proof easily follows from the Barratt-Hilton formula. Q. E. D.

Then we have the following

PROPOSITION 2.26. *For each $1 \leq h, s \leq r$,*

$$[K_h, K_s] \cong Z\{\sigma_{hs}\} \oplus Z\{\lambda_{hs}\} \oplus G_{hs} \cong Z \oplus Z \oplus G_{hs},$$

where

$$G_{hs} = p_h^* l_{s*} \pi_{n+k+1}(S^n) / p_h^* l_{s*} \alpha_{h*} \pi_{n+k+1}(S^{n+k}).$$

In particular, if $\alpha_h = \alpha_s$, we obtain

COROLLARY 2.27 (S. Oka, [6]). *For each $1 \leq h \leq r$, $[K_h, K_h] \cong Z\{id_h\} \oplus Z\{\lambda_h\} \oplus G_h \cong Z \oplus Z \oplus G_h$, where $G_h = p_h^* i_{h*} \pi_{n+k+1}(S^n)$.*

Proof of Proposition 2.26. Consider the exact sequence

$$\pi_{n+1}(K_s) \xrightarrow{E\alpha_h^*} \pi_{n+k+1}(K_s) \xrightarrow{p_h^*} [K_h, K_s] \xrightarrow{i_h^*} \pi_n(K_s) = Z\{t_s\}.$$

Applying (2.24) and (2.25) to the above sequence, we have the exact sequence

$$0 \longrightarrow Z\{\xi_s\} \oplus \hat{G}'_{hs} \xrightarrow{p_h^*} [K_h, K_s] \xrightarrow{i_h^*} Z\{t_s\} \xrightarrow{\alpha_h^*} \pi_{n+k}(K_s),$$

where $\hat{G}'_{hs} = l_{s*} \pi_{n+k+1}(S^n) / i_{s*} \alpha_{h*} \pi_{n+k+1}(S^{n+k})$.

On the other hand, since $t_{sh} \alpha_h = t_{hs} \alpha_s$ and $\text{Ker}(\alpha_h^*) = \{t_{sh} l_s\} \cong Z$, the following sequence is exact.

$$0 \longrightarrow Z\{\xi_s\} \oplus \hat{G}'_{hs} \xrightarrow{p_h^*} [K_h, K_s] \xrightarrow{i_h^*} \{t_{sh} l_s\} \cong Z \longrightarrow 0.$$

Then the proof is completed by (2.19).

Q. E. D.

Here we remark the following result which is useful for the applications.

PROPOSITION 2.28. *Let G_{hs} be the abelian group*

$$p_h^* l_{s*} \pi_{n+k+1}(S^n) / p_h^* l_{s*} \alpha_{h*} \pi_{n+k+1}(S^{n+k}).$$

Then

$$G_{hs} \cong \pi_{n+k+1}(S^n) / I_{hs},$$

where

$$I_{hs} = \alpha_{h*} \pi_{n+k+1}(S^{n+k}) + \alpha_{s*} \pi_{n+k+1}(S^{n+k}),$$

$$\alpha_{h*} \pi_{n+k+1}(S^{n+k}) = E\alpha_h^* \pi_{n+1}(S^n),$$

and

$$\alpha_{s*} \pi_{n+k+1}(S^{n+k}) = E\alpha_s^* \pi_{n+1}(S^n).$$

In particular, for the case $\alpha_h = \alpha_s$, we have

$$\begin{aligned} G_h = G_{hh} &\cong \pi_{n+k+1}(S^n) / \alpha_{h*} \pi_{n+k+1}(S^{n+k}) \\ &= \pi_{n+k+1}(S^n) / E\alpha_h^* \pi_{n+1}(S^n). \end{aligned}$$

Proof. Consider the exact sequence

$$\pi_{n+1}(K_s) \xrightarrow{E\alpha_h^*} \pi_{n+k+1}(K_s) \xrightarrow{p_h^*} [K_h, K_s].$$

It follows from the Barratt-Hilton formula that the induced homomorphism

$$p_h^* : i_{s*}\pi_{n+k+1}(S^n)/i_{s*}\alpha_h*\pi_{n+k+1}(S^{n+k}) \longrightarrow [K_h, K_s]$$

is monic. Hence we have $G_{ns} \cong i_{s*}\pi_{n+k+1}(S^n)/i_{s*}\alpha_h*\pi_{n+k+1}(S^{n+k})$. Next, consider the exact sequence

$$\begin{array}{ccccc} \pi_{n+k+2}(K_s, S^n) & \xrightarrow{\partial} & \pi_{n+k+1}(S^n) & \xrightarrow{i_{s*}} & \pi_{n+k+1}(K_s) \\ \downarrow p_{r*} \cong & & \nearrow \alpha_{s*} & & \\ \pi_{n+k+2}(S^{n+k+1}) & & & & \\ \uparrow E \cong & & & & \\ \pi_{n+k+1}(S^{n+k}) & & & & \end{array}$$

Then we have $i_{s*}\pi_{n+k+1}(S^n) \cong \pi_{n+k+1}(S^n)/\alpha_{s*}\pi_{n+k+1}(S^{n+k})$, and the statement can be easily obtained. Q. E. D.

§ 3. The additive structure.

Let K, K' and K'' denote the CW-complexes (See (2.3))

$$(3.1) \quad \begin{aligned} K &= \bigvee_{h=1}^r K_h, \\ K' &= \bigvee_{h=1}^r K'_h, \end{aligned}$$

and

$$K'' = \bigvee_{h=1}^r K''_h.$$

Then it is easy to see that

$$(3.2) \quad K = EK' = E^2K'', \text{ and } K' = EK''.$$

The aim of this section is to determine the additive structure of $[K, K]$, which is defined by the track addition.

Let α (resp. α') denote the map

$$(3.3) \quad \begin{aligned} \alpha : \bigvee^r S^{n+k} &\longrightarrow \bigvee^r S^n \\ (\text{resp. } \alpha' : \bigvee^r S^{n+k-1} &\longrightarrow \bigvee^r S^{n-1}), \end{aligned}$$

where the h -th factor of α (resp. α') is the map α_h (resp. α'_h). Since the com-

plex K (resp. K') is considered as the mapping cone of α (resp. α'), there is a cofibre sequence

$$(3.4) \quad \begin{array}{ccccccc} \bigvee^r S^{n+k-1} & \xrightarrow{\alpha'} & \bigvee^r S^{n-1} & \xrightarrow{i'} & K' & \xrightarrow{p'} & \bigvee^r S^{n+k} \\ & & \alpha & \bigvee^r S^n & \xrightarrow{i} & K & \xrightarrow{p} & \bigvee^r S^{n+k+1} & \xrightarrow{E\alpha} & \bigvee^r S^{n+1}. \end{array}$$

For each $1 \leq h \leq r$, let j_h denote the natural inclusion map to the h -th factor

$$(3.5) \quad j_h : K_h \longrightarrow K = \bigvee_{s=1}^r K_s.$$

Then from $\pi_{n+1}(K) = l_* \pi_{n+1}(\bigvee^r S^n)$ and $\pi_{n+1}(\bigvee^r S^n) \cong \bigoplus_{h=1}^r \pi_{n+1}(S^n)$, we obtain the following two lemmas.

LEMMA 3.6. $\pi_{n+k+1}(K) = l_* \pi_{n+k+1}(\bigvee^r S^n) \oplus (\bigoplus_{h=1}^r Z \{j_h \xi_h\})$.

LEMMA 3.7. Let $E\alpha_h^*$ denote the induced homomorphism

$$E\alpha_h^* : \pi_{n+1}(K) \longrightarrow \pi_{n+k+1}(K).$$

Then,

$$\text{Im}(E\alpha_h^*) = \bigoplus_{s \neq h} j_{s*} l_{s*} \alpha_{h*} \pi_{n+k+1}(S^{n+k}).$$

Thus, by replacing K instead of K_s in the proof of (2.26), we can obtain

PROPOSITION 3.8. For each $1 \leq h \leq r$,

$$[K_h, K] = \bigoplus_{s=1}^r Z \{j_s \circ \sigma_{hs}, j_s \circ \lambda_{hs}\} \oplus G_h \cong 2rZ \oplus G_h,$$

where

$$G_h = p_h^* l_* \pi_{n+k+1}(\bigvee^r S^n) / p_h^* (\bigoplus_{s \neq h} j_{s*} l_{s*} \alpha_{h*} \pi_{n+k+1}(S^{n+k})).$$

Let $\pi_h : K = \bigvee_{s=1}^r K_s \rightarrow K_h$ be the natural projection map to the h -th factor.

Then for each $1 \leq h, s \leq r$, we define two elements of $[K, K]$, $\bar{\sigma}_{hs}$ and $\bar{\lambda}_{hs}$, by the following :

$$(3.9) \quad \bar{\sigma}_{hs} = j_s \circ \sigma_{hs} \circ \pi_h \quad \text{and} \quad \bar{\lambda}_{hs} = j_s \circ \lambda_{hs} \circ \pi_h.$$

Then, from $[K, K] = \bigoplus_{h=1}^r \pi_h^* [K_h, K]$ and (3.8), we have

THEOREM A.

$$[K, K] = \bigoplus_{h, s=1}^r Z \{\bar{\sigma}_{hs}, \bar{\lambda}_{hs}\} \oplus G \cong 2r^2 Z \oplus G,$$

where

$$G = \bigoplus_{h=1}^r \pi_h^* \bar{G}_h \cong \bigoplus_{h=1}^r \bar{G}_h$$

and

$$\bar{G}_h = p_h^* i_* \pi_{n+k+1}(\sqrt{S^n}) / p_h^* (\bigoplus_{s \neq h} j_{s*} i_{s*} \alpha_{h*} \pi_{n+k+1}(S^{n+k})).$$

Now, for each $1 \leq h, s \leq r$, let A_{hs} be an abelian group. Then we define the matrix group of degree r , $\text{Mat}(r, A_{hs})$, by

$$(3.10) \quad \text{Mat}(r, A_{hs}) = \{A = (a_{hs}) : a_{hs} \in A_{hs}\},$$

where the (h, s) -entry of A goes through A_{hs} . In particular, if $A_{hs} = A$ for any (h, s) , then $\text{Mat}(r, A_{hs})$ is the usual matrix group of degree r with the coefficient A , $\text{Mat}(r, A)$.

Let j denote the inclusion map

$$(3.11) \quad j : K = \bigvee_{h=1}^r K_h \longrightarrow \prod_{h=1}^r K_h.$$

PROPOSITION 3.12 (A.J. Sieradski, [10]). *There is a split exact sequence*

$$0 \longrightarrow H \longrightarrow [K, K] \xrightarrow{j_*} \text{Mat}(r, [K_h, K_s]) \longrightarrow 0,$$

where

$$H \text{ denotes the group } \bigoplus_{h=1}^r [CK_h, K_h : \prod_{s=1}^r K_s, K].$$

Proof. This follows from the exponential law and $K = EK'$. See in detail [10]. Q. E. D.

In general, it is difficult to compute the group H . However, if $n \geq k+3$, H is trivial by the dimensional reason and we have

PROPOSITION 3.13. *Let $n \geq k+3$, then the induced homomorphism*

$$j_* : [K, K] \xrightarrow{\cong} \text{Mat}(r, [K_h, K_s])$$

is an isomorphism.

Let $n \geq k+3$. Then it follows from (3.13) that, for $\Theta \in [K, K]$, there is an element $A = (\theta_{hs}) \in \text{Mat}(r, [K_h, K_s])$ satisfying

$$(3.14) \quad j_*(\Theta) = (\theta_{hs})$$

and this expression is uniquely determined. Since θ_{hs} is an element of $[K_h, K_s]$, by Proposition 2.26, there exists uniquely determined triple (a_{hs}, b_{hs}, g_{hs}) such that,

$$(3.15) \quad \theta_{hs} = a_{hs}\sigma_{hs} + b_{hs}\lambda_{hs} + g_{hs},$$

where

$$a_{hs}, b_{hs} \in Z \quad \text{and} \quad g_{hs} \in G_{hs}.$$

We may consider $(a_{hs}), (b_{hs})$ and (g_{hs}) as an element of $\text{Mat}(r, Z), \text{Mat}(r, Z)$ and $\text{Mat}(r, G_{hs})$, respectively, and Proposition 3.13 can be restated as

COROLLARY B. *If $n \geq k+3$, then we have a group isomorphism*

$$j_* : [K, K] \longrightarrow \text{Mat}(r, Z) \oplus \text{Mat}(r, Z) \oplus \text{Mat}(r, G_{hs}).$$

§ 4. The multiplicative structure.

The purpose of this section is to investigate the multiplicative structure of $[K, K]$ which is defined by the composition of maps. In particular, for the case $n \geq k+3$, $[K, K] (\cong \text{Mat}(r, [K_h, K_s]))$ becomes a (non-commutative) ring and we will study its ring structure.

First, for each $1 \leq h, s, m \leq r$, we consider the multiplication

$$(4.1) \quad [K_h, K_s] \times [K_m, K_h] \longrightarrow [K_m, K_s]$$

which is also defined by the composition of maps. It is well-known that the left distributive law

$$\beta \circ (\gamma + \delta) = \beta \circ \gamma + \beta \circ \delta$$

holds in (4.1). But, in general, the right one does not hold. However, as is well known, we have

$$(4.2) \quad (\gamma + \delta) \circ E\beta = \gamma \circ E\beta + \delta \circ E\beta \quad \text{for } \beta \in [X, Y] \text{ and } \gamma, \delta \in [EY, Z].$$

In particular,

$$(4.3) \quad a\gamma = \gamma \circ (a(id_X)), \quad aE\gamma' = (a(id_{EY})) \circ E\gamma'$$

for $\gamma, \gamma' \in [X, Y]$ and any integer a .

Now, for each integer a , we define a homomorphism

$$(4.4) \quad \phi_a^s : [K_h, K_s] \longrightarrow [K_h, K_s]$$

by

$$\phi_a^s(\theta) = (a(id_s)) \circ \theta \quad \text{for } \theta \in [K_h, K_s].$$

Similarly, we define a homomorphism

$$(4.5) \quad \psi_a : \pi_{n+k+1}(S^n) \longrightarrow \pi_{n+k+1}(S^n)$$

by

$$\phi_a(g) = (at_n) \circ g \quad \text{for } g \in \pi_{n+k+1}(S^n).$$

Then, from the formula (4.2) we have

LEMMA 4.6.

- (a) $\phi_a^s \circ p_h^* \circ i_{s*} = p_h^* \circ i_{s*} \circ \phi_s.$
- (b) $\phi_a^s(\theta) = a\theta \quad \text{for } \theta = E\theta' \in [K_h, K_s].$
- (c) $\phi_a(g) = ag \quad \text{for } g = Eg' \in \pi_{n+k+1}(S^n).$

Since $\sigma_{hs} = E\sigma'_{hs}$ and $\lambda_{hs} = E\lambda'_{hs}$, we have

$$(4.7) \quad \phi_a^s(\sigma_{hs}) = a\sigma_{hs} \quad \text{and} \quad \phi_a^s(\lambda_{hs}) = a\lambda_{hs}.$$

In particular, the above lemma shows that the subgroup G_{hs} is closed with respect to ϕ_a^s , and that $\phi_a^s|G_{hs}$ is determined by ϕ_a .

PROPOSITION 4.8. *Let*

$$g = p_h^* i_{s*}(g_1) \in G_{hs} \quad \text{and} \quad g' = p_m^* i_{h*}(g'_1) \in G_{mh}$$

for $g_1 \in \pi_{n+k+1}(S^n)/\alpha_{h*}\pi_{n+k+1}(S^{n+k})$ *and*

$$g'_1 \in \pi_{n+k+1}(S^n)/\alpha_{m*}\pi_{n+k+1}(S^{n+k});$$

then the following relations hold:

- (1) $\lambda_{hs} \circ \lambda_{mh} = m_h \lambda_{ms}.$
- (2) $\lambda_{hs} \circ \sigma_{mh} = t_{mh} \lambda_{ms}.$
- (3) $\lambda_{hs} \circ g' = 0.$
- (4) $g \circ \lambda_{mh} = m_h(p_m^* i_{s*}(g_1)).$
- (5) $\sigma_{hs} \circ g' = \phi_{t_{sh}}(p_m^* i_{s*}(g'_1)).$
- (6) $g \circ \sigma_{mh} = t_{mh}(p_m^* i_{s*}(g_1)).$
- (7) $g \circ g' = 0.$

Proof.

$$\begin{aligned} (1) \quad \lambda_{hs} \circ \lambda_{mh} &= (\xi_s \circ p_h) \circ (\xi_h \circ p_m) \\ &= \xi_s \circ (p_h \circ \xi_h) \circ p_m \\ &= \xi_s \circ (m_h t_{n+k+1}) \circ p_m \quad (\text{by (2.9)}) \\ &= m_h(\xi_s \circ p_m) \\ &= m_h \lambda_{ms}. \end{aligned}$$

- (2) $\lambda_{hs} \circ \sigma_{mh} = (\xi_s \circ p_h) \circ \sigma_{mh}$
 $= \tilde{\xi}_s \circ (p_h \circ \sigma_{mh})$
 $= \tilde{\xi}_s \circ (t_{mh} p_m)$ (by (2.19))
 $= t_{mh} (\tilde{\xi}_s \circ p_m)$
 $= t_{mh} \lambda_{ms}$.
- (3) Since $p_h \circ i_h = 0$, $\lambda_{hs} \circ g' = (\xi_s \circ p_h) \circ (i_h \circ g'_1 \circ p_m) = 0$.
- (4) $g \circ \lambda_{mh} = (i_s \circ g_1 \circ p_h) \circ (\xi_h \circ p_m)$
 $= i_s \circ g_1 \circ (p_h \circ \xi_h) \circ p_m$
 $= i_s \circ g_1 \circ (m_h t_{n+k+1}) \circ p_m$ (by (2.9))
 $= m_h (p_m^* i_{s*}(g_1))$.

The rest of the proof can be obtained by the similar argument. Q. E. D.

PROPOSITION 4.9.

- (a) $\sigma_{hs} \circ \lambda_{mh} \equiv (m_h t_{hs} / m_s) \lambda_{ms} \pmod{G_{ms}}$.
- (b) $\sigma_{hs} \circ \sigma_{mh} \equiv \{(t_{sm} t_{mh} t_{hs} - t_{ms} t_{sh} t_{hm}) / m_s t_{sm}\} \lambda_{ms} + (t_{sh} t_{hm} / t_{sm}) \sigma_{ms} \pmod{G_{ms}}$.

Proof. It follows from Proposition 2.26 that we may assume

$$\sigma_{hs} \lambda_{mh} = a \lambda_{ms} + b \sigma_{ms} + g \quad \text{for } a, b \in Z$$

and $g = p_m^* i_{s*}(g') \in G_{ms}$.

Since

$$i_m^*(\sigma_{hs} \circ \lambda_{mh}) = \sigma_{hs} \circ \lambda_{mh} \circ i_m = \sigma_{hs} \circ \xi_h \circ p_m \circ i_m = 0$$

we have

$$\begin{aligned} 0 &= i_m^*(a \lambda_{ms} + b \sigma_{ms} + g) \\ &= (a \lambda_{ms} + b \sigma_{ms} + g) \circ i_m \\ &= a(\lambda_{ms} \circ i_m) + b(\sigma_{ms} \circ i_m) + g \circ i_m && \text{(by } i_m = E i'_m) \\ &= a(\xi_s \circ p_m \circ i_m) + b(t_{sm} i_s) + (i_s \circ g' \circ p_m \circ i_m) && \text{(by (2.19))} \\ &= (b t_{sm}) i_s. && \text{(by } p_m \circ i_m = 0) \end{aligned}$$

Hence $b=0$. Similarly, from (2.9) and (2.19) we have

$$p_{s*}(\sigma_{hs} \circ \lambda_{mh}) = (p_s \circ \sigma_{hs}) \circ \lambda_{mh} = (t_{hs} p_h) \circ (\xi_h \circ p_m) = (m_h t_{hs}) p_m.$$

On the other hand, by using $p_s \circ i_s = 0$,

$$\begin{aligned}
p_{s*}(\sigma_{hs} \circ \lambda_{mh}) &= p_{s*}(a\lambda_{ms} + g) \\
&= p_{s*}(a\xi_s \circ p_m) \\
&= (am_s)p_m. \quad (\text{by (2.9)})
\end{aligned}$$

Hence

$$(am_s)p_m = (m_h t_{hs})p_m \quad \text{in } [K_m, S^{n+k+1}] = Z\{p_m\},$$

and so

$$a = m_h t_{hs} / m_s.$$

Therefore,

$$\sigma_{hs} \circ \lambda_{mh} = (m_h t_{hs} / m_s) \lambda_{ms} \quad \text{mod } G_{hs}.$$

Thus the statement (a) is established and (b) also can be shown by the similar argument. Q. E. D.

The above proposition implies that, for each $1 \leq m, h, s \leq r$, there exists elements β_{ms}^h and $\gamma_{ms}^h \in G_{ms}$ satisfying two conditions

$$(4.10) \quad \sigma_{hs} \circ \lambda_{mh} = (m_h t_{hs} / m_s) \lambda_{ms} + \beta_{ms}^h$$

and

$$(4.11) \quad \begin{aligned} \sigma_{hs} \circ \sigma_{mh} &= \{(t_{sm} t_{mh} t_{hs} - t_{ms} t_{sh} t_{hm}) / m_s t_{sm}\} \lambda_{ms} \\ &\quad + (t_{sh} t_{hm} / t_{sm}) \sigma_{ms} + \gamma_{ms}^h. \end{aligned}$$

In particular, we note that

$$(4.12) \quad \beta_{ms}^h = 0 \quad \text{if } \alpha_h = \alpha_s$$

and

$$(4.13) \quad \gamma_{ms}^h = 0 \quad \text{if } \alpha_h = \alpha_s \text{ or } \alpha_m = \alpha_h.$$

THEOREM 4.14. *The multiplication (4.1) is given by the formula:*

$$\begin{aligned}
\Theta \circ \Theta' &= (a a' t_{sh} t_{hm} / t_{sm}) \sigma_{ms} \\
&\quad + \{a a' (t_{sm} t_{mh} t_{hs} - t_{ms} t_{sh} t_{hm}) / (m_s t_{sm}) + (a b' m_h t_{hs} / m_s) \\
&\quad + a' b t_{mh} + b b' m_h\} \lambda_{ms} \\
&\quad + (a' t_{mh} + b' m_h) p_{m's*}^*(g_1) + \phi_{a't_{sh}}^a(p_{m's*}^*(g_1')) \\
&\quad + a a' \gamma_{ms}^h + a b' \beta_{ms}^h,
\end{aligned}$$

for $\Theta = a\sigma_{hs} + b\lambda_{hs} + g \in [K_h, K_s]$ and

$$\Theta' = a'\sigma_{mh} + b'\lambda_{mh} + g' \in [K_m, K_h],$$

where $a, a', b, b' \in Z$,

$$g = p_h^* i_{s*}(g_1) \in G_{hs},$$

and

$$g' = p_m^* i_{h*}(g_1) \in G_{mh}.$$

In particular, for the case $h=s=m$, we also have

COROLLARY 4.15 (S. Oka, [6]). *The multiplication of $[K_h, K_h]$ is given by the formula:*

$$\Theta \circ \Theta' = aa' id_h + (ab' + a'b + m_h bb') \lambda_h + (a' + b' m_h) g + \phi_a^h(g')$$

$$\text{for } \Theta = a id_h + b \lambda_h + g \in [K_h, K_h]$$

and

$$\Theta' = a' id_h + b' \lambda_h + g' \in [K_h, K_h],$$

where $a, a', b, b' \in Z$ and $g, g' \in G_h$.

Proof of (4.14). It follows from the right distributive law that

$$\begin{aligned} \Theta \circ \Theta' &= \Theta \circ (a' \sigma_{mh} + b' \lambda_{mh} + g') \\ &= a' \Theta \circ \sigma_{mh} + b' \Theta \circ \lambda_{mh} + \Theta \circ g'. \end{aligned}$$

Since $\sigma_{mh} = E \sigma'_{mh}$, we have

$$\begin{aligned} a' \Theta \circ \sigma_{mh} &= aa' \sigma_{hs} \circ \sigma_{mh} + a' b \lambda_{hs} \circ \sigma_{mh} + a' (g \circ \sigma_{mh}) \\ &= (aa' t_{sh} t_{hm} / t_{sm}) \sigma_{ms} + a' \gamma_{ms}^h \\ &\quad + \{aa' (t_{sm} t_{mh} t_{hs} - t_{ms} t_{sh} t_{hm}) / m_s t_{sm}\} \lambda_{ms} \\ &\quad + (a' b t_{mh}) \lambda_{ms} + (a' t_{mh}) p_m^* i_{s*}(g_1). \\ &= (aa' t_{sh} t_{hm} / t_{sm}) \sigma_{ms} + \{aa' (t_{sm} t_{mh} t_{hs} - t_{ms} t_{sh} t_{hm}) / \\ &\quad (m_s t_{sm}) + a' b t_{mh}\} \lambda_{ms} + (a' t_{mh}) p_m^* i_{s*}(g_1) + aa' \gamma_{ms}^h. \end{aligned}$$

Similarly, by using $\lambda_{mh} = E \lambda'_{mh}$, we have

$$b' \Theta \circ \lambda_{mh} = \{(ab' m_h t_{hs} / m_s) + bb' m_h\} \lambda_{ms} + (m_h b') p_m^* i_{s*}(g_1) + ab' \beta_{ms}^h.$$

Moreover, by using $i_h = E i'_h$ we have

$$\begin{aligned} \Theta \circ g' &= (a \sigma_{ns} + b \lambda_{ns} + g) \circ (i_h \circ g'_1 \circ p_m) \\ &= (a \sigma_{ns} \circ i_h + b \lambda_{ns} \circ i_h + g \circ i_h) \circ (g'_1 \circ p_m) \\ &= (a t_{sh} i_s) \circ (g'_1 \circ p_m) \\ &= \phi_{a t_{sh}}^s (p_m^* i_{s*}(g'_1)). \end{aligned}$$

Thus the proof is completed.

Q. E. D.

Let π'_h denote the natural projection map to the h -th factor,

$$(4.16) \quad \pi'_h : \bigvee^r S^n \longrightarrow S^n.$$

Furthermore we put

$$(4.17) \quad \hat{\beta}_{ms}^h = \pi'_m{}^* j_{s*} \beta_{ms}^h \quad \text{and} \quad \hat{\gamma}_{ms}^h = \pi'_m{}^* j_{s*} \gamma_{ms}^h.$$

Now, we proceed to investigate the multiplication

$$(4.18) \quad [K, K] \times [K, K] \longrightarrow [K, K]$$

which is defined by the composition of maps.

PROPOSITION 4.19. *Let $g = \sum_{h=1}^r g_h$ and $g' = \sum_{h=1}^r g'_h$ be any elements in $G = \bigoplus_{h=1}^r \pi_h^* \bar{G}_h$, where $g_h = \pi_h^*(p_h^* i_*(\hat{g}_h))$ and $g'_h = \pi_h^*(p_h^* i_*(\hat{g}'_h)) \in \pi_h^* \bar{G}_h$ for $\hat{g}_h, \hat{g}'_h \in \pi_{n+k+1}(\bigvee^r S^n)$, $1 \leq h \leq r$.*

Then the following relations hold:

- (1) $\bar{\lambda}_{hs} \circ \bar{\lambda}_{mq} = \begin{cases} m_h \bar{\lambda}_{ms} & \text{if } h=q \\ 0 & \text{otherwise.} \end{cases}$
- (2) $\bar{\lambda}_{hs} \circ \bar{\sigma}_{mq} = \begin{cases} t_{mh} \bar{\lambda}_{ms} & \text{if } h=q \\ 0 & \text{otherwise.} \end{cases}$
- (3) $\bar{\sigma}_{hs} \circ \bar{\lambda}_{mq} = \begin{cases} (m_h t_{hs}/m_s) \bar{\lambda}_{ms} + \hat{\beta}_{ms}^h & \text{if } h=q \\ 0 & \text{otherwise.} \end{cases}$
- (4) $\bar{\sigma}_{hs} \circ \bar{\sigma}_{mq} = \begin{cases} \{(t_{sm} t_{mh} t_{hs} - t_{ms} t_{sh} t_{hm})/m_s t_{sm}\} \bar{\lambda}_{ms} + \\ (t_{sh} t_{hm}/t_{sm}) \bar{\sigma}_{ms} + \hat{\gamma}_{ms}^h & \text{if } h=q \\ 0 & \text{otherwise.} \end{cases}$
- (5) $\bar{\lambda}_{hs} \circ g' = 0.$
- (6) $g \circ \bar{\lambda}_{mq} = m_q \pi_m^*(p_m^* i_*(\hat{g}_q)).$
- (7) $\bar{\sigma}_{hs} \circ g' = t_{sh} \left(\sum_{m=1}^r \pi_m^*(j_{s*} i_{s*} \pi'_h{}^* p_m^*(\hat{g}''_m)) \right).$
- (8) $g \circ \bar{\sigma}_{mq} = t_{mq} \pi_m^*(p_m^* i_*(\hat{g}_q)).$
- (9) $g \circ g' = 0.$

Proof. Since $\pi_h \circ j_q = \delta_{hq} id_h$, the proof follows from (4.18), (4.10) and (4.11),

where δ_{hq} denotes the Kronecker delta.

Q. E. D.

Now, the multiplication of $[K, K]$ is stated as follows:

THEOREM C. *Let Θ and Θ' be any elements of $[K, K]$ of the forms*

$$(4.20) \quad \begin{aligned} \Theta &= \sum_{h,s=1}^r (a_{hs}\bar{\sigma}_{hs} + b_{hs}\bar{\lambda}_{hs}) + g, \\ \Theta' &= \sum_{h,s=1}^r (a'_{hs}\bar{\sigma}_{hs} + b'_{hs}\bar{\lambda}_{hs}) + g', \\ g &= \sum_{h=1}^r g_h, \quad g' = \sum_{h=1}^r g'_h \in G = \bigoplus_{h=1}^r \pi_h^* \bar{G}_h, \\ g_h &= \pi_h^*(p_h^* \iota_*(\hat{g}_h)), \quad g'_h = \pi_h^*(p_h^* \iota_*(\hat{g}'_h)) \in \pi_h^* \bar{G}_h, \\ a_{hs}, b_{hs}, a'_{hs}, b'_{hs} &\in Z \quad \text{for } 1 \leq h, s \leq r. \end{aligned}$$

Then the multiplication of $[K, K]$ is given by the formula:

$$(4.21) \quad \Theta \circ \Theta' = \sum_{h,s=1}^r (A_{hs}\bar{\sigma}_{hs} + B_{hs}\bar{\lambda}_{hs}) + \sum_{h=1}^r \pi_h^*(\Gamma_h),$$

where

$$(4.22) \quad \begin{aligned} A_{hs} &= \sum_{q=1}^r (a'_{hq} a_{qs} t_{sq} t_{qh} / t_{sh}), \\ B_{hs} &= \sum_{q=1}^r \{ a'_{hq} a_{qs} (t_{sh} t_{hq} t_{qs} - t_{hs} t_{sq} t_{qh}) / (m_s t_{sq}) \\ &\quad + a'_{hq} b_{qs} t_{hq} + (b'_{hq} a_{qs} m_q t_{qs} / m_s) + b'_{hq} b_{qs} m_q \}, \end{aligned}$$

and

$$(4.23) \quad \begin{aligned} \Gamma_h &= \sum_{q=1}^r (a'_{hq} t_{hq} + b'_{hq} m_q) p_h^* \iota_*(\hat{g}_q) \\ &\quad + \sum_{q,s=1}^r \{ a'_{hq} a_{qs} j_{s*} (\gamma_{hs}^q) + b'_{hq} a_{qs} j_{s*} (\beta_{hs}^q) \} \\ &\quad + p_h^* \left(\sum_{q,s=1}^r a_{qs} t_{sq} j_{s*} \iota_{s*} (\pi'_h) \right) \circ \hat{g}'_h. \end{aligned}$$

Proof. It follows from the right distributive law that

$$\Theta \circ \Theta' = \sum_{m,q=1}^r a'_{mq} \Theta \circ \bar{\sigma}_{mq} + \sum_{m,q=1}^r \Theta \circ \bar{\lambda}_{mq} + \Theta \circ g'.$$

Since $\sigma_{mq} = E\sigma'_{mq}$, by using (4.2)

$$\Theta \circ \sigma_{mq} = \sum_{h,s=1}^r a_{hs} \bar{\sigma}_{hs} \circ \bar{\sigma}_{mq} + \sum_{h,s=1}^r b_{hs} \bar{\lambda}_{hs} \circ \bar{\sigma}_{mq} + g \circ \bar{\sigma}_{mq}$$

$$\begin{aligned}
&= \sum_{s=1}^r a_{qs}(t_{sq}t_{qm}/t_{sm})\bar{\sigma}_{ms} + \sum_{s=1}^r a_{qs}\pi_m^*j_{s*}(\gamma_{ms}^q) \\
&\quad + \sum_{s=1}^r a_{qs}\{(t_{sm}t_{mq}t_{qs}-t_{mst_{sq}t_{qm}})/m_s t_{sm}\}\bar{\lambda}_{ms} \\
&\quad + \sum_{s=1}^r b_{qs}t_{mq}\bar{\lambda}_{ms} + \pi_m^*(t_{mq}p_m^*i_*(\hat{g}_q)) \\
&= \sum_{s=1}^r (a_{qs}t_{sq}t_{qm}/t_{sm})\bar{\sigma}_{ms} \\
&\quad + \sum_{s=1}^r \{a_{qs}(t_{sm}t_{mq}t_{qs}-t_{mst_{sq}t_{qm}})/(m_s t_{sm}) + b_{qs}t_{mq}\}\bar{\lambda}_{ms} \\
&\quad + \pi_m^*(t_{mq}p_m^*i_*(\hat{g}_q) + \sum_{s=1}^r a_{qs}j_{s*}(\gamma_{ms}^q)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Theta \circ \bar{\lambda}_{mq} &= \sum_{h,s=1}^r a_{hs}\bar{\sigma}_{hs} \circ \bar{\lambda}_{mq} + \sum_{h,s=1}^r b_{hs}\bar{\lambda}_{hs} \circ \bar{\lambda}_{mq} + g \circ \bar{\lambda}_{mq} \\
&= \sum_{s=1}^r \{(a_{qs}m_q t_{qs}/m_s) + b_{qs}m_q\}\bar{\lambda}_{ms} \\
&\quad + \pi_m^*(m_q p_m^*i_*(\hat{g}_q) + \sum_{s=1}^r a_{qs}j_{s*}(\beta_{ms}^q)).
\end{aligned}$$

On the other hand,

$$\Theta \circ g' = \Theta \circ \left(\sum_{m=1}^r g_m'' \right) = \sum_{m=1}^r \Theta \circ g_m''.$$

Since $i = Ei'$,

$$\begin{aligned}
\Theta \circ g_m'' &= \left(\sum_{h,s=1}^r a_{hs}\bar{\sigma}_{hs} + \sum_{h,s=1}^r b_{hs}\bar{\lambda}_{hs} + g \right) \circ (i \circ \hat{g}_m'' \circ p_m \circ \pi_m) \\
&= \left(\sum_{h,s=1}^r (a_{hs}\bar{\sigma}_{hs} \circ i + b_{hs}\bar{\lambda}_{hs} \circ i) + g \circ i \right) \circ (\hat{g}_m'' \circ p_m \circ \pi_m).
\end{aligned}$$

Furthermore, it follows from $\pi_h \circ i = i_h \circ \pi'_h$ and $p_h \circ i_h = 0$ that

$$\begin{aligned}
\bar{\sigma}_{hs} \circ i &= (j_s \circ \bar{\sigma}_{hs} \circ \pi_h) \circ i \\
&= j_s \circ \bar{\sigma}_{hs} \circ i_h \circ \pi'_h \\
&= j_s \circ (t_{sh}i_s) \circ \pi'_h = t_{sh}j_{s*}i_{s*}(\pi'_h), \\
\bar{\lambda}_{hs} \circ i &= (j_s \circ \bar{\lambda}_{hs} \circ \pi_h) \circ i \\
&= j_s \circ (\xi_s \circ p_h) \circ (i_h \circ \pi'_h) = 0,
\end{aligned}$$

and

$$\begin{aligned}
 g \circ \iota &= \left(\sum_{h=1}^r g_h \right) \circ \iota \\
 &= \sum_{h=1}^r g_h \circ \iota \quad (\text{by } \iota = E i') \\
 &= \sum_{h=1}^r (\iota \circ \hat{g}_h \circ p_h \circ \pi_h \circ i) \\
 &= \sum_{h=1}^r (\iota \circ \hat{g}_h \circ p_h \circ \iota_h \circ \pi'_h) = 0.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \Theta \circ g' &= \sum_{m=1}^r \left(\sum_{h,s=1}^r a_{hstsh} j_{s*} \iota_{s*} (\pi'_h) \right) \circ (\hat{g}''_m \circ p_m \circ \pi_m) \\
 &= \sum_{m=1}^r \pi_m^* (p_m^* \left(\sum_{h,s=1}^r a_{hstsh} j_{s*} \iota_{s*} (\pi'_h) \right) \circ \hat{g}''_m).
 \end{aligned}$$

Therefore,

$$\Theta \circ \Theta' = \sum_{h,s=1}^r (A_{hs} \bar{\sigma}_{hs} + B_{hs} \bar{\lambda}_{hs}) + \sum_{h=1}^r \pi_h^* (\Gamma_h),$$

where A_{hs} , B_{hs} and Γ_h are defined by (4.22) and (4.23).

Q. E. D.

§ 5. A representation by matrices.

Let $A = (\theta_{hs})$ and $B = (\theta'_{hs})$ be any elements of $\text{Mat}(r, [K_h, K_s])$, where $\theta_{hs}, \theta'_{hs} \in [K_h, K_s]$ for $1 \leq h, s \leq r$. Then the matrix multiplication

$$(5.1) \quad \text{Mat}(r, [K_h, K_s]) \times \text{Mat}(r, [K_h, K_s]) \longrightarrow \text{Mat}(r, [K_h, K_s])$$

is defined by

$$(5.2) \quad A \cdot B = C = (\theta''_{hs}),$$

where

$$(5.3) \quad \theta''_{hs} = \sum_{m=1}^r \theta'_{ms} \circ \theta_{hm} \quad \text{for } 1 \leq h, s \leq r.$$

In general, this matrix-multiplication does not necessarily satisfy the associative law. However, it is easy to see that the associative law holds if $j_* : [K, K] \rightarrow \text{Mat}(r, [K_h, K_s])$ is a multiplicative homomorphism since j_* is surjective. Furthermore, the element $E = (\delta_{hs} \sigma_{hs})$ is a unit element, where δ_{hs} denotes the Kronecker delta.

Here we quote

PROPOSITION 5.4 (A.J. Sieradski, [10]). *If $n \geq k + 3$, the induced homomorphism*

$$j_* : [K, K] \xrightarrow{\cong} \text{Mat}(r, [K_h, K_s])$$

is an additive and multiplicative isomorphism, where the multiplication of $[K, K]$ is defined by the composition of maps and one of $\text{Mat}(r, [K_h, K_s])$ is given by the matrix-multiplication.

If $n \geq k + 3$, then it follows from the Freudenthal suspension theorem that $\pi_{n+k+1}(\sqrt[r]{S^n}) = E\pi_{n+k}(\sqrt[r]{S^{n-1}})$ and $[K, K]$ may be considered as a (non-commutative) ring. Then, by using Theorem C and (5.2) we have

THEOREM D. *Let $n \geq k + 3$, and for each elements Θ and $\Theta' \in [K, K]$ we can put*

$$(5.5) \quad \Theta = (\theta_{hs}) \in \text{Mat}(r, [K_h, K_s])$$

and

$$\Theta' = (\theta'_{hs}) \in \text{Mat}(r, [K_h, K_s]),$$

where

$$(5.6) \quad \begin{aligned} \theta_{hs} &= a_{hs}\sigma_{hs} + b_{hs}\lambda_{hs} + g_{hs} \in [K_h, K_s], \\ \theta'_{hs} &= a'_{hs}\sigma_{hs} + b'_{hs}\lambda_{hs} + g'_{hs} \in [K_h, K_s], \\ g_{hs} &= p_h^* i_{s*}(\hat{g}_{hs}) \in G_{hs}, \\ g'_{hs} &= p_h^* i_{s*}(\hat{g}'_{hs}) \in G_{hs}, \\ a_{hs}, a'_{hs}, b_{hs}, b'_{hs} &\in Z \quad \text{for } 1 \leq h, s \leq r. \end{aligned}$$

Then the multiplicative structure of $[K, K]$ is given by the formula:

$$(5.7) \quad \Theta \circ \Theta' = (\theta''_{hs}) \in [K, K] \cong \text{Mat}(r, [K_h, K_s]),$$

where

$$(5.8) \quad \theta''_{hs} = A_{hs}\sigma_{hs} + B_{hs}\lambda_{hs} + \Gamma_{hs} \in [K_h, K_s],$$

$$(5.9) \quad A_{hs} \text{ and } B_{hs} \text{ are defined by (4.22),}$$

and

$$(5.10) \quad \begin{aligned} \Gamma_{hs} &= \sum_{q=1}^r \{(a'_{hq}t_{hq} + b'_{hq}m_q) p_h^* i_{s*}(\hat{g}_{qs}) \\ &\quad + a_{qs}t_{sq} p_h^* i_{s*}(\hat{g}'_{hq}) + b'_{hq}a_{qs}\beta_{hs}^q + a'_{hq}a_{qs}r_{hs}^q\} \quad \text{for } 1 \leq h, s \leq r. \end{aligned}$$

Remark 5.11. We suppose that $n \geq k + 3$ and

$$(5.12) \quad \alpha_h = \alpha_1 \quad \text{for } 1 \leq h \leq r.$$

Then it is easy to see that

$$(5.13) \quad \begin{aligned} \beta_{hs}^q = \gamma_{hs}^q = 0 & \quad \text{for } 1 \leq h, s, q \leq r, \\ t_{hs} = 1 & \quad \text{for } 1 \leq h, s \leq r, \end{aligned}$$

and

$$m_h = m_1 \quad \text{for } 1 \leq h \leq r.$$

Under the above assumption, by using Corollary B we have

$$(5.14) \quad [K, K] \cong \text{Mat}(r, Z) \oplus \text{Mat}(r, Z) \oplus \text{Mat}(r, G_1)$$

as an additive group.

where

$$\begin{aligned} G_1 &= \pi_{n+k+1}(S^n) / \alpha_{1*} \pi_{n+k+1}(S^{n+k}) \\ &= \pi_{n+k+1}(S^n) / E\alpha_1^* \pi_{n+1}(S^n). \end{aligned}$$

Then, by using Theorem D, the multiplication is given by

$$(5.15) \quad \begin{aligned} \Theta \circ \Theta' &= (A, B, \Gamma) \circ (A', B', \Gamma') \\ &= (A'A, A'B + B'A + m_1 B'B, (A' + m_1 B')\Gamma + \Gamma'A) \\ & \quad \text{for } \Theta = (A, B, \Gamma) \in [K, K] \text{ and } \Theta' = (A', B', \Gamma') \in [K, K]. \end{aligned}$$

Here we note the formula

$$(5.16) \quad \begin{aligned} & \begin{pmatrix} A' & 0 & 0 \\ 0 & A' + m_1 B' & \Gamma' \\ 0 & 0 & A' \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & A + m_1 B & \Gamma \\ 0 & 0 & A \end{pmatrix} \\ &= \begin{pmatrix} A'A & 0 & 0 \\ 0 & A'A + m_1(A'B + B'A + m_1 B'B) & (A' + m_1 B')\Gamma + \Gamma'A \\ 0 & 0 & A'A \end{pmatrix}. \end{aligned}$$

For each integer m and an abelian group H , let $M(r; m, H)$ denote the ring of matrices of the form

$$(5.16) \quad \begin{pmatrix} A & 0 & 0 \\ 0 & A + mB & \Gamma \\ 0 & 0 & A \end{pmatrix} \quad \begin{aligned} & \text{for } A, B \in \text{Mat}(r, Z), \text{ and} \\ & \Gamma \in \text{Mat}(r, H). \end{aligned}$$

Hence it follows from (5.14), (5.15) and (5.16) that we obtain

THEOREM E. *Let $n \geq k + 3$, and we suppose that*

$$\alpha_h = \alpha_1 \quad \text{for } 1 \leq h \leq r.$$

Then

$$[K, K] \cong M(r; m_1, G_1) \quad \text{as a ring,}$$

where

$$\begin{aligned} &\text{the integer } m_1 \text{ is the order of } \alpha_1 \text{ and} \\ &G_1 = \pi_{n+k+1}(S^n) / \alpha_1 * \pi_{n+k+1}(S^{n+k}) = \pi_{n+k+1}(S^n) / E\alpha_1^* \pi_{n+1}(S^n). \end{aligned}$$

We denote by $\text{Inv}(M)$ the group of all multiplicative invertible elements of a ring M .

Then we also have

COROLLARY F. *Under the same assumptions as above, we have an isomorphism*

$$\text{Eq}(K) \cong \text{Inv}(M(r; m_1, G_1)).$$

EXAMPLE 5.17. Let $n \geq m + 4$. Then

$$\text{Eq}(\bigvee^r (S^n \vee S^{n+m})) \cong \text{Inv}(M(r; 1, \pi_m^S)),$$

where π_m^S is the m -th stable homotopy group of spheres.

EXAMPLE 5.18. Let $m \geq 2$ and CP^2 be the two dimensional complex projective space.

Then

$$\text{Eq}(\bigvee^r E^m CP^2) \cong \text{Inv}(M(r; 2, 0)).$$

Proof. Let $\eta_2 \in \pi_3(S^2)$ be the Hopf map and we put $\eta_t = E^{t-2}\eta_2$ for $t \geq 2$. Then it is well-known that $E^m CP^2 = S^{2+m} \bigcup_{\eta_{2+m}} e^{4+m}$ and $\pi_{t+2}(S^t) = \{\eta_t \circ \eta_{t+1}\} \cong Z/2Z = Z_2$ for $t \geq 2$. Thus the assertion easily follows from Corollary F. Q.E.D.

A similar argument also shows

EXAMPLE 5.19. Let $m \geq 4$ and HP^2 be the two dimensional quaternion projective space.

Then

$$\text{Eq}(\bigvee^r E^m HP^2) \cong \text{Inv}(M(r; 24, 0)).$$

§ 6. The applications.

In this section, we will investigate the action of $\text{Eq}(K)$ on the homotopy group $\pi_{2n+k}(K)$

$$(6.1) \quad \text{Eq}(K) \times \pi_{2n+k}(K) \longrightarrow \pi_{2n+k}(K)$$

which is defined by the composition of maps, and give the another proof of Ishimoto's main results given in [3], which classify the connected sum of sphere bundles over a sphere with cross-sections up to homotopy.

First, we recall several basic results about sphere bundles over a sphere. (See in detail [4]) Let X be a n -sphere bundle over $(n+k+1)$ -sphere which admits a cross-section, and consider the part of the homotopy exact sequence of the fibration $SO(n) \rightarrow SO(n+1) \rightarrow SO(n+1)/SO(n) \cong S^n$

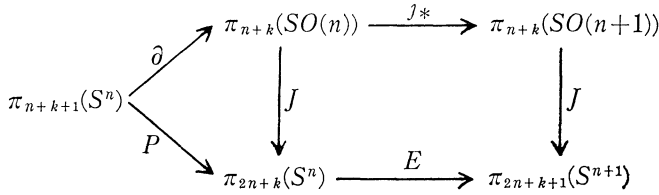


Fig. (6.2)

where J means the classical J -homomorphism and the homomorphism P is defined by Whitehead product, i. e.

$$(6.3) \quad P(\zeta) = [\zeta, \iota_n] \quad \text{for } \zeta \in \pi_{n+k+1}(S^n).$$

Furthermore, it is well-known that

$$(6.4) \quad P = -J \circ \partial \quad \text{and} \quad E \circ J = -J \circ J_*.$$

We denote by $\chi(X)$ the characteristic element of X . Since X has a cross-section, there exists some element $\gamma \in \pi_{n+k}(SO(n))$ satisfying

$$(6.5) \quad \chi(X) = j_*(\gamma).$$

Then it follows from (6.2) that the element $J(\gamma) \in J\pi_{n+k}(SO(n))$ is uniquely determined up to $P\pi_{n+k+1}(S^n)$, and so we define the invariant $\lambda(X)$ by

$$(6.6) \quad \lambda(X) = \{J(\gamma)\} \in J\pi_{n+k}(SO(n))/P\pi_{n+k+1}(S^n).$$

By using this invariant, I.M. James and J.H.C. Whitehead classified the homotopy type of sphere bundles over a sphere with cross-sections as follows:

THEOREM 6.7 (I.M. James and J.H.C. Whitehead, [4]). *Let X_1 and X_2 be n -sphere bundles over the $(n+k+1)$ -sphere which admit cross-sections, and $n \geq k+3$. Then X_1 and X_2 are of the same homotopy type if and only if $\lambda(X_1) = \pm \lambda(X_2)$.*

Let X_h be a n -sphere bundle over a $(n+k+1)$ -sphere for $1 \leq h \leq r$. We denote $\#_{h=1}^r X_h$ the connected sum of the total spaces X_h , $h=1, 2, \dots, r$. Let X'_s , $s=1, 2, \dots, r'$, be another set of such bundles. Then it is easy to see that, if $\#_{h=1}^r X_h$ and $\#_{s=1}^{r'} X'_s$ are of the same homotopy type, r' must be equal to r by the homological reason.

Then the aim of this section is to extend the above result to the case of connected sums of sphere bundles over a sphere which admit cross-sections, and our result is stated as follows :

THEOREM G (H. Ishimoto, [3]). *Let $n \geq k+3$ and $k \geq 1$. Let X_h and X'_h be n -sphere bundles over $(n+k+1)$ -spheres which admit cross-sections, for $1 \leq h \leq r$.*

Then the connected sums $\#_{h=1}^r X_h$ and $\#_{h=1}^r X'_h$ are of the same homotopy type if and only if there exists an unimodular $(r \times r)$ -matrix $A \in GL(r, Z)$ such that

$$(6.8) \quad \begin{pmatrix} \lambda(X'_1) \\ \vdots \\ \lambda(X'_r) \end{pmatrix} = A \cdot \begin{pmatrix} \lambda(X_1) \\ \vdots \\ \lambda(X_r) \end{pmatrix},$$

where the abelian group $J\pi_{n+k}(SO(n))/P\pi_{n+k+1}(S^n)$ is considered as a left Z -module.

Remark 6.9. We suppose that the attaching class $\alpha_h=0$. Then $m_h=1$ and $K_h=S^n \vee S^{n+k+1}$.

Furthermore, we may assume

(6.10) the map p_h is a retraction map to the second factor

$$p_h : S^n \vee S^{n+k+1} \longrightarrow S^n,$$

and that

(6.11) the map ξ_h is an inclusion map

$$\xi_h : S^{n+k+1} \longrightarrow S^n \vee S^{n+k+1}.$$

Proof of Theorem G. We put

$$\lambda(X_h) = \{J(\gamma_h)\} \quad \text{and} \quad \lambda(X'_h) = \{J(\gamma'_h)\} \quad \text{for } 1 \leq h \leq r,$$

where $\gamma_h, \gamma'_h \in \pi_{n+k}(SO(n))$. It is well-known that each complex X_h (resp. X'_h) has the cell-decomposition

$$(6.12) \quad X_h = K_h \cup_{\rho_h} e^{2n+k+1}, \quad (\text{resp. } X'_h = K_h \cup_{\rho_h} e^{2n+k+1}),$$

where

$$K_h = S^n \vee S^{n+k+1}, \quad \text{and}$$

$$\rho_h = i_h * J(\gamma_h) + [\xi_h, i_h] \in \pi_{2n+k}(K_h).$$

$$(\text{resp. } \rho'_h = i_h * J(\gamma'_h) + [\xi_h, i_h] \in \pi_{2n+k}(K_h).)$$

Therefore, $\#_{h=1}^r X_h$ and $\#_{h=1}^r X'_h$ have the cell-decompositions

$$(6.13) \quad X = \#_{h=1}^r X_h = \left(\bigvee_{h=1}^r K_h \right) \bigcup_{\rho} e^{2n+k+1} = K \bigcup_{\rho} e^{2n+k+1}$$

and

$$X' = \#_{h=1}^r X'_h = \left(\bigvee_{h=1}^r K'_h \right) \bigcup_{\rho'} e^{2n+k+1} = K \bigcup_{\rho'} e^{2n+k+1},$$

$$K = \bigvee_{h=1}^r K_h = \bigvee (S^n \vee S^{n+k+1}),$$

where

$$\rho = \sum_{h=1}^r j_{h*}(\rho_h) \in \pi_{2n+k}(K) \quad \text{and}$$

$$\rho' = \sum_{h=1}^r j_{h*}(\rho'_h) \in \pi_{2n+k}(K).$$

Since X and X' are of the same homotopy type it and only if there exists a homotopy equivalence $\theta \in \text{Eq}(K)$ satisfying $\theta \circ \rho = \pm \rho'$, for our purpose, it is sufficient to investigate the action (6.1).

Let $\theta = (\theta_{hs}) \in [K, K]$, where $\theta_{hs} = a_{hs}\sigma_{hs} + b_{hs}\lambda_{hs} + g_{hs}$,

$$a_{hs}, b_{hs} \in Z, \quad \text{and} \quad g_{hs} = p_h^* i_{s*}(g'_{hs}) \in G_{ns} \quad \text{for } 1 \leq h, s \leq r.$$

Then we have

$$(6.14) \quad \theta \circ \rho = \sum_{s=1}^r j_{s*} \left(\sum_{h=1}^r \theta_{hs} \circ \rho_h \right).$$

Here,

$$\begin{aligned} \theta_{hs} \circ \rho_h &= \theta_{hs} \circ i_n \circ J(\gamma_h) + \theta_{hs} \circ [\xi_h, \iota_h] \\ &= (\theta_{hs} \circ i_n) \circ J(\gamma_h) + [\theta_{hs} \circ \xi_h, \theta_{hs} \circ i_n]. \end{aligned}$$

Since $i_h = E i'_h$ and $\sigma_{ns} = id_h = id_s$, it holds that

$$\begin{aligned} \theta_{hs} \circ i_n &= a_{hs} i_s + b_{hs} \lambda_{hs} \circ i_n + g_{hs} \circ i_n \\ &= a_{hs} i_s + b_{hs} (\xi_s \circ p_h) \circ i_n + i_s \circ g'_{hs} \circ p_h \circ i_n \\ &= a_{hs} i_s. \quad (\text{by } p_h \circ i_n = 0) \end{aligned}$$

Similarly, it follows from $\xi_h = E \xi'_h$ and $p_n \circ \xi_h = \iota_{n+k+1}$ that

$$\begin{aligned} \theta_{hs} \circ \xi_h &= a_{hs} \xi_s + b_{hs} (\xi_s \circ p_h) \circ \xi_h + i_s \circ g'_{hs} \circ p_h \circ \xi_h \\ &= (a_{hs} + b_{hs}) \xi_s + i_{s*}(g'_{hs}). \end{aligned}$$

Hence we have

$$(6.15) \quad \theta_{hs} \circ \rho_h = a_{hs} i_s \circ J(\gamma_h) + a_{hs} (a_{hs} + b_{hs}) [\xi_s, i_s] + i_{s*}([g'_{hs}, \iota_n]).$$

First, we suppose that X and X' are of the same homotopy type. Then

there exists a homotopy equivalence $\Theta=(\theta_{hs})\in\text{Eq}(K)$ satisfying $\Theta\circ\rho=\pm\rho'$. Hence it follows from (6.14) and (6.15) that we have

$$(6.16) \quad J(\gamma'_s)=\pm\sum_{h=1}^r a_{hs}J(\gamma_h) \pmod{P\pi_{n+k+1}(S^n)},$$

and

$$\sum_{h=1}^r a_{hs}(a_{hs}+b_{hs})=\pm 1 \quad \text{for } 1\leq s\leq r.$$

On the other hand, since $\Theta=(\theta_{hs})\in\text{Eq}(K)$, by using (5.17) we have $A=(\pm a_{hs})\in\text{GL}(r, Z)$. Hence the unimodular matrix A satisfies the condition (6.8).

Conversely, we suppose that there exists some unimodular matrix $A=(a_{hs})\in\text{GL}(r, Z)$ satisfying the condition (6.8). However, considering (6.7), without loss of generality we may assume

$$(6.17) \quad J(\gamma'_s)=\sum_{h=1}^r a_{hs}J(\gamma_h) \quad \text{for } 1\leq s\leq r.$$

We put $\Theta=(\theta_{hs})=(a_{hs}\sigma_{hs}+b_{hs}\lambda_{hs})\in[K, K]$, and $B=(b_{hs})\in\text{Mat}(r, Z)$. Then it follows from (5.17) that

$$(6.18) \quad \Theta\in\text{Eq}(K) \quad \text{if and only if } A+B\in\text{GL}(r, Z).$$

On the other hand, by using (6.15) and (6.17), we have

$$\begin{aligned} \sum_{h=1}^r \theta_{hs}\circ\rho_h &= \sum_{h=1}^r a_{hs}i_{s*}J(\gamma_h)+\sum_{h=1}^r a_{hs}(a_{hs}+b_{hs})[\xi_s, i_s] \\ &= i_{s*}J(\gamma'_s)+\sum_{h=1}^r a_{hs}(a_{hs}+b_{hs})[\xi_s, i_s]. \end{aligned}$$

Hence it follows from (6.13), (6.14) and (6.15) that

$$(6.19) \quad \Theta\circ\rho=\rho' \quad \text{if and only if } \sum_{h=1}^r a_{hs}(a_{hs}+b_{hs})=1 \text{ for } 1\leq s\leq r.$$

Now we put $B=({}^tA)^{-1}-A\in\text{Mat}(r, Z)$. Then we also obtain ${}^tA(A+B)=E_r$, where E_r is the identity matrix of degree r . Hence, by using (6.18) and (6.19), we have

$$(6.20) \quad \Theta\in\text{Eq}(K) \quad \text{and} \quad \Theta\circ\rho=\rho'.$$

Therefore X and X' are of the same homotopy type. This completes the proof. Q. E. D.

REFERENCES

- [1] Y. ANDO AND K. YAMAGUCHI, On the homotopy self-equivalences of the product $A\times B$, Proc. Japan Acad., **58** (1982), 323-325.
- [2] W. BARCUS AND M.G. BARRATT, On the homotopy classification of a fixed map,

- Trans. Amer. Math. Soc., **88** (1958), 57-74.
- [3] H. ISHIMOTO, Homotopy classification of connected sums of sphere bundles over spheres I, Nagoya Math. J., **88** (1981), 15-36.
 - [4] I. M. JAMES AND J. H. C. WHITEHEAD, The homotopy theory of sphere bundles over spheres I, Proc. London Math. Soc., **4** (1954), 196-218.
 - [5] P. J. KAHN, Self-equivalences of $(n-1)$ -connected $2n$ -manifolds, Math. Ann., **180** (1969), 26-47.
 - [6] S. OKA, Groups of self-equivalences of certain complexes, Hiroshima Math. J., **2** (1972), 285-298.
 - [7] S. OKA, N. SAWASHITA AND M. SUGAWARA, On the group of selfequivalences of a mapping cone, Hiroshima Math. J., **4** (1974), 9-28.
 - [8] S. SASAO, Homotopy types of spherical fibre spaces over spheres, Pacific J. Math., **52** (1974), 207-219.
 - [9] S. SASAO AND Y. ANDO, On the group $\varepsilon(K(\pi, 1) \times X)$ for 1-connected CW -complexes X , Kodai Math. J., **5** (1982), 65-70.
 - [10] A. J. SIERADSKI, Twisted self-homotopy equivalences, Pacific. J. Math. **34** (1970), 789-802.
 - [11] I. TAMURA, On the classification of sufficiently connected manifolds, J. Math. Soc. Japan, **20** (1968), 371-389.
 - [12] H. TODA, Composition methods in homotopy groups of spheres, Annals of Math. Studies. **49** (1962), Princeton Univ. Press.
 - [13] C. T. C. WALL, Classification of $(n-1)$ -connected $2n$ -manifolds, Annals of Math., **75** (1962), 163-189.
 - [14] K. YAMAGUCHI, On the homotopy type of CW -complexes with the form $S^2 \cup e^1 \cup e^6$, Kodai Math. J., **5** (1982), 303-312.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO, TOKYO
JAPAN