

EINSTEIN METRICS AND FIBRED RIEMANNIAN STRUCTURES

BY TADAYUKI MATSUZAWA

§ 0. Introduction.

As is well known, there are many homogeneous Einstein spaces having fibred structures, for example, the total spaces of Hopf fibrations $S^{2n+1} \rightarrow CP^n$, $S^{4n+3} \rightarrow HP^n$ and the complex projective spaces $CP^{2n+1} \rightarrow HP^n$, where the base spaces and the fibres of these fibrations are also Einsteinian. Jenzen [3] and Ziller [5] constructed non-canonical homogeneous Einstein metrics on these standard homogeneous Einstein spaces by varying metrics homothetically along fibre directions. Following Berard-Bergery and Bourguignon [1], this type of metric variation will be called a canonical variation (See § 3, for the canonical variation). Then, in this paper, we describe how a new Einstein metric can be found by the canonical variation in a fibred Riemannian space, whose total space is Einsteinian and not necessarily homogeneous, and obtain the following theorem which will be proved in § 4.

THEOREM. *Let $\pi: M \rightarrow B$ be a fibred Riemannian structure with totally geodesic fibres where the total space is complete. Suppose the total space, base space and fibres are Einsteinian. Then the canonical variation at the parameter value equal to*

$$\frac{s^F}{f} / \left(\frac{s^B}{b} - \frac{s^F}{f} \right)$$

gives another Einstein metric to the total space if the parameter value is positive not one, where b and f are respectively dimension of the base space B and the fibres, s^B and s^F respectively the scalar curvature of the base space and the fibres.

The author wishes to thank Professor S. Ishihara and Dr. K. Sakamoto for their encouragements and helpful suggestions.

§ 1. Structure tensors of a fibred Riemannian structure.

Let $\pi: M \rightarrow B$ be a fibred Riemannian structure with totally geodesic fibres. We denote by \langle , \rangle the metric of the total space and by ∇ the Levi-Civita con-

nection determined by the metric \langle, \rangle . Following O'Neill [4], we define in M the tensors T and A for arbitrary vector fields E and F by

$$T_E F = \mathcal{C}\mathcal{V}\nabla_{\mathcal{C}\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{C}\mathcal{V}E}\mathcal{C}\mathcal{V}F$$

and

$$A_E F = \mathcal{C}\mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{C}\mathcal{V}F,$$

where $\mathcal{C}\mathcal{V}E$ and $\mathcal{H}E$ denote the vertical and horizontal parts of vector fields E respectively. The tensor T vanishes since fibres are totally geodesic submanifolds. The tensor A is called the structure tensor and has the properties stated in the following lemmas:

LEMMA 1.1. *The tensor A_E is skew-symmetric operator on the tangent space of M reversing vertical and horizontal subspaces.*

LEMMA 1.2. *The tensor A_E is horizontal i. e.*

$$A_E = A_{\mathcal{H}E}$$

LEMMA 1.3. *The structure tensor A is alternative for horizontal vector fields X and Y i. e.*

$$A_X Y = -A_Y X = \frac{1}{2}\mathcal{C}\mathcal{V}[X, Y].$$

Proofs of these lemmas are, for example, found in O'Neill [4].

By Hermann [2], completeness of the total space implies that the fibres are isometric each other, then, in the fibred Riemannian structures the Ricci tensor and the scalar curvature of the fibres make senses.

§ 2. Einstein condition and the structure tensor.

We denote by Ric the Ricci tensor of the Riemannian space (M, \langle, \rangle) . Let $\{Z_1, \dots, Z_b\}$, where $b = \dim B$, be a local orthonormal basic vector fields in M i. e. unique horizontal vector fields projected to a local orthonormal frame of the base space B by π_* . Then, we have

PROPOSITION 2.1. *For vertical vectors U, V and horizontal ones X and Y ,*

$$\text{Ric}_{UV} = \text{Ric}_{UV}^F + \sum_{i=1}^b \langle A_{Z_i} U, A_{Z_i} U \rangle,$$

$$\text{Ric}_{UX} = - \sum_{i=1}^b \langle (\nabla_{Z_i} A)_X Z_i, U \rangle,$$

$$\text{Ric}_{XY} = \text{Ric}_{XY}^R - 2 \sum_{i=1}^b \langle A_{Z_i} X, A_{Z_i} Y \rangle,$$

where Ric^F and Ric^B denote the Ricci tensor of the fibres and the base space respectively.

Proof. Let $\{U_1, \dots, U_f\}$ be a local orthonormal vertical vector fields in M , where f is the fibre dimension of the fibre. Then, by O'Neill's formula of curvature tensor, we obtain the first and the second identities and

$$\text{Ric}_{XY} = \text{Ric}_{XY}^B - 3 \sum_{i=1}^b \langle A_{Z_i} X, A_{Z_i} Y \rangle + \sum_{j=1}^f \langle A_X U_j, A_Y U_j \rangle.$$

But, using next computations, we can rewrite this into the form stated in the proposition: we have, by Lemmas 1.1, 1.2 and 1.3,

$$\begin{aligned} & \sum_{j=1}^f \langle A_X U_j, A_Y U_j \rangle \\ &= \sum_{j=1}^f \sum_{i=1}^b \langle A_X U_j, Z_i \rangle \langle A_Y U_j, Z_i \rangle = - \sum_{j=1}^f \sum_{i=1}^b \langle U_j, A_X Z_i \rangle \langle A_Y U_j, Z_i \rangle \\ &= - \sum_{j=1}^f \sum_{i=1}^b \langle A_Y A_X Z_i, Z_i \rangle = \sum_{i=1}^b \langle A_X Z_i, A_Y Z_i \rangle \\ &= \sum_{i=1}^b \langle A_{Z_i} X, A_{Z_i} Y \rangle \qquad \qquad \qquad \text{Q. E. D.} \end{aligned}$$

COROLLARY 2.2. *If M, B and the fibres are Einsteinian, then we obtain,*

$$\begin{aligned} \sum_{i=1}^b \langle A_{Z_i} U, A_{Z_i} V \rangle &= \left(\frac{s^M}{f+b} - \frac{s^F}{f} \right) \langle U, V \rangle, \\ - \sum_{i=1}^b \langle (\nabla_{Z_i} A)_X Z_i, U \rangle &= 0, \\ - 2 \sum_{i=1}^b \langle A_{Z_i} X, A_{Z_i} Y \rangle &= \left(\frac{s^M}{f+b} - \frac{s^B}{b} \right) \langle X, Y \rangle. \end{aligned}$$

Proof. Since the assumptions imply that

$$\begin{aligned} \text{Ric} &= \frac{s^M}{f+b} \langle \cdot, \cdot \rangle, \\ \text{Ric}^F &= \frac{s^F}{f} \langle \cdot, \cdot \rangle, \\ \text{Ric}^B &= \frac{s^B}{b} \langle \cdot, \cdot \rangle, \end{aligned}$$

Proposition 2, 1 implies directly this corollary. Q. E. D.

§ 3. Canonical variation.

We define new metrics \langle , \rangle_t on the total space with positive parameter t

$$\langle E, F \rangle_t = t \langle \mathcal{C}V E, UF \rangle + \langle \mathcal{A}E, \mathcal{A}F \rangle,$$

for arbitrary vectors E and F .

LEMMA 3.1. *The Levi-Civita connection ∇^t determined by the metric \langle , \rangle_t is given by the formulas*

$$\begin{aligned} \nabla_t^t V &= \nabla_V V, \\ \nabla_t^t X &= \nabla_U X + (t-1)A_X U, \\ \nabla_t^t V &= \nabla_X V + (t-1)A_X V, \\ \nabla_t^t Y &= \nabla_X Y, \end{aligned}$$

for vertical vector fields U, V and horizontal vector fields X, Y . Then, the structure tensor A^t of the new fibred Riemannian structure is given by

$$A_X^t V = t A_X V, \quad A_X^t Y = A_X Y.$$

Proof. The first and fourth formulas are trivial. To prove the second formula, we need the following computations:

$$\begin{aligned} 2 \langle \nabla_t^t X, Y \rangle &= 2 \langle \nabla_U X, Y \rangle_t \\ &= U \langle X, Y \rangle_t + X \langle U, Y \rangle_t - Y \langle U, X \rangle_t \\ &\quad + \langle [U, X], Y \rangle_t + \langle [Y, U] X \rangle_t + \langle U, [X, Y] \rangle_t \\ &= U \langle X, Y \rangle + \langle [U, X], Y \rangle + \langle [Y, U], X \rangle \\ &\quad + \langle U, [Y, X] \rangle + (t-1) \langle U, [Y, X] \rangle \\ &= 2 \langle \nabla_U X, Y \rangle + 2(t-1) \langle U, -A_X Y \rangle \\ &= 2 \langle \nabla_U X + (t-1)A_X U, Y \rangle. \end{aligned}$$

Then we have

$$\mathcal{A} \nabla_t^t X = \mathcal{A} \nabla_U X + (t-1)A_X U.$$

Having

$$\mathcal{C}V \nabla_t^t X = \mathcal{C}V \nabla_U X$$

by similar computation, we obtain the second formula. The third formula is obtained by the same way as the second. Q. E. D.

LEMMA 3.1 implies that the fibration $\pi: (M, \langle, \rangle_t) \rightarrow B$ gives a fibred Riemannian structure with totally geodesic fibres.

LEMMA 3.2. *Let Ric^t denote the Ricci tensor of the Riemannian space (M, \langle, \rangle_t) . Then,*

$$\text{Ric}_{U^t X}^t = t \text{Ric}_{UX}.$$

Thus $\text{Ric}_{U^t X}^t$ vanishes whenever Ric_{UX} vanishes.

Proof. Lemma 1.2 and 3.1 imply

$$\mathcal{C}\mathcal{V}(\nabla_{Z_i}^t A^t)_X Z_i = \mathcal{C}\mathcal{V}(\nabla_{Z_i} A)_X Z_i,$$

so the second formula of Proposition 2.1 and the definition of \langle, \rangle_t imply the lemma. Q. E. D.

§ 4. Proof of Theorem.

By Proposition 2.1, Corollary 2.2 and Lemmas 3.1, 3.2, we have

$$\begin{aligned} \text{Ric}_{UV}^t &= \frac{s^F}{f} \langle U, V \rangle + \sum_{i=1}^b \langle A_{Z_i}^t U, A_{Z_i}^t V \rangle_t \\ &= \frac{s^F}{tf} \langle U, V \rangle_t + t^2 \sum_{i=1}^b \langle A_{Z_i} U, A_{Z_i} V \rangle \\ &= \left(\frac{s^F}{tf} + t \left(-\frac{s^M}{f+b} - \frac{s^F}{f} \right) \right) \langle U, V \rangle_t, \end{aligned}$$

$$\text{Ric}_{U^t X}^t = t \text{Ric}_{UX} = 0,$$

$$\begin{aligned} \text{Ric}_{XY}^t &= \frac{s^B}{b} \langle X, Y \rangle - 2 \sum_{i=1}^b \langle A_{Z_i}^t X, A_{Z_i}^t Y \rangle_t \\ &= \frac{s^B}{b} \langle X, Y \rangle_t - 2t \sum_{i=1}^b \langle A_{Z_i} X, A_{Z_i} Y \rangle \\ &= \left(\frac{s^B}{b} + t \left(-\frac{s^M}{f+b} - \frac{s^B}{b} \right) \right) \langle X, Y \rangle_t. \end{aligned}$$

Consequently, if

$$\frac{s^F}{tf} + t \left(-\frac{s^M}{f+b} - \frac{s^F}{f} \right) = \frac{s^B}{b} + t \left(-\frac{s^M}{f+b} - \frac{s^B}{b} \right),$$

then \langle, \rangle_t gives Einstein metric. By solving this equation of t , we have two solutions, i. e.

$$t=1, \quad t = \frac{s^F}{f} / \left(\frac{s^B}{b} - \frac{s^F}{f} \right).$$

Therefore the theorem is proved.

Q. E. D.

§ 5. Examples.

For the Hopf fibring $S^{4n+3} \rightarrow \mathbf{HP}^n$, the fibres are isometric to the ordinary sphere S^3 and

$$\frac{s^F}{f} = 2, \quad \frac{s^B}{b} = 4(n+2).$$

Then

$$\frac{s^F}{f} / \left(\frac{s^B}{b} - \frac{s^F}{f} \right) = \frac{1}{2n+3}.$$

This is Jenzen's example [3]. For the fibration $CP^{2n+1} \rightarrow \mathbf{HP}^n$, the fibres are isometric to the sphere $S^2(4)$ with constant sectional curvature 4 and

$$\frac{s^F}{f} = 4, \quad \frac{s^B}{b} = 4(n+2).$$

Then

$$\frac{s^F}{f} / \left(\frac{s^B}{b} - \frac{s^F}{f} \right) = \frac{1}{n+1}.$$

This is Ziller's example [5].

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