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ON WEAKLY NONLINEAR CONTRACTIONS

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The purpose of this paper is to generalize some known fixed point theorems to cone-value metric spaces.

(I) Definitions

Let E be a normed space. A set $K\subset E$ is said to be a cone if (i) K is closed (ii) if $u, v\in K$ then $au+bv\in K$ for all $a, b\geqq 0$, (iii) $K\cap(-K)=\{\theta\}$ where θ is the zero of the space E, and (iv) $K^{\circ}\neq\emptyset$, where K° is the interior of K. We say $u\geqq v$ if and only if $u-v\in K$, and $u>v$ if and only if $u-v\in K$ and $u\neq v$. The cone K is said to be strongly normal if there is $\delta>0$ such that if $z=\sum_{i=1}^{n}b_{i}x_{i}$, $x_{i}\in K$, $\|x_{i}\|=1$, $\sum_{i=1}^{n}b_{i}=1$, $b_{i}\geqq 0$ implies $\|z\|>\delta$. The norm in E is said to be semimonotone if there is a numerical constant M such that $\theta\leqq x\leqq y$ implies $\|x\|\leqq M\|y\|$ (where the constant M does not depend on x and y).

Let X be a set and K a cone. A function $d: X\times X\rightarrow K$ is said to be a K-metric on X if and only if (i) $d(x, y)=d(y, x)$, (ii) $d(x, y)=\theta$ if and only if $x=y$, and (iii) $d(x, y)\leqq d(x, z)+d(z, y)$. A sequence $\{x_{n}\}$ in a K-metric space X is said to converge to x_{0} in X if and only if for each $u\in K^{\circ}$ there exists a positive integer N such that $d(x_{n}, x_{0})\leqq u$ for $n\geqq N$. A sequence $\{x_{n}\}$ in X is Cauchy if and only if for each $u\in K^{\circ}$ there exists a positive integer N such that $d(x_{n}, x_{m})\leqq u$ for $n, m\geqq N$. The K-metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges. Let S be a subset of X; a point $x\in X$ is adherent to S if there is a sequence of points of S converging to x. The set of the points of X adherent to S is called the closure of S. The set S is closed if and only if it is equal to its closure. A point in X is a boundary point of S if it is adherent to both S and its complement $C(S)$. The boundary of S, denoted by ∂S, is the set of its boundary points.

Throughout the rest of this paper we assume that K is strongly normal, that E is a reflexive Banach space, that (X, d) is a complete K-metric space, that $P(S)=\{d(x, y); x, y\in S\}$ where S is a subset of X, that $\bar{P}(S)$ denotes the weak closure of $P(S)$, and that $P_{1}(S)=\{z; z\in\bar{P}(S)$ and $z\neq\theta\}$.

Many preliminary results and examples which will be used in our theorems, are listed in [4, 8].

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(II) Main results

DEFINITION 1. The mapping $\emptyset: P_1(S) \to K$ is said to be upper semicontinuous if $\{u_n\}$ and $\{\emptyset u_n\}$ are both weakly convergent, then $\lim_{n\to\infty} \emptyset u_n \leqq \emptyset(\lim_{n\to\infty} u_n)$.

DEFINITION 2. Let $S \subset X$. We say that a mapping $T: S \to X$ satisfies Condition (A) if for each $x \in S$ there exists an element u of S such that $d(x, u) + d(u, Tx) = d(x, Tx)$.

Let $x_0 \in S$. We shall construct two sequences $\{x_n\}$ and $\{x'_n\}$ as follows: Define $x'_1 = Tx_0$. If $x'_1 \in S$, set $x_1 = x'_1$. If $x'_1 \notin S$, choose $x_1 \in S$ so that $d(x_0, x_1) + d(x_1, x'_1) = d(x_0, x'_1)$. Set $x'_2 = Tx_1$. If $x'_2 \in S$, set $x_2 = x'_2$. If not, choose $x_2 \in S$ so that $d(x_1, x_2) + d(x_2, x'_2) = d(x_1, x'_2)$. Continuing in this manner, we obtain $\{x_n\}$, $\{x'_n\}$ satisfying

(i) $x'_{n+1} = Tx_n$,

(ii) $x_n = x'_n$ if $x'_n \in S$, and

(iii) $x_n \in S$ and $d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x_{n-1}, x'_n)$ if $x'_n \notin S$.

Let $Q(x_0) = \{x_i \in \{x_n\}; x_i \neq x'_i\}$ and $F(x_0) = \{x_i \in \{x_n\}; x_i = x'_i\}$.

The following is our main result which is comparable to Theorem 2.2 of Caristi [9] and Theorem 1 of Park and Yoon [18].

LEMMA 1. *Let (X, d) be a complete K-metric space and S a nonempty closed subset of X. Suppose that $T: S \to X$ satisfies Condition (A), (1), (2) and (3).*

(1) $d(Tx, Ty) \leqq \emptyset(d(x, y))$, $x \neq y \in S$,

(2) *$\emptyset(t) < t$ for any $t \in P_1(S)$, where $\emptyset: P_1(S) \to K$ is upper semicontinuous,*

(3) *$x_n \in Q(x_0)$ implies $x_{n-1}, x_{n+1} \in F(x_0)$, where the sequence $\{x_n\}$ defined as above. Then T has a unique fixed point in S.*

Proof. If there exists an integer j such that x_n lies in S for all $n \geqq j$, Chung [8] showed that this sequence of iterates converges to a fixed point of T. Hence we may assume that $Q(x_0)$ contains infinitely many points. Let $Q(x_0) = \{x_{n(k)}\}$.

We assert that

(B) $\{d(x_n, x_{n+1})\}$ weakly converges to θ as $n \to \infty$,

and

(C) $\{d(T(x_n), x_n)\}$ weakly converges to θ as $n \to \infty$.

To prove (B) and (C) we first prove that

(G) $d(x_{n(k)-1}, x'_{n(k)})$ weakly converges to θ as $k \to \infty$.

If we put $n(k+1) = s$, $n(k) = r$, then it follows that

$$\begin{aligned} d(x_{s-1}, x'_s) &= d(T(x_{s-2}), T(x_{s-1})) \\ &\leqq \phi(d(x_{s-2}, x_{s-1})) \\ &\leqq d(x_{s-2}, x_{s-1}) \leqq \cdots \leqq d(x_r, x_{r+1}) \\ &\leqq d(x_r, x'_r) + d(x'_r, x_{r+1}) \\ &\leqq d(x_r, x'_r) + d(x_{r-1}, x_r) \\ &= d(x_{r-1}, x'_r)\,. \end{aligned}$$

Therefore $\{d(x_{n(k)-1}, x'_{n(k)})\}$ and $\{d(x_{n(k)-2}, x_{n(k)-1})\}$ are decreasing and bounded. Let $\{d(x_{m(i)}, x'_{m(i)+1})\}$ be a subsequence of $\{d(x_{n(k)-1}, x'_{n(k)})\}$. There exist subsequences $\{d(x_{s(i)}, x'_{s(i)+1})\}$ of $\{d(x_{m(i)}, x'_{m(i)+1})\}$ and $\{d(x_{s(i)-1}, x_{s(i)})\}$ of $\{d(x_{m(i)-1}, x_{m(i)})\}$ such that $\{d(x_{s(i)}, x'_{s(i)+1})\}$ weakly converges to $z \in K$ and $\{d(x_{s(i)}, x_{s(i)-1})\}$ to $t \in K$. From the fact $d(x_{s-1}, x'_s) \leqq d(x_{s-2}, x_{s-1}) \leqq d(x_{r-1}, x'_r)$, we see that $z=t$.

Because $\phi(d(x_{s-2}, x_{s-1})) \geqq d(x_{s-1}, x'_s)$ we see that $\{\phi(d(x_{s-1}, x_{s-2}))\}$ is bounded. For convenience, we can assume that $\{\phi(d(x_{s(i)}, x_{s(i)-1}))\}$ has a weak limit. By the upper semicontinuity, we have $\phi(z) \geqq z$. Therefore $z=\theta$ and (G) holds.

If $n(k) < n \leqq n(k+1)$, we have

$$d(x_{n(k+1)-1}, x'_{n(k+1)}) \leqq d(x_n, x_{n+1}) \leqq d(x_{n(k)-1}, x'_{n(k)})\,.$$

Therefore (B) holds. From (B) and (G), we see that (C) holds, too.

Now we show that the sequence $\{x_n\}$ is Cauchy. Suppose not. Then there is an $\varepsilon \in K^\circ$ such that for every integer $\imath$, there exist integers $\underline{n}(i)$, $\underline{m}(i)$ with $\imath \leqq \underline{n}(i) < \underline{m}(i)$ such that

(4) $d(x_{\underline{n}(i)}, x_{\underline{m}(i)}) \not\leqq \varepsilon\,.$

Let, for each integer i, $\underline{m}(i)$ be the least integer exceeding $\underline{n}(i)$ satisfying (4); that is

(5) $d(x_{\underline{n}(i)}, x_{\underline{m}(i)}) \not\leqq \varepsilon$ and $d(x_{\underline{n}(i)}, x_{\underline{m}(i)-1}) \leqq \varepsilon\,.$

Since K is semimonotone, the sequence $\{d(x_{\underline{n}(i)}, x_{\underline{m}(i)-1})\}$ is bounded. For convenience, we let $\{d(x_{\underline{n}(i)}, x_{\underline{m}(i)})\}$ weakly converges to z. Since

$$\text{(E)}\quad \begin{cases} d(T(x_{\underline{n}(i)}), T(x_{\underline{m}(i)})) \leqq d(x_{\underline{n}(i)}, T(x_{\underline{n}(i)})) + d(x_{\underline{n}(i)}, x_{\underline{m}(i)}) + d(x_{\underline{m}(i)}, T(x_{\underline{m}(i)}))\,, \\ d(x_{\underline{n}(i)}, x_{\underline{m}(i)}) \leqq d(x_{\underline{n}(i)}, T(x_{\underline{n}(i)})) + d(T(x_{\underline{n}(i)}), T(x_{\underline{m}(i)})) + d(T(x_{\underline{m}(i)}), x_{\underline{m}(i)})\,, \end{cases}$$

we see that $\{d(T(x_{\underline{n}(i)}), T(x_{\underline{m}(i)}))\}$ weakly converges to z. If $z \neq \theta$, we have

(6) $d(T(x_{\underline{n}(i)}), T(x_{\underline{m}(i)})) \leqq \phi(d(x_{\underline{n}(i)}, x_{\underline{m}(i)})) < d(x_{\underline{n}(i)}, x_{\underline{m}(i)})\,.$

Let $\{\phi(d(x_{\underline{n}(i)}, x_{\underline{m}(i)}))\}$ have a weak limit. Therefore we have $\phi(z) \geqq z$. We obtain $z=\theta$. The rest of the proof of the theorem is the same as that of theorem

1 [8]. Therefore $\{x_n\}$ is a Cauchy sequence. By completeness, there is a $u \in S$ such that $\{x_n\}$ converges to u in S, and $Tu = u$. This completes the proof.

DEFINITION 3. Let $S \subset X$. We say that a mapping $T: S \to X$ is metrically inward if for each $x \in S$ there exists an element u of S such that $d(x, u) + d(u, Tx) = d(x, Tx)$ where $u = x$ if and only if $x = Tx$.

It is clear that if T is metrically inward, then T satisfies Condition (A).

DEFINITION 4. Let (X, d) be a complete K-metric space. We call (X, d) a complete K-metric convex space if for any real number c, $0 < c < 1$, and any $x, y \in X$, there exists $z \in X$ such that $d(x, z) = cd(x, y)$, and $d(z, y) = (1-c)d(x, y)$.

LEMMA 2. *If S is a nonempty closed subset of the complete and convex K-metric space (X, d) and if $p_0 \in S$, and $p_1 \notin S$, then there exists a point p in the boundary ∂S of S such that $d(p_0, p) + d(p, p_1) = d(p_0, p_1)$.*

Proof. By Definition 4, we can choose a point $p_2 \in X$ such that

$d(p_0, p_2) = d(p_2, p_1) = 2^{-1}d(p_0, p_1)$ and $d(p_0, p_2) + d(p_2, p_1) = d(p_0, p_1)$.

Case 1: If $p_2 \in S$, we choose $p_3 \in X$ such that $d(p_2, p_3) = d(p_3, p_1) = 2^{-2}d(p_0, p_1)$ and $d(p_2, p_3) + d(p_3, p_1) = d(p_2, p_1)$. Since $d(p_0, p_2) + d(p_2, p_3) + d(p_3, p_1) = d(p_0, p_1)$, and $d(p_0, p_1) \leqq d(p_0, p_3) + d(p_3, p_1)$, we have $d(p_0, p_2) + d(p_2, p_3) \leqq d(p_0, p_3)$ and $d(p_0, p_2) + d(p_2, p_3) = d(p_0, p_3)$. We get $d(p_0, p_3) + d(p_3, p_1) = d(p_0, p_1)$.

Case 2: If $p_2 \notin S$, we choose $p_3 \in X$ such that $d(p_0, p_3) = d(p_3, p_2) = 2^{-2}d(p_0, p_1)$ and $d(p_0, p_3) + d(p_3, p_2) = d(p_0, p_2)$. Since $d(p_0, p_3) + d(p_3, p_2) + d(p_2, p_1) = d(p_0, p_1)$ and $d(p_0, p_1) \leqq d(p_0, p_3) + d(p_3, p_1)$, we have $d(p_3, p_2) + d(p_2, p_1) \leqq d(p_3, p_1)$, and $d(p_3, p_2) + d(p_2, p_1) = d(p_3, p_1)$. We get $d(p_0, p_3) + d(p_3, p_1) = d(p_0, p_1)$.

Continuing the above process, we can choose a sequence $\{p_n\} \subset X$ such that $d(p_n, p_{n+1}) = 2^{-n}d(p_0, p_1)$ and $d(p_0, p_n) + d(p_n, p_1) = d(p_0, p_1)$. Let $p_{k(n)}$ be another point such that $p_{k(n)} \neq p_n$ and $d(p_{k(n)}, p_{n+1}) = 2^{-n}d(p_0, p_1)$. Then either $p_{k(n)} \in S$ and $p_n \notin S$ or $p_{k(n)} \notin S$ and $p_n \in S$. By the construction of $\{p_n\}$, we see that $\{p_n\}$ is Cauchy. There exists a point $p \in X$ such that $\{p_n\}$ converges to p. We also know that $p \in \partial S$. Since $d(p_0, p_n) + d(p_n, p_1) = d(p_0, p_1)$ for all $n \geqq 1$, we have $d(p_0, p_1) \geqq d(p_0, p_n)$, $d(p_0, p_1) \geqq d(p_n, p_1)$. Sequences $\{d(p_0, p_n)\}$ and $\{d(p_n, p_1)\}$ are bounded. Since E is a reflexive Banach space, for convenience, let

$$\begin{cases} d(p_0, p_n) \text{ weakly converge to } x, \text{ and} \\ d(p_n, p_1) \text{ weakly converge to } y. \end{cases}$$

According to the triangular inequality, we have

(7) $d(p_0, p_n) \leqq d(p_0, p) + d(p, p_n)$,

(8) $d(p, p_0) \leqq d(p_0, p_n) + d(p_n, p)$,

(9) $d(p_n, p_1) \leqq d(p_1, p) + d(p, p_n)$,

(10) $d(p_1, p) \leqq d(p_1, p_n) + d(p_n, p)$.

From (7), (8), (9), and (10), we see that $x \leqq d(p_0, p)$, $d(p_0, p) \leqq x$, $y \leqq d(p_1, p)$, and $d(p_1, p) \leqq y$. By (j) [8], we see that $d(p_0, p) + d(p, p_1) = d(p_0, p_1)$. This completes the proof.

THEOREM 1. *Let (X, d) be a complete, convex, K-metric space and S a nonempty closed subset of X. Suppose that $T: S \to X$ satisfies* (1), (2) *and* (11).
(11) $Tx \in S$ *for every* $x \in \partial S$.
Then T has a unique fixed point in S.

Proof. We construct a sequence $\{p_n\}$ in S as follows: Let p_0 be an arbitrary point in S. Let $p_1' = T(p_0)$. If $p_1' \in S$, then $p_1 = p_1'$, otherwise, by lemma 2, we choose $p_1 \in \partial S$ so that $d(p_0, p_1) + d(p_1, p_1') = d(p_0, p_1')$. Suppose that $\{p_i\}$, $\{p_i'\}$, $i = 1, \cdots, N$ have been chosen so that

(i) $p_i' = T(p_{i-1})$, $i = 1, \cdots, N$;

(ii) either $p_i = p_i' \in S$ or $p_i \in \partial S$ and satisfies the relation:

$$d(p_{i-1}, p_i) + d(p_i, p_i') = d(p_{i-1}, p_i').$$

Now set $p_{N+1}' = T(p_N)$. If $p_{N+1}' \in S$ we put $p_{N+1} = p_{N+1}'$, otherwise we choose $p_{N+1} \in \partial S$ so that $d(p_N, p_{N+1}') = d(p_N, p_{N+1}) + d(p_{N+1}, p_{N+1}')$. Thus by induction we are finished.

By the construction of $\{p_n\}$, (11) implies that the sequence $\{p_n\}$ satisfies (3). Lemma 1 is applicable. Hence T has a unique fixed point in S.

THEOREM 2. *Let (X, d) be a complete K-metric space and S a nonempty closed subset of X. Suppose that $T: S \to X$ is metrically inward and that T satisfies* (1), (2) *and* (3). *Then T has a unique fixed point in S.*

If E is the set of all real numbers and if K is the set of all nonnegative reals, then, from (4) and (6), Theorems 1 and 2 may now be restated in the following forms.

THEOREM 3. *Let (X, d) be a complete, convex K-metric space and S a nonempty closed subset of X. Suppose that $T: S \to X$ satisfies* (1), $(\bar{2})$ *and* (11). $(\bar{2})$ $\emptyset(t) < t$ *for any* $t \in P_1(S)$, *where $\emptyset$ is upper semicontinuous from the right on $P_1(S)$. Then T has a unique fixed point in S.*

THEOREM 4. *Let (X, d) be a complete metric space and S a nonempty closed subset of X. Suppose that $T: S \to X$ is metrically inward and that T satisfies* (1), $(\bar{2})$, *and* (3). *Then T has a unique fixed point in S.*

Utilizing the way of the proof of Lemma 2 [19], we have the following result.

THEOREM 5. *Let* (X, d) *be a complete metric space and* S *a nonempty closed subset of* X. *Suppose that* T *is a mapping from* S *into* X. *Then the following conditions are equivalent*:

(i) *For any* $\varepsilon>0$, *there exists* $\delta(\varepsilon)>0$ *such that* $d(Tx, Ty)<\varepsilon$ *whenever*

$$\varepsilon \leqq d(x, y)<\varepsilon+\delta(\varepsilon) \quad and \quad x, y \in S.$$

(ii) *There exists a self-mapping* $\emptyset$ *of* $[0\ \infty)$ *into* $[0\ \infty]$ *such that* $\emptyset(s)<s$ *for all* $s>0$, *where* $\emptyset$ *is upper semicontinuous from the right on* $[0\ \infty)$ *and* $d(Tx, Ty) \leqq \emptyset(d(x, y))$, $x, y \in S$.

From Theorem 5, we have the following results.

THEOREM 6. *Let* (X, d) *be a complete metrically convex space and* S *a nonempty closed subset of* X. *Suppose that* $T: S \rightarrow X$ *satisfies* (i) *in Theorem* 5 *and* (11). *Then* T *has a unique fixed point in* S.

Theorem 6 was proved in [1] by Assad, but it is a special case of our Theorem 1.

THEOREM 7. *Let* (X, d) *be a complete metric space and* S *a nonempty closed subset of* X. *Suppose that* $T: S \rightarrow X$ *is a metrically inward mapping satisfying* (i) *in Theorem* 5 *and* (3). *Then* T *has a unique fixed point in* S.

Theorem 7 was proved in [18] by Park and Yoon, but it is a special case of our Theorem 2.

Many related papers can be found in [2], [4], [7], [8], [9], and [18]. In [11, 12, 13], it is required that the mapping $\emptyset: P_1(S) \rightarrow K$ be monotone but in our paper it isn't.

The mapping $\emptyset: P_1(S) \rightarrow K$ is said to be lower semicontinuous if $\{u_n\}$ and $\{\emptyset u_n\}$ are both weakly convergent, then $\lim_{n\to\infty} \emptyset u_n \geqq \emptyset(\lim_{n\to\infty} u_n)$.

The idea of lower semicontinuity is used in many areas. We would like to have the following result.

THEOREM 8. *Let* (X, d) *be a complete* K*-metric space and* S *a nonempty closed subset of* X. *Suppose that* $T: S \rightarrow X$ *satisfies* (12), (13), (3) *and Condition* (A).
(12) $\emptyset(d(Tx, Ty)) \leqq d(x, y)$, $x \neq y \in S$,
(13) $\emptyset(t)>t$ *for any* $t \in P_1(S)$, *where* $\emptyset: P_1(S) \rightarrow K$ *is lower semicontinuous. Then* T *has a unique fixed point in* S.

Proof. The proof is almost the same as that of Lemma 1. We omit it.
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