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ON A NEW CLASS OF ULTRAHYPERELLIPTIC SURFACES

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1. **Introduction.** Let R be an ultrahyperelliptic surface defined by $y^2=g(x)$ with an entire function $g(x)$ having only an infinite number of simple zeros. Let $\mathcal{M}(R)$ be the class of non-constant meromorphic functions on R. Let $P(f)$ be the number of lacunary values of f in $\mathcal{M}(R)$. Let $P(R)$ be $\sup_{f\in\mathcal{M}(R)} P(f)$. This quantity is called the Picard constant of R. In the ultrahypere lliptic case $2\leqq P(R)\leqq 4$. Surfaces with $P(R)=2$ or 4 are completely determined and those with $P(R)=3$ are still undetermined except for those of finite order [2], [8]. Let S be another ultrahyperelliptic surface defined by $Y^2=G(X)$ with a similar entire function $G(X)$. Let ϕ be a non-trivial analytic mapping of R into S. Then $P(R)\geqq P(S)$. The existence of ϕ is equivalent to the existence of entire h and meromorphic f satisfying

$$f(z)^2g(z)=G(h(z)).$$

Here h is called the projection of ϕ and is defined by

$$S\circ\phi\circ\mathcal{P}_R^{-1}$$

with $\mathcal{P}_R:(x, y)\to x$ and $\mathcal{P}_S:(X, Y)\to X$. This h is one-valued which is equivalent to the rigidity of ϕ [6], [7]. The above functional equation gives a powerful tool to get several criteria for the existence of analytic mappings [1], [2], [7], [8], [9], [10].

In this paper we shall introduce a new class of surfaces. Let R be an ultrahyperelliptic surface defined by $y^2=g(x)$ with entire $g(x)$ having only an infinite number of simple zeros. Let $\mathcal{E}(R)$ be the set of non-constant regular function on R. If there is a member f in $\mathcal{E}(R)$ satisfying the following conditions, then R is called of maximal B type:

(1) There are constants $a\neq\infty$, $c\neq 0$ satisfying

$$a^2-2af_1+f_1^2-f_2^2g=c$$

when $f\circ\mathcal{P}_R^{-1}(x)$ is represented as $f_1(x)+f_2(x)\sqrt{g(x)}$.

(2) There are systems $(a_1, \cdots, a_t)$ and $(n_1, \cdots, n_t)$ such that for each j all the roots of $f=a_j$ have their orders n_jp_{jk} with integers $n_j\geqq 2$ and $p_{jk}\geqq 1$. Further $(n_1, \cdots, n_t)$ satisfies

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$$\sum_{j=1}^{t}\left(1-\frac{1}{n_j}\right)=2 .$$

We shall decide the surfaces of maximal B type and discuss the existence problem of analytic mappings.

2. In order to go further we need several preparations. We firstly remark that

$$\sum_{j=1}^{t}\left(1-\frac{1}{n_j}\right)\leqq 2$$

in general. The Nevanlinna-Selberg theory [11] of two-valued algebroid functions gives

$$(q-4)T(r, f)<\sum_{1}^{q} N(r, w_\nu)-N(r, W_f)+O(\log rT(r, f)) .$$

Our function f satisfies $N(r, \infty)=0$, $N(r, a)=0$. Further

$$N(r, W_f)\geqq \sum_{\substack{f(z_0)\neq\infty \\ f'(z_0)\neq\infty}} \{m(z_0)-1\}$$

with the multiplicity $m(z_0)$ at z_0. Hence

$$(q-4)T(r, f)<\sum_{1}^{q-2} \overline{N}(r, w_\nu)+O(\log rT(r, f)) , \qquad w_\nu\neq\infty,\ a .$$

Now we put $w_\nu=a_\nu$, $q-2=t$. Then

$$(t-2)T(r, f)<\sum_{j=1}^{t}\frac{1}{n_j}N(r, a_j)+O(\log rT(r, f)) .$$

Hence

$$\sum_{j=1}^{t}\left(1-\frac{1}{n_j}\right)\leqq 2 .$$

Recently Toda [12] had proved the following fact:

Let $f_0, \cdots, f_p$ ($p\geqq 1$) be $p+1$ non-constant entire functions and let $a_0, \cdots, a_p$ be $p+1$ meromorphic functions ($\not\equiv 0$) in $|z|<\infty$ such that $T(r, a_j)=o(T(r, f_j))$, $j=0, \cdots, p$. Then, if

$$\sum_{j=0}^{p} a_j(z)f_j^{n_j}(z)=1$$

for some integers $n_0, \cdots, n_p$ ($\geqq 1$),

$$\sum_{j=0}^{p}\frac{1}{n_j}\geqq\frac{1}{p} .$$

3. In this section we shall decide all the surfaces of maximal B type.

Firstly we may assume that $a=0$ and hence $a_j\neq 0$. by

$$\sum_{j=1}^{t}\left(1-\frac{1}{n_j}\right)=2$$

we have the following four possibilities:

i) $t=3,\quad n_1=2,\quad n_2=3,\quad n_3=6;$
ii) $t=3,\quad n_1=2,\quad n_2=4,\quad n_3=4;$
iii) $t=3,\quad n_1=3,\quad n_2=3,\quad n_3=3;$
iv) $t=4,\quad n_1=2,\quad n_2=2,\quad n_3=2,\quad n_4=2.$

Case i). Let us consider the two-valued entire algebroid function satisfying

$$F(z,\ y)\equiv y^2+2Ay+c=0\,.$$

Then $F(z,\ 0)=c$. Further with entire $g_1,\ g_2,\ g_3$

$$F(z,\ a_1)=g_1^2\,,$$

$$F(z,\ a_2)=g_2^3\,,$$

$$F(z,\ a_3)=g_3^6\,.$$

Hence

$$a_3g_2^3-a_2g_3^6=(a_3-a_2)(c-a_2a_3)\,.$$

If $c\neq a_2a_3$, then Toda's result gives a contradiction. If $c=a_2a_3$, then $a_2g_3^6=a_3g_2^3$. This shows that g_2 has only zeros of even order $\geqq 2$, that is, g_2 can be written as g_4^2. Hence we may put $n_2=6$, which is a contradiction by

$$\sum_{j=1}^{3}\left(1-\frac{1}{n_j}\right)=\frac{13}{6}>2\,.$$

When g_2 has no zero, then we may put $n_2=\infty$ and $n_3=\infty$. This is again impossible.

Case ii). This case is impossible by the similar reasoning as in case i).

Case iii). In this case

$$F(z,\ a_j)=g_j^3\,,\qquad j=1,\ 2,\ 3\,.$$

Hence

$$a_1g_2^3-a_2g_1^3=(a_1-a_2)(c-a_1a_2)\,.$$

If $c\neq a_1a_2$, then this is impossible by Toda's result. If $c=a_1a_2$, then $c\neq a_1a_3$ and hence

$$a_1g_3^3-a_3g_1^3=(a_1-a_3)(c-a_1a_3)$$

implies a contradiction.

Case iv). In this case

$$F(z,\ a_j)=g_j^2\,,\qquad j=1,\ 2,\ 3,\ 4\,.$$

We firstly prove that $c\neq a_1a_2$, $c\neq a_2a_3$ implies $c=a_1a_3$. By $c\neq a_1a_2$, $c\neq a_2a_3$ we have

$$\gamma_2^2g_2^2-\gamma_1^2g_1^2=1\,,\quad \gamma_2^2=\frac{a_1}{(a_1-a_2)(c-a_1a_2)}\,,\quad \gamma_1^2=\frac{a_2}{(a_1-a_2)(c-a_1a_2)}\,,$$

$$\gamma_3^2g_3^2-\gamma_2^{*2}g_2^2=1\,,\quad \gamma_3^2=\frac{a_2}{(a_2-a_3)(c-a_2a_3)}\,,\quad \gamma_2^{*2}=\frac{a_3}{(a_2-a_3)(c-a_2a_3)}\,.$$

Hence

$$\gamma_2g_2-\gamma_1g_1=\beta_1e^{H_1}\,,$$

$$\gamma_2g_2+\gamma_1g_1=\frac{1}{\beta_1}e^{-H_1}$$

with entire H_1, $H_1(0)=0$ and a constant $\beta_1\neq 0$. Thus

$$\gamma_2g_2=\frac{1}{2}\Big(\beta_1e^{H_1}+\frac{1}{\beta_1}e^{-H_1}\Big)\,.$$

Similarly we have

$$\gamma_2^*g_2=\frac{1}{2}\Big(\frac{1}{\beta_2}e^{-H_2}-\beta_2e^{H_2}\Big)$$

with entire H_2, $H_2(0)=0$ and a non-zero constant β_2. Hence

$$\gamma_2^*\beta_1d^{H_1}+\frac{\gamma_2^*}{\beta_1}e^{-H_1}=\frac{\gamma_2}{\beta_2}e^{-H_2}-\beta_2\gamma_2e^{H_2}\,.$$

By the impossibility of Borel's identity we have two possibilities

$$\begin{cases}H_2=H_1\\ \gamma_2^*\beta_1=-\beta_2\gamma_2\\ \gamma_2^*\beta_2=\beta_1\gamma_2\,,\end{cases}\qquad \begin{cases}H_2=-H_1\\ \gamma_2^*\beta_1\beta_2=\gamma_2\\ \gamma_2^*=-\beta_1\beta_2\gamma_2\,.\end{cases}$$

In both cases we have

$$\gamma_2^{*2}+\gamma_2^2=0\,,$$

which gives $c=a_1a_3$.

The above fact gives the following possibilities:

$$\begin{pmatrix}c=a_1a_2\\ c=a_3a_4\,,\end{pmatrix}\quad \begin{pmatrix}c=a_1a_3\\ c=a_2a_4\,,\end{pmatrix}\quad \begin{pmatrix}c=a_1a_4\\ c=a_2a_3\,.\end{pmatrix}$$

We may restrict to the first case. Hence

$$a_1g_2^2=a_2g_1^2\,,\qquad a_4g_3^2=a_3g_4^2\,.$$

Since $c\neq a_2a_3$

$$\gamma_3g_3=\frac{1}{2}\Big(\frac{1}{\beta}e^{-H}-\beta e^{H}\Big)$$

with entire H, $H(0)=0$ and a constant $\beta\neq 0$ and

$$\gamma_3^2=\frac{1}{(a_3-a_2)(a_1-a_3)}.$$

Hence

$$A=\frac{1}{2a_3}\left\{\frac{1}{4\gamma_3^2}\left(\beta e^H-\frac{1}{\beta e^H}\right)^2-a_1a_2-a_3\right\},$$

$$C-A^2=-\frac{1}{64a_3^2\gamma_3^4}\left\{\left(\beta e^H-\frac{1}{\beta e^H}\right)^4-2u\left(\beta e^H-\frac{1}{\beta e^H}\right)^2+v^2\right\}$$

with

$$u=\frac{4(a_1a_2+a_3^2)}{(a_3-a_2)(a_1-a_2)},\qquad v^2=\frac{16(a_3^2-a_1a_2)^2}{(a_3-a_2)^2(a_1-a_2)^2}.$$

Further with a constant K

$$C-A^2=K\left\{\left(\beta e^H-\frac{1}{\beta e^H}\right)^2-\delta_1\right\}\left\{\left(\beta e^H-\frac{1}{\beta e^H}\right)^2-\delta_2\right\}.$$

Here $\delta_1\delta_2(\delta_1-\delta_2)\neq 0$ and $(\delta_1+4)(\delta_2+4)\neq 0$. In fact $\delta_1=0$ gives $u^2=u^2-v^2$, $v=0$, that is, $a_3^2=a_1a_2$. Hence $a_1a_2=a_3a_4$ gives $a_2=a_4$ ($a_3\neq 0$). This is impossible. If $\delta_1=\delta_2$, $u^2=v^2$ and hence $a_3=0$ or $a_1a_2=0$, which is again impossible. $\delta_1=-4$ implies $16+8u+v^2=0$. This gives $a_3=0$ or $a_1=a_2$, which is impossible.

We may write

$$C-A^2=\frac{K}{\beta^4e^{4H}}\prod_{j=1}^{8}(\beta e^H-\lambda_j).$$

Hence

$$\lambda_1\lambda_2=\lambda_3\lambda_4=\lambda_5\lambda_6=\lambda_7\lambda_8=-1,$$

$$\lambda_1+\lambda_2=-\lambda_3-\lambda_4=-\sqrt{\delta_1},\quad \lambda_5+\lambda_6=-\lambda_7-\lambda_8=-\sqrt{\delta_2}.$$

$\delta_1\neq -4$, $\delta_2\neq -4$ imply $\lambda_1\neq\lambda_2$, $\lambda_3\neq\lambda_4$, $\lambda_5\neq\lambda_6$, $\lambda_7\neq\lambda_8$. Further $\lambda_i\neq\lambda_j$ if $i\neq j$ and $\lambda_j\neq 0$.

LEMMA. *Let $N_1(r,\gamma,e^H)$ be the counting function of multiple zeros of $e^H-\gamma$, $\gamma\neq 0$. Then*

$$N_1(r,\gamma,e^H)=o(m(r,e^H)).$$

Let $N_2(r,\gamma,e^H)$ be the counting function of simple zeros of $e^H-\gamma$, $\gamma\neq 0$. Then

$$N_2(r,\gamma,e^H)\sim m(r,e^H).$$

This was proved in [5]. It is evident that $e^H-\lambda_1=0$, $e^H-\lambda_2=0$ have no common root if $\lambda_1\neq\lambda_2$, $\lambda_1\lambda_2\neq 0$. These facts imply that $C-A^2$ has infinitely many simple zeros.

Since every f in $\mathcal{E}(R)$ can be represented as

$$-A+f_2\sqrt{g},$$

f satisfies

$$y^2+2Ay+A^2-f_2^2g=0.$$

Hence

$$-f_2^2g=c-A^2.$$

Let us put

$$\beta e^H-\lambda_j=m_j(z)^2L_j(z),$$

where L_j has only simple zeros. Then

$$c-A^2=\frac{K}{\beta^4e^{4H}}\prod_{j=1}^{8}m_j(z)^2\prod_{j=1}^{8}L_j(z).$$

Hence we may put

$$g=\prod_{j=1}^{8}L_j(z).$$

However there does not occur any trouble even if we adopt

$$\prod_{j=1}^{8}(\beta e^H-\lambda_j), \qquad H(0)=0$$

as g, since the structure of R is invariant under this change and $\mathcal{E}(R)$ is too. Hence we put

$$g=\prod_{j=1}^{8}(\beta e^H-\lambda_j), \qquad H(0)=0.$$

Here $\lambda_i\neq\lambda_j$ for $i\neq j$, $\lambda_j\neq 0$ and further

$$\lambda_1\lambda_2=\lambda_3\lambda_4=\lambda_5\lambda_6=\lambda_7\lambda_8=-1,$$

$$\lambda_1+\lambda_2=-\lambda_3-\lambda_4=-\sqrt{\delta_1}, \quad \lambda_5+\lambda_6=-\lambda_7-\lambda_8=-\sqrt{\delta_2}.$$

Another representation of g is

$$g=\beta^8e^{8H}-A_2\beta^6e^{6H}+A_4\beta^4e^{4H}-A_2\beta^2e^{2H}+1$$

with entire H, $H(0)=0$, a constant $\beta\neq 0$ and

$$A_2=4+2u, \qquad A_4=6+4u+v^2.$$

For u, v we have

$$v\neq 0, \quad u^2\neq v^2, \quad 16+8u+v^2\neq 0.$$

4. In §3 we have gotten the representation of g and hence the surface R defined by $y^2=g(x)$. We shall now prove that this is really of maximal B type. We may adopt

$$g=-\frac{(a_3-a_2)^2(a_1-a_2)^2}{64a_3^2\beta^4e^{4H}}\prod_{j=1}^{8}(\beta e^H-\lambda_j).$$

Let us consider the following function

$$f_1+\sqrt{g}\,, \qquad f_1^2=a_1a_2-g\,.$$

This belongs to $\mathcal{E}(R)$. We now put

$$f_1=\frac{1}{2a_3}\left\{\frac{(a_3-a_2)(a_1-a_2)}{4}\left(\beta e^H-\frac{1}{\beta e^H}\right)^2-a_1a_2-a_3^2\right\}.$$

This gives

$$a_3^2-2a_3f_1+a_1a_2=-\frac{(a_3-a_2)(a_1-a_2)}{4}\left(\beta e^H-\frac{1}{\beta e^H}\right)^2\equiv g_3^2\,,$$

$$a_4^2-2a_4f_1+a_3a_4=\frac{a_1a_2}{a_3^2}(a_1a_2-2a_3f_1+a_3^2)$$

$$=\frac{a_1a_2}{a_3^2}g_3^2\equiv g_4^2\,,$$

$$a_2^2-2a_2f_1+a_1a_2$$

$$=a_2^2+\frac{a_2}{a_3}\left\{\frac{(a_3-a_2)(a_1-a_2)}{4}\left(\beta e^H-\frac{1}{\beta e^H}\right)^2-a_3^2-a_1a_2\right\}+a_1a_2$$

$$=\frac{a_2}{4a_3}(a_3-a_2)(a_1-a_3)\left(\beta e^H+\frac{1}{\beta e^H}\right)^2\equiv g_2^2\,,$$

$$a_1^2-2a_1f_1+a_1a_2$$

$$=\frac{a_3a_4}{a_2^2}(a_2^2-2a_2f_1+a_1a_2)=\frac{a_3a_4}{a_2^2}g_2^2\equiv g_1^2\,.$$

Thus our R belongs to the class of maximal B type.

5. Let S be another ultrahyperelliptic surface defined by $Y^2=G(X)$ with entire G having only infinitely many simple zeros. Let ϕ be a non-trivial analytic mapping of S into R. Then we have the following fact: If R is of maximal B type, then S is also of maximal B type if ϕ exists.

We shall prove this. Let h be the projection of ϕ, that is, $h=\mathcal{P}_R\circ\phi\circ\mathcal{P}_S^{-1}$. Let f be a member of $\mathcal{E}(R)$ such that f satisfies two conditions of maximal B type. Then

$$f\circ\mathcal{P}_R^{-1}=f_1+f_2\sqrt{g}\,,$$

$$a^2-2af_1+f_1^2-f_2^2g=c$$

for some $a\neq\infty$ and for a non-zero constant c. Transplanting f on S by ϕ, that is,

$$f\circ\phi\circ\mathcal{P}_S^{-1}$$

we have

$$f\circ\phi\circ\mathcal{P}_S^{-1}=f\circ\mathcal{P}_R^{-1}\circ\mathcal{P}_R\circ\phi\circ\mathcal{P}_S^{-1}$$
$$=f\circ\mathcal{P}_R^{-1}\circ h=f_1\circ h+f_2\circ h\sqrt{g\circ h}\,.$$

Hence

$$a^2-2af_1\circ h+(f_1\circ h)^2-(f_2\circ h)^2 g\circ h=c\,.$$

However by [7]

$$f^{*2}G=g\circ h$$

with meromorphic f^*. However g and G have only simple zeros. Hence f^* is entire. Thus

$$a^2-2af_1\circ h+(f_1\circ h)^2-(f_2\circ h)^2 f^{*2}G=c\,.$$

Let

$$f\circ\mathcal{P}_S^{-1}=f_1\circ h+(f_2\circ h)f^*\sqrt{G}\,.$$

Then $\hat{f}\in\mathcal{E}(S)$ and a is the desired lacunary value of f. The condition (2) in the definition of maximal B type holds for f with the same a_1, a_2, a_3, a_4. Thus we have the desired result.

6. Let R be of maximal B type. We shall consider the existence problem of analytic mappings of R into another S or of S into R.

Assume that $P(S)=4$. Consider a non-trivial analytic mapping ϕ of R into S. Then there exist an entire function h and a meromorphic function f such that

$$g\equiv\beta^8e^{8H}-A_2\beta^6e^{6H}+A_4\beta^4e^{4H}-A_2\beta^2e^{2H}+1$$
$$=f(z)^2(e^{L\circ h}-\delta_1)(e^{L\circ h}-\delta_2)$$

with constants δ_1, δ_2, $\delta_1\delta_2(\delta_1-\delta_2)\neq0$. For simplicity's sake we put $M=L\circ h-L\circ h(0)$, $c=\exp L\circ h(0)$. The right hand side is

$$f(z)^2(ce^M-\delta_1)(ce^M-\delta_2)\,.$$

Then

$$N_2(r,0,g)=N_2(r,0,(ce^M-\delta_1)(ce^M-\delta_2))$$
$$\sim2m(r,e^M)$$

and

$$N_2(r,0,g)\sim8m(r,e^H)$$

with a negligible exceptional set of r. Hence

$$4m(r,e^H)\sim m(r,e^M)\,.$$

Further

$$2N(r,0,f)\leqq N_1(r,0,g)+\bar{N}_1(r,0,g)=o(m(r,e^H))\,,$$

$$2N(r, \infty, f)\leqq \overline{N}_1(r, 0, (ce^M-\delta_1)(ce^M-\delta_2))$$
$$+\overline{N}_1(r, 0, (ce^M-\delta_1)(ce^M-\delta_2))$$
$$=o(m(r, e^M)).$$

By differentiation of

$$g=f^2(ce^M-\delta_1)(ce^M-\delta_2)$$

and by elimination of f^2 we have

$$a_1e^{2M+8H}+a_2e^{2M+6H}+a_3e^{2M+4H}+a_4e^{2M+2H}+a_5e^{2M}$$
$$+a_6e^{M+8H}+a_7e^{M+6H}+a_8e^{M+4H}+a_9e^{M+2H}+a_{10}e^M$$
$$+a_{11}e^{8H}+a_{12}e^{6H}+a_{13}e^{4H}+a_{14}e^{2H}+a_{15}=0,$$

$$a_1=\Big(\frac{2f'}{f}+2M'-8H'\Big)\beta^8c^2, \qquad a_2=\Big(\frac{2f'}{f}+2M'-6H'\Big)(-A_2)\beta^6c^2,$$

$$a_3=\Big(\frac{2f'}{f}+2M'-4H'\Big)A_4\beta^4c^2, \qquad a_4=\Big(\frac{2f'}{f}+2M'-2H'\Big)(-A_2)\beta^2c^2,$$

$$a_5=\Big(\frac{2f'}{f}+2M'\Big)c^2, \qquad a_6=\Big(-\frac{2f'}{f}-M'+8H'\Big)\beta^8(\delta_1+\delta_2)c,$$

$$a_7=\Big(\frac{2f'}{f}+M'-6H'\Big)A_2\beta^6(\delta_1+\delta_2)c, \quad a_8=\Big(-\frac{2f'}{f}-M'+4H'\Big)\beta^4A_4(\delta_1+\delta_2)c,$$

$$a_9=\Big(\frac{2f'}{f}+M'-2H'\Big)A_2\beta^2(\delta_1+\delta_2)c, \qquad a_{10}=\Big(-\frac{2f'}{f}-M'\Big)(\delta_1+\delta_2)c,$$

$$a_{11}=\Big(2\frac{f'}{f}-8H'\Big)\beta^8\delta_1\delta_2, \qquad a_{12}=\Big(-\frac{2f'}{f}+6H'\Big)A_2\beta^6\delta_1\delta_2,$$

$$a_{13}=\Big(2\frac{f'}{f}-4H'\Big)A_4\beta^4\delta_1\delta_2, \qquad a_{14}=\Big(-\frac{2f'}{f}+2H'\Big)A_2\beta^2\delta_1\delta_2,$$

$$a_{15}=2\frac{f'}{f}\delta_1\delta_2.$$

Evidently $T(r, a_j)=N(r, \infty, a_j)+m(r, a_j)=o(m(r, e^M))+o(m(r, e^H))$ for every j, $1\leqq j\leqq 15$. Now we can make use of Nevanlinna's proof [3] of the impossibility of Borel's identity. By $m(r, e^{4H})\sim m(r, e^M)$ we can save our consideration and conclude either $M=4H$ or $M=-4H$. Indeed we have firstly the existence of $(c_j)_{j=1,\dots,14}$ such that

$$c_1a_1e^{2M+6H}+c_2a_2e^{2M+4H}+c_3a_3e^{2M+2H}+c_4a_4e^{2M}$$
$$+c_5a_5e^{2M-2H}+c_6a_6e^{M+6H}+c_7a_7e^{M+4H}+c_8a_8e^{M+2H}$$
$$+c_9a_9e^M+c_{10}a_{10}e^{M-2H}+c_{11}a_{11}e^{6H}+c_{12}a_{12}e^{4H}$$

$$+c_{13}a_{13}e^{2H}+c_{14}a_{14}=0\,.$$

If $c_ic_j=0$ $(i\neq j,\ i,\ j=1,\cdots,13)$, then we have only one possible case

$$c_7a_7e^{M+4H}+c_{14}a_{14}=0\,,$$

which gives $M+4H=0$. If there is at least one $c_ic_j\neq 0$ $(i,\ j=1,\cdots,13,\ i\neq j)$, then we have the existence of $(c'_j)_{j=1,\dots,13}$ such that

$$c'_1a_1e^{2M+4H}+c'_2a_2e^{2M+2H}+\cdots+c'_{12}a_{12}e^{2H}+c'_{13}a_{13}=0\,.$$

If $c'_ic'_j=0$ $(i\neq j,\ i,\ j=1,\cdots,12)$, then we have two possible cases

$$c'_6a_6e^{M+4H}+c'_{13}a_{13}=0$$

and

$$c'_{10}a_{10}e^{M-4H}+c'_{13}a_{13}=0\,.$$

These give either $M+4H=0$ or $M-4H=0$. If there is at least one $c'_ic'_j\neq 0$ $(i\neq j,\ i,\ j=1,\cdots,12)$, we continue the same process repeatedly. In each step we have the desired result: $M=4H$ or $M=-4H$.

The case $M=4H$. Then we have

$$a_1e^{16H}+a_2e^{14H}+(a_3+a_6)e^{12H}+(a_4+a_7)e^{10H}$$
$$+(a_5+a_8+a_{11})e^{8H}+(a_9+a_{12})e^{6H}+(a_{10}+a_{13})e^{4H}$$
$$+a_{14}e^{2H}+a_{15}=0\,.$$

By our earlier result in [2] this gives

$$a_1=a_2=a_3+a_6=a_4+a_7=a_5+a_8+a_{11}$$
$$=a_9+a_{12}=a_{10}+a_{13}=a_{14}=a_{15}=0\,.$$

Hence f is a constant and $A_2=0$,

$$cA_4=-\beta^4(\delta_1+\delta_2)\,,\qquad \beta^8\delta_1\delta_2=c^2\,.$$

The case $M=-4H$. Then we have

$$a_5e^{-8H}+a_4e^{-6H}+(a_3+a_{10})e^{-4H}+(a_2+a_9)e^{-2H}$$
$$+a_1+a_8+a_{15}+(a_7+a_{14})e^{2H}+(a_6+a_{13})e^{4H}$$
$$+a_{12}e^{6H}+a_{11}e^{8H}=0\,.$$

This gives

$$a_5=a_4=a_3+a_{10}=a_2+a_9=a_1+a_8+a_{15}$$
$$=a_7+a_{14}=a_6+a_{13}=a_{12}=a_{11}=0\,.$$

Hence

$$\frac{f'}{f}=-M', \qquad A_2=0,$$

$$A_4\beta^4c=-\delta_1-\delta_2, \qquad \beta^8c^2=\delta_1\delta_2.$$

Hence we have the following

THEOREM 1. *Let R be of maximal B type and let S be the surface of $P(S)=4$. Assume that there is a non-trivial analytic mapping ϕ of R into S. Then, with entire projection h of ϕ, $A_2=0$ and either*

$$4H=L\circ h-L\circ h(0),$$

$$A_4=-e^{-L\circ h(0)}\beta^4(\delta_1+\delta_2),$$

$$e^{2L\circ h(0)}=\beta^8\delta_1\delta_2$$

or

$$4H=-L\circ h+L\circ h(0),$$

$$A_4\beta^4=-e^{-L\circ h(0)}(\delta_1+\delta_2),$$

$$\beta^8e^{2L\circ h(0)}=\delta_1\delta_2.$$

If the conditions hold, then ϕ exists.

The inverse statement is trivial by [7].

COROLLARY 1. *Let R be of maximal B type. If $P(R)=4$, then $A_2=0$, that is, on assuming that 0 is lacunary*

$$2a_3^2+a_1a_2+a_3a_1-a_3a_2+a_2^2=0$$

and vice versa.

THEOREM 2. *Let R be of maximal B type and let S be the surface of $P(S)=4$. Assume that there is a non-trivial analytic mapping ϕ of S into R. Then $A_2=0$ and either*

$$4H\circ h-4H\circ h(0)=L,$$

$$A_4=-\beta^4e^{4H\circ h(0)}(\delta_1+\delta_2),$$

$$\beta^8e^{8H\circ h(0)}\delta_1\delta_2=1$$

or

$$4H\circ h-4H\circ h(0)=-L,$$

$$A_4\beta^4e^{4H\circ h(0)}=-(\delta_1+\delta_2),$$

$$\beta^8e^{8H\circ h(0)}=\delta_1\delta_2.$$

If the conditions hold, then ϕ exists.

There is an ultrahyperelliptic surface R of maximal B type and with $P(R)=3$. It is known that $P(R)\geqq 3$ implies

$$g=1-2\beta_1 e^H-2\beta_2 e^L+\beta_1^2 e^{2H}-2\beta_1\beta_2 e^{H+L}+\beta_2^2 e^{2L}$$

with two entire functions H, L ($H(0)=L(0)=0$) and non-zero constants β_1, β_2. Let us put $2H=L$. Then we have

$$g=1-2\beta_1 e^H+(\beta_1^2-2\beta_2)e^{2H}-2\beta_1\beta_2 e^{3H}+\beta_2^2 e^{4H}.$$

If we put

$$-2\beta_1=-A_2\beta^2,$$

$$\beta_1^2-2\beta_2=A_4\beta^4,$$

$$2\beta_1\beta_2=A_2\beta^6,$$

$$\beta_2^2=\beta^8,$$

then g has the form of maximal B type. In this case

$$\beta^4=\beta_2,\quad 4\beta_1^2=A_2^2\beta_2,\quad A_2^2=4A_4+8.$$

Hence a_1, a_2, a_3, a_4 must satisfy

$$a_1a_2=a_3a_4,\qquad 16a_1a_2a_3^2=(a_3-a_2)^2(a_1-a_2)^2.$$

Next we shall prove that

$$y^2=1-2\beta_1e^H+(\beta_1^2-2\beta_2)e^{2H}-2\beta_1\beta_2e^{3H}+\beta_2^2e^{4H}\equiv g_1$$

determine a surface of $P(S)=3$, when $16\beta_2\neq\beta_1^2$.

If $16\beta_2\neq\beta_1^2$, it is easy to prove

$$N_2(r, 0, g_1)\sim 4m(r, e^H).$$

Assume that $P(S)=4$. Then

$$g_1=f^2(e^L-\delta_1)(e^L-\delta_2),\qquad \delta_1\delta_2(\delta_1-\delta_2)\neq 0.$$

Then the similar consideration as in the proof of Theorem 1 does work. And we have either $L=2H$ or $L=-2H$. If $L=2H$, then $a_{15}=0$ implies the constancy of f. Thus

$$1-2\beta_1e^H+(\beta_1^2-2\beta_2)e^{2H}-2\beta_1\beta_2e^{3H}+\beta_2^2e^{4H}$$
$$=c^2(e^{4H}-(\delta_1+\delta_2)e^{2H}+\delta_1\delta_2).$$

This gives $\beta_1=0$, which is a contradiction. If $L=-2H$, then $a_5=a_{11}=0$. Hence we have

$$\frac{f'}{f}=4H',\qquad f=f(0)e^{4H}.$$

Thus

$$1-2\beta_1e^H+(\beta_1^2-2\beta_2)e^{2H}-2\beta_1\beta_2e^{3H}+\beta_2^2e^{4H}$$
$$=c^2e^{8H}(e^{-4H}-(\delta_1+\delta_2)e^{-2H}+\delta_1\delta_2)\,.$$

This gives $\beta_1=0$, which is a contradiction. Therefore $P(S)=3$.

Assume that $16\beta_2=\beta_1^2$. Then $N_2(r, 0, g_1)\sim 2m(r, e^H)$. However $N_2(r, 0, g)\sim 4m(r, e^H)$ if

$$g=1-A_2\beta^2e^H+A_4\beta^4e^{2H}-A_2\beta^6e^{3H}+\beta^8e^{4H}$$

with $\beta^4=\beta_2$, $4\beta_1^2=A_2^2\beta_2$, $A_2^2=4A_4+8$. This is a contradiction. Therefore $16\beta_2\neq\beta_1^2$. Thus $P(R)=3$.

7. Let R and S be of maximal B type. Let ϕ be a non-trivial analytic mapping of R into S. Then

$$g\equiv\beta^4e^{4H}-A_2\beta^3e^{3H}+A_4\beta^2e^{2H}-A_2\beta e^H+1$$
$$=f^2[\gamma^4e^{4L\circ h}-B_2\gamma^3e^{3L\circ h}+B_4\gamma^2e^{2L\circ h}-B_2\gamma e^{L\circ h}+1]$$
$$\equiv f^2G\circ h\,.$$

Let $L\circ h-L\circ h(0)$ be M and let c be $\exp L\circ h(0)$. Then

$$g=f^2[\gamma^4c^4e^{4M}-B_2\gamma^3c^3e^{3M}+B_4\gamma^2c^2e^{2M}-B_2\gamma ce^M+1]\,.$$

By differentiation of this equation and by elimination of f^2 we have

$$\begin{aligned}a_1e^{4H+4M}&+a_2e^{3H+4M}+a_3e^{2H+4M}+a_4e^{H+4M}+a_5e^{4M}\\&+a_6e^{4H+3M}+a_7e^{3H+3M}+a_8e^{2H+3M}+a_9e^{H+3M}+a_{10}e^{3M}\\&+a_{11}e^{4H+2M}+a_{12}e^{3H+2M}+a_{13}e^{2H+2M}+a_{14}e^{H+2M}+a_{15}e^{2M}\\&+a_{16}e^{4H+M}+a_{17}e^{3H+M}+a_{18}e^{2H+M}+a_{19}e^{H+M}+a_{20}e^M\\&+a_{21}e^{4H}+a_{22}e^{3H}+a_{23}e^{2H}+a_{24}e^H+a_{25}=0\,,\end{aligned}$$

$$a_1=\left(4H'-\frac{2f'}{f}-4M'\right)\beta^4\gamma^4c^4\,,\qquad a_2=A_2\left(-3H'+\frac{2f'}{f}+4M'\right)\beta^3\gamma^4c^4\,,$$

$$a_3=A_4\left(2H'-\frac{2f'}{f}-4M'\right)\beta^2\gamma^4c^4\,,\qquad a_4=A_2\left(-H'+\frac{2f'}{f}+4M'\right)\beta\gamma^4c^4\,,$$

$$a_5=\left(-\frac{2f'}{f}-4M'\right)\gamma^4c^4\,,\qquad a_6=-B_2\left(4H'-\frac{2f'}{f}-3M'\right)\beta^4\gamma^3c^3\,,$$

$$a_7=-B_2A_2\left(-3H'+\frac{2f'}{f}+3M'\right)\beta^3\gamma^3c^3\,,\quad a_8=-B_2A_4\left(2H'-\frac{2f'}{f}-3M'\right)\beta^2\gamma^3c^3\,,$$

$$a_9=-B_2A_2\Big(-H'+\frac{2f'}{f}+3M'\Big)\beta\gamma^3c^3\,,\qquad a_{10}=-B_2\Big(-\frac{2f'}{f}-3M'\Big)\gamma^3c^3\,,$$

$$a_{11}=B_4\Big(4H'-\frac{2f'}{f}-2M'\Big)\beta^4\gamma^2c^2\,,\qquad a_{12}=B_4A_2\Big(-3H'+\frac{2f'}{f}+2M'\Big)\beta^3\gamma^2c^2\,,$$

$$a_{13}=B_4A_4\Big(2H'-\frac{2f'}{f}-2M'\Big)\beta^2\gamma^2c^2\,,\qquad a_{14}=B_4A_2\Big(-H'+\frac{2f'}{f}+2M'\Big)\beta\gamma^2c^2\,,$$

$$a_{15}=B_4\Big(-\frac{2f'}{f}-2M'\Big)\gamma^2c^2\,,\qquad a_{16}=-B_2\Big(4H'-\frac{2f'}{f}-M'\Big)\beta^4\gamma c\,,$$

$$a_{17}=B_2A_2\Big(3H'-\frac{2f'}{f}-M'\Big)\beta^3\gamma c\,,\qquad a_{18}=-B_2A_4\Big(2H'-\frac{2f'}{f}-M'\Big)\beta^2\gamma c\,,$$

$$a_{19}=B_2A_2\Big(H'-\frac{2f'}{f}-M'\Big)\beta\gamma c\,,\qquad a_{20}=B_2\Big(\frac{2f'}{f}+M'\Big)\gamma c\,,$$

$$a_{21}=\Big(4H'-\frac{2f'}{f}\Big)\beta^4\,,\qquad a_{22}=-A_2\Big(3H'-\frac{2f'}{f}\Big)\beta^3\,,$$

$$a_{23}=A_4\Big(2H'-\frac{2f'}{f}\Big)\beta^2\,,\qquad a_{24}=-A_2\Big(H'-\frac{2f'}{f}\Big)\beta\,,\qquad a_{25}=-\frac{2f'}{f}\,.$$

In the present case we have

$$4m(r,\,e^H)\sim N_2(r,\,0,\,g)=N_2(r,\,0,\,G\circ h)\sim 4m(r,\,e^M)$$

and

$$N_1(r,\,\infty,\,f)=o(m(r,\,e^H))\,.$$

Hence

$$T(r,\,a_j)=o(m(r,\,e^H))\,.$$

Thus we can make use of Nevanlinna's method of proof of the impossibility of Borel's identity. In our case $m(r, e^H)\sim m(r, e^M)$ brings us a simplicity. By a similar consideration as in § 6 we only have two possibilities: a) $H=M$ or b) $H=-M$.

Case a). We have

$$a_1e^{8H}+(a_2+a_6)e^{7H}+(a_3+a_7+a_{11})e^{6H}+(a_4+a_8+a_{12}+a_{16})e^{5H}$$
$$+(a_5+a_9+a_{13}a+a_{17}+a_{21})e^{4H}+(a_{10}+a_{14}+a_{18}+a_{22})e^{3H}$$
$$+(a_{15}+a_{19}+a_{23})e^{2H}+(a_{20}+a_{24})e^H+a_{25}=0\,.$$

Hence $a_{25}=0$ and hence f is a constant. Therefore

$$\beta^4=f^2\gamma^4c^4\,,\quad A_2\beta^2=f^2B_2\gamma^3c^3\,,\quad A_4\beta^2=B_4\gamma^2c^2f^2\,,$$
$$A_2\beta=f^2\gamma cB_2\,,\quad f^2=1\,.$$

These give $\beta^4=\gamma^4c^4$. If $\beta^2=\gamma^2c^2$, we have $A_4=B_4$, $A_2=B_2$ or $A_2=-B_2$. If $\beta^2=-\gamma^2c^2$, we have $B_2=A_2=0$ and $A_4=-B_4$.

Case b). We have

$$a_{21}e^{8H}+(a_{16}+a_{22})e^{7H}+(a_{11}+a_{17}+a_{23})e^{6H}+(a_6+a_{12}+a_{18}+a_{24})e^{5H}$$
$$+(a_1+a_7+a_{13}+a_{19}+a_{25})e^{4H}+(a_2+a_8+a_{14}+a_{20})e^{3H}$$
$$+(a_3+a_9+a_{15})e^{2H}+(a_4+a_{10})e^H+a_5=0\,.$$

Hence $a_{21}=0$ and $f=f(0)e^{2H}$. Thus

$$\beta^4e^{4H}-A_2\beta^3e^{3H}+A_4\beta^2e^{2H}-A_2\beta e^H+1$$
$$=d^2[\gamma^4c^4-B_2\gamma^3c^3e^H+B_4\gamma^2c^2e^{2H}-B_2\gamma ce^{3H}+e^{4H}]\,.$$

Therefore

$$\beta^4=d^2\,,\quad A_2\beta^3=B_2\gamma cd^2\,,\quad A_4\beta^2=B_4d^2\gamma^2c^2\,,$$
$$A_2\beta=B_2\gamma^3c^3d^2\,,\quad 1=d^2\gamma^4c^4\,.$$

Hence

$$A_2\beta^3\gamma^3c^3=B_2\,,\quad A_2\beta\gamma c=B_2\,,\quad \beta^4\gamma^4c^4=1\,.$$

If $\beta\gamma c=1$, then $A_2=B_2$ and $A_4=B_4$. If $\beta\gamma c=-1$, then $A_2=-B_2$ and $A_4=B_4$. If $\beta\gamma c=i$, then $A_2=B_2=0$ and $A_4=-B_4$. If $\beta\gamma c=-i$, then $A_2=B_2=0$ and $A_4=-B_4$. Therefore we have the following

THEOREM 3. *Let R and S be of maximal B type. Assume that there is a non-trivial analytic mapping ϕ of R into S. Then there exists an entire function h such that either $H=L\circ h-L\circ h(0)$ and one of the following four holds:*

$$\begin{cases}\beta=\gamma\exp(L\circ h(0))\\ A_4=B_4\\ A_2=B_2\,,\end{cases}\qquad \begin{cases}\beta=-\gamma\exp(L\circ h(0))\\ A_4=B_4\\ A_2=-B_2\,,\end{cases}$$
$$\begin{cases}\beta=i\gamma\exp(L\circ h(0))\\ A_4=-B_4\\ A_2=B_2=0\,,\end{cases}\qquad \begin{cases}\beta=-i\gamma\exp(L\circ h(0))\\ A_4=-B_4\\ A_2=B_2=0\,,\end{cases}$$

or $H=-L\circ h+L\circ h(0)$ and one of the following four holds:

$$\begin{cases}\beta\gamma\exp(L\circ h(0))=1\\ A_4=B_4\\ A_2=B_2\,,\end{cases}\qquad \begin{cases}\beta\gamma\exp(L\circ h(0))=-1\\ A_4=B_4\\ A_2=-B_2\,,\end{cases}$$

$$\begin{cases}\beta\gamma\exp(L\circ h(0))=i\\ A_4=-B_4\\ A_2=B_2=0,\end{cases}\qquad \begin{cases}\beta\gamma\exp(L\circ h(0))=-i\\ A_4=-B_4\\ A_2=B_2=0.\end{cases}$$

The inverse statement is also true.

8. We here mention some remarks.

We can prove that the following types do not occur: The condition (1) holds and in the condition (2)

$$(n_1,\ n_2,\ n_3)=(2,\ 3,\ 5),\ (2,\ 3,\ 4),\ (2,\ 3,\ 3),\ (2,\ 2,\ 3)$$

instead of

$$\sum_{j=1}^{t}\left(1-\frac{1}{n_j}\right)=2.$$

We can also prove that, if the condition (1) holds,

$$(n_1,\ n_2,\ n_3)=(2,\ 2,\ 2)$$

implies the existence of another value a_4 defined by $a_1a_2=a_3a_4$, when $a=0$, say, and the function g_4 as in our result mentioned already, that is, R belongs to the class of maximal B type.

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