K. SHIGA
KODAI MATH. J.
6 (1983), 204—219

MANIFOLDS AND DISCRETE STRUCTURES

BY KOJI SHIGA

Let M be a smooth manifold and let $C(M)$ be the algebra of smooth functions on M. It is well-known that the smooth structure of M is completely determined by the algebraic structure of $C(M)$, while the latter structure is determined by the behaviors of smooth functions restricted on an everywhere dense subset of M. Hence we might say that an everywhere dense subset of M has already sufficiently many informations on the smooth structure of M. This rather perspective view seems to give a plausible reason to the validity of the following fact. Let $\{a_n\}$ and $\{b_n\}$ $(n=1, 2, \cdots)$ be two sequences on M, each of which forms an everywhere dense subset of M. Then there is a diffeomorphism φ of M such that $\varphi\left(\bigcup_{n=1}^{\infty} a_n\right)=\bigcup_{n=1}^{\infty} b_n$. Actually, this will be proved without difficulty (Section 3). We note that in this case we have $\varphi(a_n)=b_{\sigma(n)}$, where σ is a bijective map of the set of positive integers.

In order to obtain more strict relation between diffeomorphisms and everywhere dense sequences on M, it is natural to ask under what condition there exists a diffeomorphism φ with $\varphi(a_n)=b_n$ $(n=1, 2, \cdots)$. This problem-setting will be approved if we consider that any manifold M is obtained by first taking a sequence $a_1, a_2, \cdots$ successively so as to make a dense set and then doing completion of this set. Thus, $\{a_n\}$ is, in a sense, regarded as a generating set of the manifold. Then the above problem implies that, if we have two generating sets $\{a_n\}$ and $\{b_n\}$ of M, under what condition we can find such a diffeomorphism φ of M, that keeps the orders of these generating sets. Really, it seems to be very difficult to approach this problem in general. However, we hope that the problem may turn our attention to various aspects of manifolds which relate continuous or smooth structures with discrete structures.

Besides, we like to make a remark that, in case M is compact, to give an everywhere dense sequence $\{a_n\}$ on M allows us to regard M as a compactification of the set of positive integers via the correspondence of n to a_n $(n=1, 2, \cdots)$. Hence in such a way the set of positive integers will be able to acquire a kind of notion on density through the geometric structure of M.

We say that two sequences $\{a_n\}$ and $\{b_n\}$, everywhere dense on M, define the same discrete structure on M, if there are a diffeomorphism φ on M and an integer n_0 such that $\varphi(a_n)=b_n$ for $n \geqq n_0$.

In the present paper, we first want to clarify a fact that there exists an in-

Received September 6, 1982

timate relation between the set of discrete structures on M and a certain coset space of the symmetric group $\mathfrak{S}_\infty$ based on the positive integers. Since the group structure of $\mathfrak{S}_\infty$ seems interesting in itself, we give some results on the structure of $\mathfrak{S}_\infty$ (Section 1). In Section 2, we give examples of everywhere dense sequences on M, which will show the full of variety and complexity of these objects. In Section 3 we establish a basic relation between discrete structures on M and the group $\mathfrak{S}_\infty$.

Next, we put a problem how to distinguish two discrete structures on a given manifold. Perhaps, there are many ways to approach this problem. For example, the local or global density might be available for that purpose, if one succeeds in introducing such a notion in an appropriate way. But our approach adapted here is somewhat different. We try to construct function spaces on M, closely connected with discrete structures. Specifically, if two dense sequences on M are given, we will get function spaces on M, canonically associated to them. If the supports of these function spaces, being defined as closed sets of M, are not homeomorphic, then these sequences give rise to different discrete structures to M.

We now explain what the notion of support means. Actually, the notion of support was obtained from the following intuitive idea. Let us consider the situation where a physical observation about some quantity spreading on the earth is taken place. If the number of times of observation on a certain area is comparatively small, then the observation will provide only little information on that area. In an extreme case, we will find an area on which we cannot get any information from the observation. According to our view, such area is just regarded as lying outside the support. Sections 4 and 5 concern such subjects.

1. Symmetric group based on the positive integers

Let N be the set of positive integers. Let $\mathfrak{S}_\infty$ be the symmetric group based on N. Hence any element of $\mathfrak{S}_\infty$ is given by a bijective map of N to itself. The cardinality of $\mathfrak{S}_\infty$ is of continuum and $\mathfrak{S}_\infty$ contains a subgroup isomorphic to a countable group which is given arbitrarily beforehand. Thus we may say that $\mathfrak{S}_\infty$ is a huge discrete group. We will give some examples of subgroups of $\mathfrak{S}_\infty$ and discuss related topics.

EXAMPLE 1. If we denote by $\mathfrak{S}_n$ $(n=1, 2, \cdots)$ the symmetric group based on $\{1, \cdots, n\}$, then the natural injection $\{1, \cdots, n\} \to \{1, \cdots, n+1\}$ yields the injection $\mathfrak{S}_n \to \mathfrak{S}_{n+1}$, so that we obtain the inductive limit group

$$\mathfrak{S}_{\lim} = \varinjlim \mathfrak{S}_n .$$

In a similar way, if we start from alternating groups $\mathfrak{A}_n$ $(n=1, 2, \cdots)$, we obtain

$$\mathfrak{A}_{\lim} = \varinjlim \mathfrak{A}_n .$$

Both $\mathfrak{S}_{\lim}$ and $\mathfrak{A}_{\lim}$ are normal subgroups of $\mathfrak{S}_\infty$ with $\mathfrak{A}_{\lim} \subset \mathfrak{S}_{\lim}$. It is known

[1; p. 306] that a proper normal subgroup of $\mathfrak{S}_\infty$ is either $\mathfrak{A}_{\mathrm{lim}}$ or $\mathfrak{S}_{\mathrm{lim}}$. It follows that if we put

$$G_\infty = \mathfrak{S}_\infty / \mathfrak{S}_{\mathrm{lim}}\,,$$

then G_∞ is a simple group. As a result, we can conclude that $\mathfrak{S}_\infty$ is not isomorphic to a subgroup of any direct product of countable groups. Really, this follows from the observation that the image of $\mathfrak{S}_\infty$ by any non-trivial homomorphism has necessarily cardinality $\aleph$.

EXAMPLE 2. Let $\boldsymbol{N} = \coprod_{i=1}^{\infty} S^{(i)}$ be a subdivision of $\boldsymbol{N}$, where each $S^{(i)}$ is an infinite subset of $\boldsymbol{N}$. Considering the symmetric group based on $S^{(i)}$ $(i=1, 2, \cdots)$, we find that there exists a subgroup G of $\mathfrak{S}_\infty$ which is isomorphic to the direct product $\mathfrak{S}_\infty \times \mathfrak{S}_\infty \times \cdots$. Let $\boldsymbol{Q}$ be the additive group of rational numbers. Since each component $\mathfrak{S}_\infty$ of the direct product contains a subgroup isomorphic to $\boldsymbol{Q}$, it follows that G, a fortiori $\mathfrak{S}_\infty$ contains a subgroup which is isomorphic to a countable direct product of $\boldsymbol{Q}$'s. This subgroup is also characterized as the vector space over the rational number field with $\aleph$ dimension. Since $\boldsymbol{R}$ has the same characterization, $\boldsymbol{R}$ is imbedded in $\mathfrak{S}_\infty$ as a subgroup. This interesting fact is remarked by I. Amemiya. Related to this, we propose

Problem. What kind of discrete groups with cardinality $\aleph$ is imbedded in $\mathfrak{S}_\infty$? For example, is it true $GL(n;\boldsymbol{R}) \subsetneq \mathfrak{S}_\infty$?

EXAMPLE 3. Let G be a subgroup of $\mathfrak{S}_\infty$ generated by the elements with the form

$$(2n_1-1,\ 2n_1)(2n_2-1,\ 2n_2)\cdots(2n_k-1,\ 2n_k)\cdots,$$

where $n_1 < n_2 < \cdots < n_k < \cdots$ is any increasing sequence of $\boldsymbol{N}$ and $(2n_k-1,\ 2n_k)$ denote the transposition of $2n_k-1$ and $2n_k$. Then G is a commutative group, each element of which has order 2. Also the cardinality of G is of continuum.

EXAMPLE 4. Consider the totality of $\sigma \in \mathfrak{S}_\infty$ such that

$$0 < \liminf \frac{\sigma(n)}{n}\,, \qquad \limsup \frac{\sigma(n)}{n} < \infty\,.$$

Then this becomes a subgroup of $\mathfrak{S}_\infty$.

EXAMPLE 5. The subgroups of Examples 3 and 4 do not extremely disturb the order of $\boldsymbol{N}$. In order to get subgroups of $\mathfrak{S}_\infty$ with more complicated structure, it is available to use the imbedding φ of $\boldsymbol{N}$ to a compact space X as a dense subset. Then the set of homeomorphisms of X, having the dense set as an invariant set forms a subgroup G_φ of Homeo(X). It is easily seen that, through the operation of G_φ on this set, G_φ is faithfully represented in $\mathfrak{S}_\infty$. Hence G_φ is canonically identified with a subgroup of $\mathfrak{S}_\infty$. Subgroups of $\mathfrak{S}_\infty$ constructed in such a way seem to have complicated structures. We will treat such subgroups in Section 3 in connection with discrete structures on a manifold.

Now we try to represent the groups $\mathfrak{S}_\infty$ and G_∞ in the homeomorphism groups of some compact spaces. Let $C(\boldsymbol{N})$ be the normed algebra of bounded functions on $\boldsymbol{N}$. Let X be the Gel'fand representation space of $C(\boldsymbol{N})$. Then X is a compact Hausdorff space and $C(\boldsymbol{N})$ forms a dense set of $C(X)$, where $C(X)$ is the algebra of continuous functions on X. The points of X consist of the maximal ideals of $C(\boldsymbol{N})$, which in turn are canonically identified with ultrafiltres of $\boldsymbol{N}$.

We have then an isomorphism

$$\mathfrak{S}_\infty \cong \mathrm{Homeo}(X).$$

This follows from the observation that any $\sigma \in \mathfrak{S}_\infty$ naturally induces an automorphism σ^* of $C(\boldsymbol{N})$, whence σ^* defines a homeomorphism of X. Hence we have $\mathfrak{S}_\infty \subsetneq \mathrm{Homeo}(X)$. In order to show the surjectivity, take $\tilde{\tau} \in \mathrm{Homeo}(X)$. Then $\tilde{\tau}^*$ gives an isomorphism from $C(X)$ to $C(X)$. Note that $\boldsymbol{N}$ is canonically imbedded in X as an open set and the complement of $\boldsymbol{N}$ has no interior points. Hence the Dirac function at a point $x \in X$ belongs to $C(X)$ if and only if $x \in \boldsymbol{N}$. But Dirac function δ is characterized as an idempotent element of $C(X)$ such that $\{f \mid \delta f = 0\}$ forms a maximal ideal. It follows that $\tilde{\tau}^*$ sends Dirac functions to Dirac functions and that $\tilde{\tau}$ is induced by $\tau \in \mathfrak{S}_\infty$. This proves the surjectivity.

Next we want to show

$$G_\infty \subsetneq \mathrm{Homeo}(X - \boldsymbol{N}).$$

We identify $\mathfrak{S}_\infty$ with $\mathrm{Homeo}(X)$. Then from the above discussion, we find that $\sigma \in \mathfrak{S}_\infty$ induces a homeomorphism of $X - \boldsymbol{N}$. Hence we have a map

$$\Psi : \mathfrak{S}_\infty \longrightarrow \mathrm{Homeo}(X - \boldsymbol{N}).$$

It is sufficient to show that $\mathrm{Ker}\,\Psi = \mathfrak{S}_{\lim}$. It is easy to see $\mathfrak{S}_{\lim} \subset \mathrm{Ker}\,\Psi$. In order to prove the converse implication, take $\sigma \notin \mathfrak{S}_{\lim}$. Then there is a subsequence $\{n_s\}$ $(s=1, 2, \cdots)$ of $\boldsymbol{N}$ with $\sigma(n_s) \neq n_s$. Using this fact, we can find an increasing sequence $\{m_s\}$ such that

$$\{\sigma(m_1), \sigma(m_2), \cdots\} \cap \{m_1, m_2, \cdots\} = \emptyset$$

Let ω be any ultrafiltre containing $\{m_s\}$. Then we have $\omega \in X - \boldsymbol{N}$ and $\sigma(\omega) \neq \omega$. Hence we have $\sigma \notin \mathrm{Ker}\,\Psi$, as desired.

We note that $X - \boldsymbol{N}$ is a compact space. By the compact-open topology, $\mathrm{Homeo}(X)$ and $\mathrm{Homeo}(X - \boldsymbol{N})$ become topological groups, whence $\mathfrak{S}_\infty$ and G_∞ inherit structure of topological groups from these groups. Nevertheless, in what follows, we will regard $\mathfrak{S}_\infty$ and G_∞ only as discrete groups.

2. Examples of dense sequences on a manifold

The general method of constructing dense sequences on a manifold will be discussed in the next section. Examples below are rather instructive in character.

Let M be a compact smooth manifold. Take a smooth triangulation Δ of M and apply barycentric (or standard) subdivisions successively to it. Then we can obtain a sequence of triangulations

$$\Delta,\ \Delta^{(1)},\ \Delta^{(2)},\ \Delta^{(3)},\ \cdots$$

on M. We arrange all the vertices appearing in these triangulations in order in such a way that first we put the vertices in Δ, and next put the vertices in $\Delta^{(1)}$ as successors which do not belong to Δ and so on. Then we get an everywhere dense sequence

$$\{a_1,\ a_2,\ \cdots,\ a_n,\ \cdots\}\,. \tag{1}$$

The way of distribution of this sequence may be intuitively understood as something like what is called 'equi-distributed'.

However, the following give examples of dense sequences on M with more complicated structure. Let $p_1, p_2, \cdots, p_s \cdots$ be a dense sequence of distinct points on M. To each point p_s take a positive continuous function f_s on $M-p_s$ such that $f_s(x)\to\infty$ when $x\to p_s$. Take

$$a_n^{(s)}\in f_s^{-1}(n)\,.$$

Then we obtain sequences of points of M

$$\{a_1^{(1)},\ a_2^{(1)},\ \cdots,\ a_1^{(2)},\ \cdots,\ a_1^{(s)},\ a_2^{(s)},\ \cdots\}\,. \tag{2}$$

We assume that these points are distinct from each other and also from each a_n in (1). We note that if $f_s(x)$ tends slowly to infinity when $x\to p_s$, then $a_1^{(s)}$, $a_2^{(s)}$, $\cdots$ accumulate densely towards p_s.

First take a rapidly increasing sequence $n_1<n_2<\cdots<n_s<\cdots$ (for example $n_s=e^s$) and put in order

$$\{a_1^{(1)},\ \cdots,\ a_{n_1}^{(1)},\ a_1,\ a_{n_1+1}^{(1)},\ \cdots,\ a_{n_2}^{(1)},\ a_2,\ \cdots\}\,. \tag{3}$$

Then we obtain an everywhere dense sequence on M. Observe that any truncated finite sequence of (3) always shows a strong density near the point p_1. Hence, from the point of view of approximation, the property that a sequence is everywhere dense cannot be generally foreseen from the step of approximation.

More generally, combining (1) with (2) and arrange them in some order, we obtain another everywhere dense sequence on M. We want to point out that, in that case, there is no general rule in introducing the order so that the resulting sequence is well distributed. Each order brings utterly different aspect of approximation to dense sequence and individual properties of $\{a_1^{(s)}, a_2^{(s)}, \cdots\}$ will disapper in this stage.

3. Discrete structures on a compact manifold

Let M be a compact smooth manifold. Let $\{a_n\}$ and $\{b_n\}$ $(n=1, 2, \cdots)$ be everywhere dense sequenses on M, where we always assume that $a_m \neq a_n$ and $b_m \neq b_n$ for $m \neq n$. Then we have

PROPOSITION 1. *There exists a diffeomorphism φ of M such that*

$$\varphi\left(\bigcup_{n=1}^{\infty} a_n\right) = \bigcup_{n=1}^{\infty} b_n .$$

Proof. We may assume that M is connected. Let φ_1 be a diffeomorphism of M such that $\varphi_1(a_1) = b_1$. Inductively, we will construct a sequence of diffeomorphism φ_n on M with the properties described below. Suppose that we have obtained a diffeomorphism φ_n such that

$$\varphi_n(a_1) = b_{i_1}, \cdots, \varphi_n(a_n) = b_{i_n}$$

and

$$\varphi_n^{-1}(b_1) = a_{j_1}, \cdots, \varphi_n^{-1}(b_n) = a_{j_n},$$

where $\{i_1, \cdots, i_n\}$ and $\{j_1, \cdots, j_n\}$ are suitable subsets of N.

First consider the case where $\{i_1, \cdots, i_n\} \ni n+1$, say $i_n = n+1$. We have thus $\varphi_n(a_n) = b_{n+1}$. Then we can find a diffeomorphism $\tilde{\varphi}_n$ sufficiently close to φ_n such that

$$\tilde{\varphi}_n(a_i) = \varphi_n(a_i), \qquad i = 1, \cdots, n-1,$$

$$\tilde{\varphi}_n(a_n) = b_{i'_n}, \qquad \text{where} \quad i'_n \neq n+1,$$

$$\tilde{\varphi}_n^{-1}(b_i) = \varphi_n^{-1}(b_i), \qquad i = 1, \cdots, n.$$

Hence if we replace φ_n by $\tilde{\varphi}_n$, we may assume from the outset that $\{i_1, \cdots, i_n\} \not\ni n+1$. In a similar way, we may also assume $\{j_1, \cdots, j_n\} \not\ni n+1$.

In order to obtain φ_{n+1}, first deform φ_n slightly only on a neighborhood of a_{n+1} so that the resulting diffeomorphism φ'_{n+1} satisfies the condition $\varphi'_{n+1}(a_{n+1}) = b_{i_{n+1}}$, $\varphi'_{n+1}(a_i) = \varphi_n(a_i)$ and $\varphi'^{-1}_{n+1}(b_i) = \varphi_n^{-1}(b_i)$ $(i = 1, \cdots, n)$. This procedure is possible since $\{b_n\}$ is dense. Next, deform φ'^{-1}_{n+1} only on a small neighborhood of b_{n+1}. Then, in view of the density of $\{a_n\}$, we can obtain a diffeomorphism φ_{n+1} which satisfies

$$\varphi_{n+1}(a_{n+1}) = b_{i_{n+1}}, \qquad \varphi_{n+1}^{-1}(b_{n+1}) = a_{j_{n+1}},$$

$$\varphi_{n+1}(a_i) = \varphi_n(a_i), \qquad \varphi_{n+1}^{-1}(b_i) = \varphi_n^{-1}(b_i) \quad (i = 1, \cdots, n).$$

This completes the inductive procedure. Put $\varphi = \lim \varphi_n$. If we take each deformation from φ_n to φ_{n+1} sufficiently small, then the limit exists and $\varphi \in \mathrm{Diff}(M)$. Moreover, from the construction we have

$$\varphi_n\left(\bigcup_{i=1}^{n} a_i\right) \subset \bigcup_{i=1}^{\infty} b_i,$$

$$\varphi_n^{-1}\left(\bigcup_{i=1}^{n} b_i\right) \subset \bigcup_{i=1}^{\infty} a_i .$$

Hence, if we take the limit as $n\to\infty$, we find that φ is a desired diffeomorphism. This completes the proof.

From Proposition 1 we know the way how to construct everywhere dense sequences on M. Actually, take first an everywhere dense sequence $\{a_n\}$ on M and apply a diffeomorphism φ to it. Put $b_n=\varphi(a_n)$. After that, take any $\sigma\in\mathfrak{S}_\infty$ and put $c_n=b_{\sigma(n)}$. Then $\{c_n\}$ is an everywhere dense sequence on M and all everywhere dense sequences on M can be obtained in this way. Note that in this case we have $c_n=\varphi(a_{\sigma(n)})$.

Let $\boldsymbol{a}=\{a_n\}$ be an everywhere dense sequence on M and fix this sequence in the following discussion. Let

$$\mathrm{Diff}\,(\boldsymbol{a})=\left\{\varphi\,|\,\varphi\in\mathrm{Diff}\,(M),\ \varphi\Big(\bigcup_{n=1}^{\infty}a_n\Big)=\bigcup_{n=1}^{\infty}a_n\right\}.$$

Then $\mathrm{Diff}\,(\boldsymbol{a})$ is a subgroup of $\mathrm{Diff}\,(M)$. We note that the cardinality of $\mathrm{Diff}\,(\boldsymbol{a})$ is $\aleph$. This follows from the fact that in the proof of Proposition 1 the choice of $\varphi_n(a_n)$ has $\aleph_0$ possibility at each step. Moreover, in view of Proposition 1, for any $\boldsymbol{b}=\{b_n\}$ $\mathrm{Diff}\,(\boldsymbol{b})$ is conjugate to $\mathrm{Diff}\,(\boldsymbol{a})$.

Here we make a simple remark. Let

$$\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \cdots,\ \boldsymbol{a}_n,\ \cdots,\ \boldsymbol{a}_\omega,\ \cdots,\ \boldsymbol{a}_{2\omega},\ \cdots$$

(at most till an ordinal of the second number class) be a sequence of everywhere dense sequences on M, where, if α is not a limit ordinal, $\boldsymbol{a}_\alpha$ is obtained from $\boldsymbol{a}_{\alpha-1}$ by adding countable points. To this sequence we can associate a sequence of subgroups of $\mathrm{Diff}\,(M)$

$$G_{\boldsymbol{a}_1},\ G_{\boldsymbol{a}_2},\ \cdots,\ G_{\boldsymbol{a}_n},\ \cdots,\ G_{\boldsymbol{a}_\omega},\ \cdots,\ G_{\boldsymbol{a}_{2\omega}},\ \cdots\qquad (G_{\boldsymbol{a}_\alpha}=\mathrm{Diff}\,(\boldsymbol{a}_\alpha)).$$

Then these subgroups are conjugate to each other and moreover the cardinality of $G_{\boldsymbol{a}_\beta}/G_{\boldsymbol{a}_\alpha}\cap G_{\boldsymbol{a}_\beta}$ $(\alpha<\beta)$ is always of continuum.

For $\varphi\in\mathrm{Diff}\,(\boldsymbol{a})$, we define $\tilde{\varphi}\in\mathfrak{S}_\infty$ by

$$\varphi(a_n)=a_{\hat{\varphi}(n)},$$

which gives rise to a representation of $\mathrm{Diff}\,(\boldsymbol{a})$ to $\mathfrak{S}_\infty$. Since $\{a_n\}$ is dense, this representation is faithful. Hence $\mathrm{Diff}\,(\boldsymbol{a})$ can be identified with a subgroup of $\mathfrak{S}_\infty$ with cardinality $\aleph$. Consider the coset space

$$\mathrm{Diff}\,(\boldsymbol{a})\backslash\mathfrak{S}_\infty.$$

Take τ and τ' from $\mathfrak{S}_\infty$ and apply them to $\{a_n\}$. So we have two sequences $\{a_{\tau(n)}\}$ and $\{a_{\tau'(n)}\}$. Then in order that there exists a diffeomorphism φ with $\varphi(a_{\tau(n)})=a_{\tau'(n)}$, it is necessary and sufficient that τ and τ' belong to the same coset.

Let $\{b_n\}$ be another dense sequence on M. By Proposition 1 we can find $\sigma\in\mathfrak{S}_\infty$ and $\varphi\in\mathrm{Diff}\,(M)$ such that $b_n=\varphi\circ\sigma(a_n)$, where $\sigma(a_n)$ means $a_{\sigma(n)}$. For simplicity, we denote this by $\boldsymbol{b}=\varphi\sigma(\boldsymbol{a})$. Then the relation

$$\varphi\sigma(\boldsymbol{a})=\varphi'\sigma'(\boldsymbol{a})$$

is equivalent to

$$\varphi'^{-1}\varphi\in\mathrm{Diff}(\boldsymbol{a}) \quad \text{or} \quad \sigma'\sigma^{-1}\in\mathrm{Diff}(\boldsymbol{a}).$$

Hence if we denote by $[\sigma]$ the coset of $\mathrm{Diff}(\boldsymbol{a})\backslash\mathfrak{S}_\infty$ containing σ, we can write

$$\boldsymbol{b}=\varphi[\sigma](\boldsymbol{a}), \tag{4}$$

where the coset $[\sigma]$ is uniquely determined by $\boldsymbol{b}$. This leads to the following definition.

DEFINITION 1. Two sequences $\boldsymbol{a}=\{a_n\}$ and $\boldsymbol{b}=\{b_n\}$, everywhere dense on M, are called *smoothly equivalent* if there exists a diffeomorphism φ such that $\varphi(a_n)=b_n$ $(n=1, 2, \cdots)$.

Then (4) involves that, for any $[\sigma]\in\mathrm{Diff}(\boldsymbol{a})\backslash\mathfrak{S}_\infty$, if we take a representative σ of $[\sigma]$ and put $c_n^{[\sigma]}=a_{\sigma(n)}$, then the set

$$\{\boldsymbol{c}^{[\sigma]} \mid \boldsymbol{c}^{[\sigma]}=\{c_n^{[\sigma]}\},\ [\sigma]\in\mathrm{Diff}(\boldsymbol{a})\backslash\mathfrak{S}_\infty\}$$

gives a complete set of the smooth equivalence classes. Thus we proved

THEOREM 1. *The smoothly-equivalent classes of the sequences on M bijectively correspond to the cosets of* $\mathrm{Diff}(\boldsymbol{a})\backslash\mathfrak{S}_\infty$.

Next, for any subgroup G of $\mathfrak{S}_\infty$, consider the double coset space

$$\mathrm{Diff}(\boldsymbol{a})\backslash\mathfrak{S}_\infty/G.$$

We can easily see that each double coset canonically corresponds to an equivalence class of everywhere dense sequences on M, where the equivalence relation is introduced in the following way:

$\{b_n\}$ and $\{c_n\}$ are equivalent if and only if there are $\varphi\in\mathrm{Diff}(M)$ and $g\in G$ such that $\varphi(b_n)=c_{g(n)}$.

In a particular case where $G=\mathfrak{S}_{\mathrm{lim}}$, G becomes a normal subgroup so that we have

$$\mathrm{Diff}(\boldsymbol{a})\backslash\mathfrak{S}_\infty/\mathfrak{S}_{\mathrm{lim}}=\mathrm{Diff}(\boldsymbol{a})\backslash G_\infty.$$

Hence each coset of $\mathrm{Diff}(\boldsymbol{a})\backslash G_\infty$ represents an equivalence class of dense sequences on M, where the equivalence relation is given by

$\{b_n\}\sim\{c_n\}\Leftrightarrow$ There is an integer N such that $\varphi(b_n)=c_c$ for $n\geqq N$, where $\phi\in\mathrm{Diff}(M)$.

Let $[\{b_n\}]$ denote the equivalence class including $\{b_n\}$.

DEFINITION 2. A manifold M endowed with an equivalence class $[\{b_n\}]$ is called a *manifold with discrete structure*. We say that $[\{b_n\}]$ gives a discrete structure to M.

Summarizing the discussion above, we have

THEOREM 2. *The set of discrete structures on M can be identified with the coset space* $\mathrm{Diff}(\boldsymbol{a})\backslash G_\infty$.

The group $\mathrm{Diff}(M)$ is a subgroup of $\mathrm{Homeo}(M)$, whence we can consider the coset space $\mathrm{Diff}(M)\backslash\mathrm{Homeo}(M)$. This coset space is a huge space. But we have a natural injection

$$\mathrm{Diff}(M)\backslash\mathrm{Homeo}(M) \hookrightarrow \mathrm{Diff}(\boldsymbol{a})\backslash G_\infty . \tag{5}$$

This is seen as follows. For $\psi\in\mathrm{Homeo}(M)$, set $b_n=\psi(a_n)$. We want to show that the map $\psi\to\{b_n\}$ induces the injection (5). Suppose $\psi\notin\mathrm{Diff}(M)$. Then $\boldsymbol{a}$ and $\boldsymbol{b}=\psi(\boldsymbol{a})$ define different discrete structures, since the behaviors of ψ is completely determined on a set $\bigcup_{n\geqq N} a_n$. On the other hand, any $\{\psi(a_n)\}$ and $\{\psi'(a_n)\}$ define the same discrete structure if ψ and ψ' belong to the same coset of $\mathrm{Diff}(M)\backslash\mathrm{Homeo}(M)$. As a consequence, we find that the space of discrete structures on M is also huge.

We will give an additional remark. Even in order that two sequences $\{a_n\}$ and $\{b_n\}$ on M are transformed by a homeomorphism of M in keeping the order fixed, they have to satisfy a rather strong condition that, if a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ tends to some a_m as $n_i\to\infty$, then $\{b_{n_i}\}$ must tend to b_m. This condition, however, is not sufficient to yield a homeomorphism with the property above. A sufficient condition for this is given by

$$L'\rho(b_m, b_n)\leqq\rho(a_m, a_n)\leqq L\,\rho(b_m, b_n)$$

$(m, n=1, 2, \cdots)$ where L and L' are positive integers and ρ is a metric on M. On the other hand, in order that $\{a_n\}$ and $\{b_n\}$ are smoothly-equivalent, it is necessary that the above condition holds, but it is not sufficient.

3. Function spaces

Let X be a compact metric space. If X is a compact manifold, we always assume that the metric is induced from a Riemannian metric. We consider a family of functions ε defined on $X\times\boldsymbol{N}$ with the following properties.

i) The values of ε lie in $\boldsymbol{R}^+$.

ii) For each $x\in X$,

$$\lim_n \varepsilon(x, n)=0 .$$

Let $\Gamma=\Gamma(X\times\boldsymbol{N})$ be the set consisting of functions which satisfy i) and ii). For any ε and $\eta\in\Gamma$, we define a function space $C(X;\varepsilon,\eta)$ which consists of real-valued functions on X, satisfying the following condition:

$$C(X;\varepsilon,\eta)=\{f\,|\sup_{y\in V_{\varepsilon(x,n)}(x)}|f(y)-f(x)|\leqq A\eta(x, n)\quad\text{for } n\geqq n_0\},$$

where $V_{\varepsilon(x,n)}(x)$ denotes the $\varepsilon(x, n)$-neighborhood of x, A is a positive constant independent of (x, n) and n_0 is a suitable positive integer independent of x. Dependence of n_0 in the above definition shows that $C(X; \varepsilon, \eta)$ is considered as an inductive limit space with respect to n_0. In what follows, we will simply write $V_{\varepsilon(x,n)}$ for $V_{\varepsilon(x,n)}(x)$, if there arises no confusion.

The following proposition will be proved immediately.

PROPOSITION 2. *$C(X; \varepsilon, \eta)$ is a subalgebra of the algebra formed by the continuous functions on X.*

DEFINITION 3. If there is a positive constant K independent of (x, n) such that $\varepsilon(x, n) \leqq K\eta(x, n)$, then we say that η *dominates* ε.

Henceforce M is always assumed to be a compact smooth manifold.

PROPOSITION 3. *If η dominates ε, then $C(M; \varepsilon, \eta) \supset C^1(M)$, where $C^1(M)$ denotes the space of C^1-functions on M.*

Proof. If $f \in C^1(M)$, then f satisfies the Lipschitz condition. This means that, for $x, y \in M$, we have $|f(x)-f(y)| \leqq L\rho(x, y)$, where L is a Lipschitz constant and ρ is a distance function on M. Hence we have

$$\sup_{y \in V_{\varepsilon(x,n)}} |f(y)-f(x)| \leqq L\varepsilon(x, n).$$

The right-hand side then is estimated by $LK\eta(x, n)$ from the above, whence we have $f \in C(M; \varepsilon, \eta)$.

DEFINITION 4. If ε and η satisfy

$$C(M; \varepsilon, \eta) + C(M; \eta, \varepsilon) \supset C^1(M),$$

then we say that ε and η are complementary. Here $C(M; \varepsilon, \eta)$ and $C(M; \eta, \varepsilon)$ are both regarded as subspaces of $C^0(M)$, the space of continuous functions on M.

PROPOSITION 4. *If the following condition is fulfilled, then ε and η are complementary.*

There are a finite open covering $\{U_\alpha\}$ of M and continuous functions k_α (>0) defined on U_α such that on each U_α either

$$\varepsilon(x, n) < k_\alpha(x)\eta(x, n)$$

or

$$\eta(x, n) < k_\alpha(x)\varepsilon(x, n)$$

holds for $n \geqq n_0$.

This can be easily proved by localization.

We are interested in the mutual relation of ε and η. The following simple discussions treat with a case where $C(X; \varepsilon, \eta)$ presents a remarkable contrast to

$C(X;\eta,\varepsilon)$. Let I be the unit interval $[0, 1]$. Let x_i $(i=1, 2, \cdots)$ be an everywhere dense sequence on I. For ε, $\eta\in\Gamma(I\times\boldsymbol{N})$, put

$$\varepsilon^{(n)}=\inf_i \varepsilon(x_i, n), \qquad \eta^{(n)}=\sup_i \eta(x_i, n).$$

PROPOSITION 5. (i) *If $\varepsilon(x, n)$ is lower semi-continuous in x and*

$$\liminf \frac{\eta^{(n)}}{\varepsilon^{(n)}}=0,$$

then $C(I;\varepsilon,\eta)$ consists only of constant functions.

(ii) *If ε and η are continuous in x and*

$$\lim \frac{\eta^{(n)}}{\varepsilon^{(n)}}=0,$$

then $C(I;\eta,\varepsilon)\supset C^1(I)$.

Proof. (i) First we note that, under the assumption of lower semi-continuity of ε, $\varepsilon^{(n)}$ is always positive. The variation of f on $V_{(x_i,n)}$ is at most $2\eta(x_i, n)\leqq 2\eta^{(n)}$. On the other hand, if n is fixed, I is covered by at most l open sets $V_{\varepsilon(x_{i_s},n)}$ $(s=1, 2, \cdots, l)$, where $l=1/\varepsilon^{(n)}$. Hence the variation of f on I is dominated by at most $2\eta^{(n)}/\varepsilon^{(n)}$. Hence if there exists a subsequence $\{n_s\}$ such that $\eta^{(n_s)}/\varepsilon^{(n_s)}\to 0$, then f must be constant.

(ii) This follows directly from the Lipschitz condition of $f\in C^1(I)$.

We are now going to study the effect of diffeomorphism upon the space $C(M;\varepsilon,\eta)$. For $\varphi\in\mathrm{Diff}(M)$ and $\varepsilon\in\Gamma(M\times\boldsymbol{N})$, put

$$(\varphi^*\varepsilon)(x, n)=\varepsilon(\varphi(x), n).$$

Then $\varphi^*\varepsilon\in\Gamma(M\times\boldsymbol{N})$.

First we note that the relation

$$\varphi(y)\in V_{\varepsilon(\varphi(x),n)}(\varphi(x)) \tag{6}$$

is rewritten as

$$\rho(\varphi(x),\varphi(y))<\varepsilon(\varphi(x), n)=(\varphi^*\varepsilon)(x, n). \tag{6$'$}$$

Let L^{-1} be a Lipschitz constant of φ. Then we have

$$\rho(x, y)>L\rho(\varphi(x),\varphi(y)). \tag{7}$$

Consider for a pair of points x and y a relation

$$\rho(x, y)<L(\varphi^*\varepsilon)(x, n). \tag{8}$$

If (8) holds, we have by (7)

$$L\rho(\varphi(x),\varphi(y))<L(\varphi^*\varepsilon)(x, n),$$

whence we obtain (6)$'$. Thus we have proved the implication (8)$\Rightarrow$(6).

Now we impose the following condition (P) on ε:
(P) For any $\alpha>0$, there exist $n_1\in\boldsymbol{N}$ and $k\in\boldsymbol{N}$ such that

$$\varepsilon(x, n+k)<\alpha\varepsilon(x, n)$$

holds for every $x\in M$ when $n\geqq n_1$.

Under the condition (P), we can find k and n_1 for which we have

$$\varepsilon(\varphi(x), n+k)<L(\varphi^*\varepsilon)(x, n) \quad \text{for} \quad n\geqq n_1 .$$

This, combined with (8), yields that, if y satisfies

$$\rho(x, y)<\varepsilon(\varphi(x), n+k)$$

for some n, $n\geqq n_1$, then we have (6).

This being understood, for $\varepsilon\in\Gamma(M\times\boldsymbol{N})$ put

$$\varepsilon_k(x, n)=\varepsilon(x, n+k) \quad (k=1, 2, \cdots) .$$

Then $\varepsilon_k\in\Gamma(M\times\boldsymbol{N})$ and $\varepsilon_{k+1}(x, n)=\varepsilon_k(x, n+1)$. What we have shown is then reformulated

$$y\in V_{(\varphi^*\varepsilon_k)(x,n)}(x)\Rightarrow\varphi(y)\in V_{\varepsilon(\varphi(x),n)}(\varphi(y)) \quad \text{for} \quad n\geqq n_1 . \tag{9}$$

DEFINITION 5. For ε and $\eta\in\Gamma(M\times\boldsymbol{N})$, we define

$$\tilde{C}(M; \varepsilon, \eta)=\bigcup_{k=1}^{\infty}C(M; \varepsilon_k, \eta) .$$

THEOREM 3. *If ε satisfies the condition* (P), *then we have*

$$f\in\tilde{C}(M; \varepsilon, \eta)\Leftrightarrow\varphi^*f\in\tilde{C}(M; \varphi^*\varepsilon, \varphi^*\eta) .$$

Proof. It is sufficient to show the implication from the left to the right, since $\varphi^*\varepsilon$ also satisfies (P). Let $f\in\tilde{C}(M; \varepsilon, \eta)$, say $f\in C(M; \varepsilon_{k_0}, \eta)$. Then by the definition we have

$$|f(y)-f(x)|\leqq A\eta(x, n)$$

when $y\in V_{\varepsilon_{k_0}(x,n)}(x)$. This amounts to saying that

$$|f(\varphi(y))-f(\varphi(x))|\leqq A\eta(\varphi(x), n)$$

when $\varphi(y)\in V_{\varepsilon_{k_0}(\varphi(x),n)}(\varphi(x))$. Hence by (9) we have

$$y\in V_{(\varphi^*\varepsilon_{k+k_0})(x,n)}\Rightarrow|\varphi^*f(y)-\varphi^*f(x)|\leqq A(\varphi^*\eta)(x, n)$$

for $n\geqq\text{Max}(n_1-k_0, 1)$. This involves that $\varphi^*f\in C(M; \varphi^*\varepsilon_{k+k_0}, \varphi^*\eta)$, which completes the proof.

DEFINITION 6. The *support* of $\tilde{C}(M; \varepsilon, \eta)$ is defined as the smallest closed set outside of which every $f \in \tilde{C}(M; \varepsilon, \eta)$ becomes locally constant.

COROLLARY. *The support of* $\tilde{C}(M; \varphi^*\varepsilon, \varphi^*\eta)$ *is just the image of the support of* $\tilde{C}(M; \varepsilon, \eta)$ *via* φ^{-1}.

The following theorem makes a situation clear where the support is not a total space.

THEOREM 4. *Let* $\varepsilon \in \Gamma(M \times \boldsymbol{N})$ *satisfy the condition* (P). *Moreover, we assume that* $\varepsilon(x, n)$ *is lower semi-continuous in* x. *Let* U *be a connected open set of* M. *Suppose that there exists an everywhere dense sequence* $\{x_1, x_2, \cdots\}$ *in* U, *for which we have*

$$\liminf_n \frac{\eta^{(n)}}{\varepsilon_k^{(n)}} = 0, \qquad n, k = 1, 2, \cdots,$$

where $\varepsilon_k^{(n)} = \inf_i \varepsilon_k(x_i, n)$, $\eta^{(n)} = \sup_i \eta(x_i, n)$. *Then, for every* $f \in \tilde{C}(M; \varepsilon, \eta)$, *the restriction of* f *to* U *becomes constant.*

Proof. Take f from $C(M; \varepsilon_k, \eta)$. Let x and y be any distinct two points in U and let $c(t)$ $(0 \leqq t \leqq 1)$ be a smooth curve in U with $c(0) = x$, $c(1) = y$. We choose a dense set $\{y_1, y_2, \cdots\}$ on this curve and consider an everywhere dense set $\{x_1, x_2, \cdots, y_1, y_2, \cdots\}$ on U. We may and do assume that these points are all distinct. It follows from Proposition 1 that there exists a diffeomorphism $\varphi \in \mathrm{Diff}(U)$ such that $\varphi(x) = x$, $\varphi(y) = y$ and $\varphi(\cup x_i) = \cup x_i \cup y_j$. To each y_s, let $\varphi(x_{s'}) = y_s$. Then

$$(\varphi^{-1})^* f(y_s) = f(x_{s'}).$$

By restricting $(\varphi^{-1})^* f$ to the curve c, we can regard $(\varphi^{-1})^* f$ as a function on $I = [0, 1]$, which in turn belongs to $C(I; (\varphi^{-1})^*\varepsilon, (\varphi^{-1})^*\eta)$ by Theorem 3. Then it is easily checked that $(\varphi^{-1})^* f$ satisfies the conditions corresponding to those which are stated in Proposition 5 (i). Hence the proposition applies to this case, which yields $(\varphi^{-1})^* f(c(0)) = (\varphi^{-1})^* f(c(1))$, that is, $f(x) = f(y)$. From this we can deduce that $f|U$ is constant.

The space $\tilde{C}(M; \varepsilon, \eta)$ itself seems to deserve our attention. Nevertheless, when we apply the above results to discrete structures, the space $\tilde{C}(M; \varepsilon, \eta)$ turns out to be not so appropriate. It is better to consider a larger function space

$$\tilde{\tilde{C}}(M; \varepsilon, \eta) = \bigcup_{k=-\infty}^{\infty} \bigcup_{l=-\infty}^{\infty} C(M; \varepsilon_k, \eta_l),$$

where for non-positive k we set

$$\varepsilon_k(x, n) = \begin{cases} \varepsilon(x, n+k), & \text{if } n > -k \\ \varepsilon(x, 1), & \text{if } n \leqq -k; \end{cases}$$

η_l is defined similarly for $l \leqq 0$. Really, the next proposition is useful.

PROPOSITION 6. *If* ε' *and* $\eta' \in \Gamma(M \times \boldsymbol{N})$ *satisfy*

$$\varepsilon'_k(x, n) \leqq \varepsilon(x, n) \leqq \varepsilon'_{k'}(x, n)$$

$$\eta'_l(x, n) \leqq \eta(x, n) \leqq \eta'_{l'}(x, n)$$

for suitable k, k', l *and* l' *independent of* (x, n) *for sufficiently large* n, *then we have*

$$\tilde{\tilde{C}}(M; \varepsilon, \eta) = \tilde{\tilde{C}}(M; \varepsilon', \eta') .$$

This follows immediately from the definitions. For any $\tilde{\tilde{C}}(N; \varepsilon, \eta)$, we can also define the notion of the support similar to Definition 6. Then we have

$$\operatorname{supp} \tilde{\tilde{C}}(M; \varepsilon, \eta) \supset \bigcap_{l=1}^{\infty} \operatorname{supp} \tilde{C}(M; \varepsilon, \eta_l) .$$

4. Discrete structures and function spaces

Let M be a compact Riemannian manifold. The simplest function belonging to $\Gamma(M \times \boldsymbol{N})$ is defined by

$$\varepsilon(x, n) = \gamma^n ,$$

where $0 < \gamma < 1$. Then it is clear that $\varepsilon(x, n)$ satisfies the condition (P). Besides, $\varepsilon(x, n)$ is trivially lower semi-continuous in x. Putting γ for this ε, we know that the results described in Section 3 can be applied to $\tilde{C}(M; \gamma, \eta)$ and $\tilde{\tilde{C}}(M; \gamma, \eta)$ for any η.

Let $\boldsymbol{a} = \{a_n\}$ be an everywhere dense sequence on M. To this $\boldsymbol{a}$, we try to associate certain functions belonging to $\Gamma(M \times \boldsymbol{N})$. There is no standard way in constructing such functions. First observe the speed of convergence of $\boldsymbol{a}$ approaching every point of M. Then the following procedure gives a function $\eta_{\delta, A}(\boldsymbol{a})$, depending on the parameters $\boldsymbol{\delta}$ and $\boldsymbol{A}$. Here $\boldsymbol{\delta}$ and $\boldsymbol{A}$ are decreasing positive sequences given by

$$\boldsymbol{\delta} : \delta > \delta^2 > \cdots > \delta^n > \cdots \qquad \text{for} \quad 0 < \delta < 1$$

and

$$\boldsymbol{A} : A_1 > A_2 > \cdots > A_n > \cdots \longrightarrow 0 .$$

For any $n \in \boldsymbol{N}$ and $x \in M$, let n_x be the integer defined by the relation

$$a_1, \cdots, a_{n_x - 1} \notin V_{\delta^n}(x) , \qquad a_{n_x} \in V_{\delta^n}(x) - \{x\} .$$

Hence a_{n_x} is just the first point of $\boldsymbol{a}$, entering $V_{\delta^n}(x)$. We define

$$\eta_{\delta, A}(\boldsymbol{a})(x, n) = A_{n_x} .$$

Then it is clear that $\eta_{\delta, A}(\boldsymbol{a}) \in \Gamma(M \times \boldsymbol{N})$, so that we obtain the function space $\tilde{\tilde{C}}(M; \gamma, \eta_{\delta, A}(\boldsymbol{a}))$. We note that if $\boldsymbol{a} = \{a_n\}$ and $\boldsymbol{a}' = \{a'_n\}$ coincide with each other for $n \geqq n_0$, then

$$\tilde{\tilde{C}}(M;\gamma,\ \eta_{\delta,A}(\boldsymbol{a}))=\tilde{\tilde{C}}(M;\gamma,\ \eta_{\delta,A}(\boldsymbol{a}')).$$

Let $\varphi\in \mathrm{Diff}(M)$ and consider the discrete structure $\varphi(\boldsymbol{a})=\{\varphi(a_n')\}$. Since φ satisfies the Lipschitz condition, we can find positive integers l and l' with

$$\eta_{\delta,A}(\boldsymbol{a})_l(x,\ n)\leqq\eta_{\delta,A}(\varphi(\boldsymbol{a}))(\varphi(x),\ n))\leqq\eta_{\delta,A}(\boldsymbol{a})_{l'}(x,\ n)$$

or

$$(\varphi^{-1})^*\eta_{\delta,A}(\boldsymbol{a})_l\leqq\eta_{\delta,A}(\varphi(\boldsymbol{a}))\leqq(\varphi^{-1})^*\eta_{\delta,A}(\boldsymbol{a})_{l'}.$$

Accordingly, in view of Proposition 6, we have

$$C(M;\gamma,\ (\varphi^{-1})^*\eta_{\delta,A}(\boldsymbol{a}))=C(M;\gamma,\ \eta_{\delta,A}(\varphi(\boldsymbol{a}))).$$

Hence, referring to Corollary to Theorem 3, we have

THEOREM 5. *In order that* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *give rise to the same discrete structure, it is necessary that* supp $\tilde{\tilde{C}}(M;\gamma,\ \eta_{\delta,A}(\boldsymbol{a}))$ *is diffeomorphic to* supp $\tilde{\tilde{C}}(M;\gamma,\ \eta_{\delta,A}(\boldsymbol{b}))$ *through a diffeomorphism of* M.

In this theorem, γ plays a role of assigning the range of 'observations', which is independent of the choice of $\boldsymbol{a}$. If we want to study the relative situation $\boldsymbol{a}$ and $\boldsymbol{b}$, it might be better to consider the space

$$\tilde{\tilde{C}}(M;\ \eta_{\delta,A}(\boldsymbol{a}),\ \eta_{\delta,A}(\boldsymbol{b})).$$

However, if we demand that this space should have a definite meaning for discrete structures, $\eta_{\delta,A}(\boldsymbol{a})$ have to satisfy the condition (P). These conditions will impose some restriction upon $\boldsymbol{a}=\{a_n\}$ about the distribution over M.

On the other hand, we can introduce another function which shows to some extent the degree of approximation to 'density' defined by $\boldsymbol{a}$. For any open set U of M, put

$$\kappa(U,\ N)=\frac{1}{N}{}^{\#}\{i\mid a_i\in U,\ i\leqq N\}.$$

For $0<\delta<1$, let $\boldsymbol{\delta}:\delta>\delta^2>\cdots$ be fixed. Consider the δ^n-neighborhood $V_{\delta^n}(x)$ of each point x of M. Let μ_n be the least number which satisfies the condition that each $V_{\delta^n}(x)$ $(x\in M)$ contains at least one a_i $(1\leqq i\leqq\mu_n)$. Moreover let Ψ be a real valued function defined on N such that $\Psi(n)\to\infty$ as $n\to\infty$. Now we put

$$\kappa_{\delta,\Psi}(\boldsymbol{a})(x,\ n)=\frac{1}{\Psi(n)}\kappa(V_{\delta^n}(x),\ \mu_n).$$

Then $\kappa_{\delta,\Psi}(\boldsymbol{a})>0$. If we can find Ψ such that $\kappa_{\delta,\Psi}(\boldsymbol{a})(x,\ n)\to 0$ as $n\to\infty$, we say that $\boldsymbol{a}=\{a_n\}$ has the degree of unform density at most Ψ. Then $\kappa_{\delta,\Psi}(\boldsymbol{a})\in\Gamma(M\times\boldsymbol{N})$ so that we can get the function space $\tilde{\tilde{C}}(M;\gamma,\ \kappa_{\delta,\Psi}(\boldsymbol{a}))$. Hence we will be able to formulate a theorem corresponding to Theorem 5. However we do not repeat it here. We note that in this case it is not so easy to put some other interpretation on the condition (P).

The above examples of the functions associated to a discrete structure are rather special. There are many ways of constructing function spaces, each of which furnishes some information on a given discrete structure. From this fact, we may infer that the discrete structure is so complicated that any standard (for example, axiomatic) treatment might not be expected.

REFERENCES

[1] W. R. SCOTT, "Group Theory," Prentice-Hall, New Jersey, U. S. A., 1964.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO
TOKYO, 152
JAPAN