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$\mathcal{E}(X)$ FOR NON-SIMPLY CONNECTED H-SPACES

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§ 0. Introduction.

Let X be a path-connected H-space with a unit x_0 and let $\mathcal{E}(X)$ be the group of homotopy classes of homotopy equivalences: $(X, x_0)\to(X, x_0)$. In the case of X being simply connected, D.M. Sunday, J.R proved that if $\mathrm{rank}(\pi_i(X))\geqq 2$, for some i, then $\varepsilon(X)$ contains a non abelian free subgroup (Theorem B-(2) of [3]). In this paper we investigate the case of an associative H-space X being not simply connected and having the homotopy type of a CW-complex.

THEOREM A. *There exists a splitting exact sequence:*

$$\{1\}\longrightarrow \nu^{-1}(1)\longrightarrow \varepsilon(X)\underset{\nu}{\longrightarrow} GL(n, Z)\longrightarrow\{1\},$$

where n is the rank of $\pi_1(X, x_0)$.

Especially, since $GL(n, Z)$ is not of finite rank for $n\geqq 2$ we have

COROLLARY. *If $\mathrm{rank}(\pi_1(X, x_0))\geqq 2$ then $\mathcal{E}(X)$ is not of finite rank.*

Next let $\mathcal{E}_H(X)$ be the subgroup of $\mathcal{E}(X)$ consisting of homotopy-homomorphisms (H-maps), then we have

THEOREM B. *If the natural homomorphism*

$$\pi_1(Z(X), x_0)\longrightarrow \pi_1(X, x_0)/Torsion$$

is onto, where $Z(X)$ denotes the homotopy-centre of X, then $\mathcal{E}_H(X)$ contains $GL(n, Z)$ as a semi-direct factor.

In addition, if we assume that $\pi_1(X, x_0)$ is torsion free we have

COROLLARY. *$\mathcal{E}(X)$ is isomorphic to the direct sum $GL(n, Z)\oplus K(X)$, where $K(X)$ denotes the kernel of the natural representation:*

$$\mathcal{E}_H(X)\longrightarrow \mathrm{Aut}(\pi_1(X, x_0))=GL(n, Z).$$

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§ 1. Definitions of homomorphisms ν and μ.

Let $\pi_1(X, x_0)$ be isomorphic to the direct sum $Z^n \oplus F$, where F is the torsion subgroup, and let $\{\alpha_i\}$ $(i=1, \cdots, n)$ be a system of generators of Z^n. For a map $f: (X, x_0) \to (X, x_0)$ we define a homomorphism:

$$\tilde{f}: Z^n \underset{i}{\longrightarrow} \pi_1(X, x_0) \underset{f_*}{\longrightarrow} \pi_1(X, x_0) \underset{j}{\longrightarrow} Z^n$$

by the composition $jf_*\imath$, $\imath$ is the inclusion and j is the projection.

LEMMA 1. $\widetilde{(hf)}=\tilde{h}\tilde{f}$, $\widetilde{\imath d}=id$

Proof. Consider the diagram:

$$\begin{array}{ccccccc} Z^n & \underset{i}{\longrightarrow} & \pi_1(X, x_0) & \underset{f_*}{\longrightarrow} & \pi_1(X, x_0) & \underset{h_*}{\longrightarrow} & \pi_1(X, x_0) \\ & & & & j\Big\downarrow \Big\uparrow i & & \Big\downarrow j \\ & & & & Z^n & & Z^n \end{array}$$

Then, for $x \in Z^n$, we have

$$f_*i(x)=ijf_*i(x)+x' \qquad (x' \in F)$$

$$h_*f_*i(x)=h_*\imath\tilde{f}(x)+h_*(x')$$

$$jh_*f_*i(x)=jh_*i\tilde{f}(x)+jh_*(x') \qquad (jh_*(x')=0)$$

thus we have $\widetilde{(hf)}(x)=\tilde{h}\tilde{f}(x)$ i.e. $\widetilde{hf}=\tilde{h}\tilde{f}$.

Since we may regard $\tilde{f}$ as an element of $GL(n, Z)$ by using the system of generators $\{\alpha_i\}$ we define $\nu(f)$ by the matrix $\tilde{f}$ and get a homomorphism,

$$\nu: \mathcal{E}(X) \longrightarrow GL(n, Z).$$

Conversely, let $A=(a_{ij})$ be a matrix of degree n with Z-coefficient. Now, from isomorphisms:

$$[X, S^1] \cong H^1(X, Z) \cong \mathrm{Hom}(H_1(X, Z), Z) \cong \mathrm{Hom}(\pi_1(X, x_0), Z) \cong \mathrm{Hom}(Z^n, Z),$$

we obtain a map

$$f_i: X \longrightarrow S^1 \qquad (i=1, \cdots, n)$$

which satisfies $f_{i*}(\alpha_j)=a_{ji}$.

Define a map $f_A: X \to X$ by

$$f_A(x)=(\alpha_1 f_1(x)) \copyright (\alpha_2 f_2(x)) \copyright \cdots \copyright (\alpha_n f_n(x)),$$

where © denotes the multiplication of X.

The following lemma can be easily deduced from definitions.

LEMMA 2. *$f_{A*}: \pi_m(X, x_0) \to \pi_m(X, x_0)$ satisfies*

(1) *$f_{A*}=0$-homomorphism if $m \geqq 2$*
(2) *$f_{A*}|F=0$ and $f_{A*}|Z^n=A$ if $m=1$.*

Moreover if we define a map $\hat{f}_A: X \to X$ by $\hat{f}_A(x)=x \copyright f_A(x)$, then we can transform lemma 2 to the following

LEMMA 3. *$\hat{f}_{A*}: \pi_m(X, x_0) \to \pi_m(X, x_0)$ satisfies*

(1) *$\hat{f}_{A*}$ is the identity if $m \geqq 2$*
(2) *$\hat{f}_{A*}|F$ is the identity and $\hat{f}_{A*}|Z^n=I_n+A$ if $m=1$,*

where I_n denotes the unit matrix of degree n.

Thus, from Whitehead's theorem, we obtain

LEMMA 4. *$\hat{f}_A$ is a homotopy equivalence if and only if the matrix I_n+A is contained in $GL(n, Z)$.*

Now we define a correspondence $\mu: GL(n, Z) \to \mathcal{E}(X)$ by

$$\mu(A)=\hat{f}_{A-I_n}.$$

Then, by considering the induced homomorphism, we have, from lemma 3,

LEMMA 5. *The correspondence $\nu\mu$ is the identy.*

§ 2. The homomorphism μ.

In this section we prove

LEMMA 6. *μ is an anti-homomorphism ($\mu(A)\mu(B)=\mu(BA)$)*

For the proof we need some sub-lemmas. We denote $A-I_n$ by A' for short. First we note the equality:

$$\begin{aligned}\hat{f}_{A'}\hat{f}_{B'} &= (1_X \copyright f_{A'})(1_X \copyright f_{B'}) \\ &= 1_X(1_X \copyright f_{B'}) \copyright f_{A'}(1_X \copyright f_{B'}) \\ &= (1_X \copyright f_{B'}) \copyright f_{A'}(1_X \copyright f_{B'}) \\ &= 1_X \copyright \{f_{B'} \copyright f_{A'}(1_X \copyright f_{B'})\}.\end{aligned}$$

Sub-lemma 1. *$f_P(1_X \copyright f_Q)=f_{P+QP}$ for any matrices P and Q.*

Proof. Consider the diagram for $f_{i*}(\alpha_j)=p_{ji}$, $P=(p_{ij})$:

$$\begin{array}{ccccc} X & \xrightarrow[(1_X,\ f_Q)]{} & X\times X & \xrightarrow[\copyright]{} & X \\ & & & & \Big\downarrow (f_1, \cdots, f_n) \\ & & & & T^n=S^1\times\cdots\times S^1 \\ & & & & \Big\downarrow \alpha_1\times\cdots\times\alpha_n \\ & & & & X\times\cdots\times X \xrightarrow[\copyright\cdots\copyright]{} X. \end{array}$$

Since we have the equality for the induced homomorphisms:

$$\begin{aligned}(f_1\cdots f_n)_*(1_X\copyright f_Q)_*\{(\alpha_i)\} &= (f_1\cdots f_n)_*\{(\alpha_i)+Q(\alpha_i)\} \\ &= (f_1\cdots f_n)_*\{(I_n+Q)(\alpha_i)\} \\ &= (I_n+Q)P\{(\alpha_i)\}\end{aligned}$$

the proof follows from definitions of maps.

Sub-lemma 2. *We can take any element of* $\pi_1(X, x_0)$ *as an H-map, i.e.* $\alpha(xy)\cong\alpha(x)\copyright\alpha(y)$ *for* $\alpha:(S^1, 1)\to(X, x_0)$.

Proof. Since the obstraction is given by the separation element of two maps:

$$S^1\times S^1 \longrightarrow X,\quad \alpha(xy) \text{ and } \alpha(x)\copyright\alpha(y),$$

the proof follows from $\pi_2(X, x_0)=\{0\}$.

Analogously we have

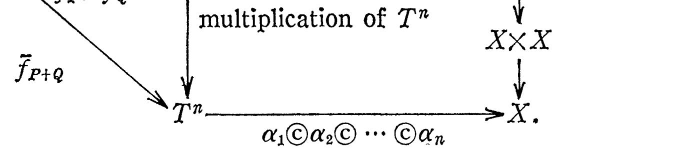
Sub-lemma 3. *For* α, $\beta\in\pi_1(X, x_0)$ *two maps,*

$$\alpha(x)\copyright\beta(y) \text{ and } \beta(y)\copyright\alpha(x):\ S^1\times S^1\longrightarrow X$$

are homotopic.

Sub-lemma 4. $f_P\copyright f_Q=f_{P+Q}$: $X\to X$.

Proof. Consider the diagram:



Here the square of maps in the right hand means that

$$\begin{array}{ccl}
(x_i)\times(y_i) & \longrightarrow & (\alpha_i(x_i))\times(\alpha_i(y_i)) \\
\Big\downarrow & & \quad\Big\downarrow \\
 & & (\alpha_1(x_1)© \cdots ©\alpha_n(x_n))\times(\alpha_1(y_1)© \cdots ©\alpha_n(y)) \\
 & & \quad\Big\downarrow \\
 & & \alpha_1(x_1)© \cdots ©\alpha_n(x_n)©\alpha_1(y_1) \cdots ©\alpha_n(y_n) \\
\Big\downarrow & & \\
(x_i y_i) & \longrightarrow & \alpha_1(x_1 y_1)© \cdots ©\alpha_n(x_n y_n),
\end{array}$$

and $\bar{f}_P$ is defined by $f_P(x)=(f_1(x) \cdots f_n(x))$ for $f_i: X\to S^1$ such that $f_{i*}(\alpha_j)=p_{ji}$, $P=(p_{ij})$.

Then sub-lemma 2 and 3 shows that the square is commutative up to homotopy and the triangle in the left hand is also commutative up to homotopy by the definitions of $\bar{f}_P$, $\bar{f}_Q$, and $\bar{f}_{P+Q}$. Thus the proof is completed by $f_P=(1_X©1_X© \cdots ©1_X)(\alpha_1\times \cdots \times\alpha_n)\bar{f}_P$, $\cdots$ etc.

Now the proof of lemma 6 completes from above lemmas as follows,

$$\begin{aligned}
\hat{f}_{A'}\hat{f}_{B'} &= 1_X©\{f_{B'}©f_{A'}(1_X©f_{B'})\} \\
&\cong 1_X©(f_{B'}©f_{A'+B'A'}) \\
&\cong 1_X©f_{B'+A'+B'A'} \\
&\cong 1_X©f_{B-I_n+A-I_n+BA-B-A+I_n} \\
&= 1_X©f_{BA-I_n}=\hat{f}_{BA-I_n}
\end{aligned}$$

Thus the proof of Theorem A is easily obtained from lemma 5 and 6.

§ 3. The Proof of Theorem B.

For the proof it is sufficient to show that the map:

$$\hat{f}_A: X \longrightarrow X \quad (A: \text{a matrix of degree } n)$$

is an H-map. Recall the definition of $\hat{f}_A$,

$$\begin{aligned}
\hat{f}_A(x) &= x©f_A(x) \\
&= x©(\alpha_1(f_1(x))©\alpha_2(f_2(x)) \cdots ©\alpha_n(f_n(x))),
\end{aligned}$$

where $f_i: X\to S^1$ is given by $f_{i*}(\alpha_j)=a_{ji}$ for $A=(a_{ij})$ and $\{\alpha_i\}$ is a system of generators of the free part of $\pi_1(X, x_0)$.

Under the assumption we may consider that each α_i can be taken as an H-map: $S^1\to Z(X)\to X$ by sub-lemma 2. Then we have

$$\begin{aligned}\hat{f}_A(x©y)&=x©y©(\alpha_1(f_1(x©y))©\alpha_2(f_2(x©y))©\cdots©\alpha_n(f_n(x©y))\\ &\cong x©y©\alpha_1(f_1(x))©\alpha_1(f_1(y))\cdots©\alpha_n(f_n(x))©\alpha_n(f_n(y))\\ &\cong x©y©\alpha_1(f_1(x))©\alpha_2(f_2(x))©\cdots©\alpha_n(f_n(x))©\alpha_1(f_1(y))©\cdots©\alpha_n(f_n(y))\\ &\cong x©\alpha_1(f_1(x))©\cdots©\alpha_n(f_n(x))©y©\alpha_1(f_1(y))©\cdots©\alpha_n(f_n(y))\\ &\cong\hat{f}_A(x)©\hat{f}_A(y)\,.\end{aligned}$$

Here, we note that the second equality is derived from sub-lemma 2 and the following

LEMMA 7. *For any map $f:(X, x_0)\to(S^1, 1)$, f is an H-map.*

Proof. Consider two maps:

$$X\times X \underset{f_2}{\overset{f_1}{\rightrightarrows}} S^1; \qquad \begin{aligned} &f_1(x, y)=f(x©y)\\ &f_2(x, y)=f(x)©f(y)\,.\end{aligned}$$

Clearly we have $f_1|X\vee X=f_2|X\vee X$. Then the proof is completed by $\pi_i(S^1, 1)=\{0\}$ $(i>1)$.

Now, let $h:(X, x_0)\to(X, x_0)$ be a map contained in $\mathcal{E}_H(X)$ such that $\nu(h)=I_n$, and consider two compositions $h\hat{f}_A$ and $\hat{f}_A h$ for a matrix A.

Then the following lemma completes the proof of Corollary of Theorem B.

LEMMA 8. *$\hat{f}_A h$ is homotopic to $h\hat{f}_A$.*

Proof. Since $\hat{f}_A h(x)=h(x)©f_A(h(x))\sim h(x)©\alpha_1(f_1(h(x))©\cdots©\alpha_n(f_n(h(x))$ and $h\hat{f}_A(x)=h(x©f_A(x))\sim h(x)©h(f_A(x))\sim h(x)©h(\alpha_1 f_1(x))©\cdots©h(\alpha_n f_n(x))$ it is sufficient to show $h\alpha_i f_i\sim\alpha_i f_i h$, i.e. the homotopy-commutativity of the diagram:

$$\begin{array}{ccccc} X & \xrightarrow{f_i} & S^1 & \xrightarrow{\alpha_i} & X\\ \uparrow{\scriptstyle h} & & & & \uparrow{\scriptstyle h}\\ X & \xrightarrow{f_i} & S^1 & \xrightarrow{\alpha_i} & X \end{array}$$

Since $h^*=$identity: $H^1(X;Z)\to H^1(X;Z)$ follows from the condition $\nu(h)=I_n$ we have $(hf_i)^*=f_i^*h^*=f_i^*$, i.e. $hf_i\sim f_i$. Moreover, $h\alpha_i\sim\alpha_i$ is clear from $\nu(h)=I_n$ and the additional assumption. Hence these complete the proof.

Addendum. Another proof of theorem A was informed to the author from M. Mimura and A. Kono in preparation of the paper.

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