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CRITICAL RIEMANNIAN METRICS ON SASAKIAN MANIFOLDS

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1. Introduction. Let g be a Riemannian metric which is defined on a compact orientable differentiable manifold M of dimension n and makes its volume V_g equal to 1, that is, $\int_M dV_g=1$, where dV_g is the volume element of M measured by g. We denote the set of all such metrics by $\mathfrak{M}$. When g is fixed we have a Riemannian manifold (M, g). Let us take a covering $\{U\}$ of M by coordinate neighborhoods and denote the local coordinates in U by $\{x^a\}$, where $a, b, c, \cdots$ run over the range $\{1, 2, 3, \cdots, n\}$. In each U, g is expressed by its components g_{ab}. We adopt summation convention so that the contravariant components g^{ab} of g satisfy $g_{ac}g^{bc}=\delta_a{}^b$. By $R_{abc}{}^d$, R_{ab} and R we denote the components of the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of (M, g), respectively. Now let us consider the integral

$$F_M[g]=\int_M f(R)dV_g\,,$$

where $f(R)$ is a scalar field on M determined by g as the contraction of a tensor product of the curvature tensor. This integral defines a mapping $F:\mathfrak{M}\to R$. A critical point of F is denoted by g_F and is called a critical Riemannian metric with respect to the field $f(R)$ or the integral $F_M[g]$. The following four kinds of critical Riemannian metrics have been studied by M. Berger [1] and Y. Mutō [5, 6, 7, 8, 9]:

$$A_M[g]=\int_M R\,dV_g\,,\qquad B_M[g]=\int_M R^2\,dV_g\,,$$

$$C_M[g]=\int_M R_{ab}R^{ab}\,dV_g\,,\quad D_M[g]=\int_M R_{abcd}R^{abcd}\,dV_g\,.$$

The equations of the critical Riemannian metric are written as follows:

$$A_{ab}=C_A g_{ab}\,,\quad B_{ab}=C_B g_{ab}\,,\quad C_{ab}=C_C g_{ab}\,,\quad D_{ab}=C_D g_{ab}\,, \tag{1.1}$$

where C_A, C_B, C_C and C_D are undetermined constants and A_{ab}, B_{ab}, C_{ab} and D_{ab} are given by

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$$A_{ab}=-R_{ab}+\frac{1}{2}Rg_{ab}, \tag{1.2A}$$

$$B_{ab}=2\nabla_a\nabla_b R-2\nabla_r\nabla^r Rg_{ab}-2RR_{ab}+\frac{1}{2}R^2g_{ab}, \tag{1.2B}$$

$$C_{ab}=\nabla_a\nabla_b R-\nabla_r\nabla^r R_{ab}-\frac{1}{2}\nabla_r\nabla^r Rg_{ab}-2R_{asrb}R^{sr}+\frac{1}{2}R_{sr}R^{sr}g_{ab}, \tag{1.2C}$$

$$\begin{aligned} D_{ab}=&2\nabla_a\nabla_b R-4\nabla_r\nabla^r R_{ab}+4R_{ar}R_b{}^r-4R_{asrb}R^{sr}\\ &-2R_{atsr}R_b{}^{tsr}+\frac{1}{2}R_{tsrq}R^{tsrq}g_{ab}, \end{aligned} \tag{1.2D}$$

where ∇ means covariant differentiation with respect to the connection induced by g. It is well known that a critical point g of $A_M[g]$ in $\mathfrak{M}$ is an Einstein metric.

Although critical Riemannian metrics were first defined on a compact manifold, it is to generalize the definition when M is not compact. The resulting equations are the same as the foregoing ones and (1.1) and (1.2) are valid.

The purpose of this paper is to study the results obtained by Y. Mutō [9] more deeply. §2 is recalled the definition of a Sasakian manifold. In §3 a Sasakian manifold is studied when g turns out to be a critical Riemannian metric and moreover we consider critical Riemannian metrics in a Sasakian manifold with vanishing C-Bochner curvature tensor in §4. In §5 we investigate some conditions to be a critical Riemannian metric g_C or g_D. §6 is devoted to the study of a Sasakian submersion $\Pi:(\tilde{M}, \tilde{g}, \tilde{\eta})\to(B, {}^Bg)$ where $\tilde{g}$ and ${}^Bg$ become critical Riemannian metrics simultaneously.

2. Sasakian manifold.

Let (M, g, η) be an $n(n\geqq 3)$-dimensional Sasakian manifold (connected and C^∞) with metric tensor g. Then there exist a Killing 1-form η satisfying

$$\begin{aligned} &\phi_{ab}=\nabla_a\eta_b, \quad \phi_{ab}=-\phi_{ba}, \quad \eta^a=g^{ar}\eta_r, \quad \eta_a\eta^a=1,\\ &\phi_a{}^r\phi_r{}^b=-\delta_a{}^b+\eta_a\eta^b, \quad \nabla_a\phi_{bc}=\eta_b g_{ac}-\eta_c g_{ab}. \end{aligned} \tag{2.1}$$

On a Sasakian manifold (M, g, η), the following identities are well known:

$$R_{abc}{}^r\eta_r=\eta_a g_{bc}-\eta_b g_{ac}, \qquad R_a{}^r\eta_r=(n-1)\eta_a, \tag{2.2}$$

$$R_{absr}\phi_c{}^s\phi_d{}^r-R_{abcd}=g_{ac}g_{bd}-g_{bc}g_{ad}-\phi_{ac}\phi_{bd}+\phi_{bc}\phi_{ad}, \tag{2.3}$$

$$R_{sarb}\phi^{sr}=-S_{ab}+(n-1)\phi_{ab}, \tag{2.4}$$

where we put $S_{ab}=\phi_a{}^rR_{rb}$.

We introduce the tensor field $U_{abc}{}^d$ and G_{ab} in a Sasakian manifold (M, g, η), defined by

$$U_{abc}{}^{d}=R_{abc}{}^{d}-\frac{H+3}{4}(g_{bc}\delta_{a}{}^{d}-g_{ac}\delta_{b}{}^{d}) \tag{2.5}$$

$$-\frac{H-1}{4}(g_{ac}\eta_{b}\eta^{d}+\eta_{a}\eta_{c}\delta_{b}{}^{d}-g_{bc}\eta_{a}\eta^{d}-\eta_{b}\eta_{c}\delta_{a}{}^{d}$$

$$+\phi_{bc}\phi_{a}{}^{d}-\phi_{ac}\phi_{b}{}^{d}-2\phi_{ab}\phi_{c}{}^{d}),$$

$$G_{ab}=R_{ab}-\frac{R-n+1}{n-1}g_{ab}-\frac{n(n-1)-R}{n-1}\eta_{a}\eta_{b}, \tag{2.6}$$

where we put $H=\dfrac{4R-(n-1)(3n-1)}{(n-1)(n+1)}$.

A Sasakian manifold (M, g, η) is called a space of constant ϕ-holomorphic sectional curvature H or a locally C-Fubinian manifold if its $U_{abc}{}^{d}$ vanishes identically. A Sasakian manifold (M, g, η) is called an η-Einstein manifold or a C-Einstein manifold if its G_{ab} vanishes identically. In an η-Einstein manifold $(n>3)$, the scalar curvature R is necessarily constant. A 3-dimensional η-Einstein manifold means that R is constant.

The equations (2.5) and (2.6) yield the followings:

$$|G|^{2}=G_{ab}G^{ab}=|R_{1}|^{2}-\frac{R^{2}}{n-1}+2R-n(n-1), \tag{2.7}$$

$$|U|^{2}=U_{abcd}U^{abcd}=|R|^{2}+\frac{2}{(n-1)(n+1)}\{-4R^{2}+2(n-1)(3n-1)R \tag{2.8}$$

$$-n(n-1)^{2}(3n-1)\},$$

where we put $|R|^{2}=R_{abcd}R^{abcd}$, $|R_{1}|^{2}=R_{ab}R^{ab}$.

Next we introduce the C-Bochner curvature tensor $B_{abc}{}^{d}$ in an n-dimensional Sasakian manifold given by

$$B_{abcd}=R_{abcd}+\frac{1}{n+3}(R_{ac}g_{bd}-R_{bc}g_{ad}+g_{ac}R_{bd}-g_{bc}R_{ad} \tag{2.9}$$

$$+S_{ac}\phi_{bd}-S_{bc}\phi_{ad}+\phi_{ac}S_{bd}-\phi_{bc}S_{ad}+2S_{ab}\phi_{cd}+2\phi_{ab}S_{cd}$$

$$-R_{ac}\eta_{b}\eta_{d}+R_{bc}\eta_{a}\eta_{d}-\eta_{a}\eta_{c}R_{bd}+\eta_{b}\eta_{c}R_{ad})$$

$$+\frac{R+(n-1)(n+2)}{(n+1)(n+3)}(-\phi_{ac}\phi_{bd}+\phi_{bc}\phi_{ad}-2\phi_{ab}\phi_{cd})$$

$$+\frac{R-3n-5}{(n+1)(n+3)}(-g_{ac}g_{bd}+g_{bc}g_{ad})$$

$$+\frac{R+n-1}{(n+1)(n+3)}(g_{ac}\eta_{b}\eta_{d}+\eta_{a}\eta_{c}g_{bd}-g_{bc}\eta_{a}\eta_{d}-\eta_{b}\eta_{c}g_{ad}),$$

which is constructed from the Bochner curvature tensor in a Kählerian manifold by fibering of Boothby-Wang [3].

By straight forward computations the following identities are obtained:

$$B_{abcd}=-B_{bacd}\,,\quad B_{abcd}=B_{cdab}\,,\quad B_{abc}{}^{d}+B_{bca}{}^{d}+B_{cab}{}^{d}=0\,,\quad B_{abc}{}^{a}=0\,,$$

$$B_{abc}{}^{d}\eta_{d}=0\,,\quad \phi_{a}{}^{r}B_{rbc}{}^{d}=\phi_{b}{}^{r}B_{rac}{}^{d}\,,\quad \phi^{ab}B_{abc}{}^{d}=0\,,\quad g^{dr}B_{abcr}=B_{abc}{}^{d}\,.$$

In the rest of the present section, we assume that a Sasakian manifold (M, g, η) has the vanishing C-Bochner curvature tensor with a constant scalar curvature R. Then we have following identity [3].

$$\nabla_{d}R_{ab}=(n-1)(\eta_{a}\phi_{db}+\eta_{b}\phi_{da})-(\eta_{a}S_{db}+\eta_{b}S_{da})\,. \tag{2.10}$$

Operating ∇_{e} to (2.10) and using the Ricci identity, we have

$$R_{asrb}R^{sr}=R_{ar}R_{b}{}^{r}-R_{ab}+\{R-n(n-1)\}\eta_{a}\eta_{b}+(n-1)g_{ab}\,. \tag{2.11}$$

Also, applying ∇^{d} to (2.10), we get

$$\nabla_{r}\nabla^{r}R_{ab}=-2R_{ab}+2(n-1)g_{ab}+2\{R-n(n-1)\}\eta_{a}\eta_{b}\,. \tag{2.12}$$

Contracting the vanishing C-Bochner curvature tensor with R^{bc} and $R_{e}{}^{bcd}$, we find respectively

$$\begin{aligned}(n+1)(n-1)R_{ar}R_{b}{}^{r}&=(n-1)(R-3n-5)R_{ab}\\&+\{(n+1)|R_{1}|^{2}-R^{2}+4(n+1)R-(n-1)(n^{2}+3n+4)\}g_{ab}\\&+\{-(n+1)|R_{1}|^{2}+R^{2}-(n^{2}+2n+5)R+n(n-1)(n^{2}+3n+4)\}\eta_{a}\eta_{b}\,,\end{aligned} \tag{2.13}$$

$$\begin{aligned}(n+1)(n+3)R_{atsr}R_{e}{}^{tsr}&=16(n+1)R_{er}R_{a}{}^{r}-4(3n^{2}-2n-9+2R)R_{ea}\\&+2\{(n-1)^{2}(3n+7)-8R\}g_{ea}+8(n+1)\{R-n(n-1)\}\eta_{e}\eta_{a}\,,\end{aligned} \tag{2.14}$$

where we used (2.11). These will be needed later.

3. Sasakian manifolds with critical Riemannian metrics.

Let (M, g, η) be an n-dimensional Sasakian manifold. If the Riemannian metric g is a critical Riemannian metric g_{B}, g_{C} or g_{D}, then the undetermined constants C_{B}, C_{C} and C_{D} given by (1.1) are determined as follows [9]:

$$C_{B}=\frac{1}{2}R^{2}-2(n-1)R\,, \tag{3.1B}$$

$$C_{C}=\frac{1}{2}|R_{1}|^{2}-4R+2(n-1)(n+1)\,, \tag{3.1C}$$

$$C_{D}=\frac{1}{2}|R|^{2}-12R+4(n-1)(3n-1)\,. \tag{3.1D}$$

Moreover Y. Mutō [9] has proved

THEOREM A. *If g is a critical Riemannian metric g_A, g_B, g_C or g_D in a Sasakian manifold (M, g, η), then scalar curvature is constant.*

We first consider the case when g is a critical Riemannian metric g_B in a Sasakian manifold (M, g, η). By the aid of (1.1), (1.2B) and Theorem A, we can easily see that (M, g, η) is an Einstein manifold. Conversely, if (M, g, η) is an Einstein manifold, then we can see from (1.2B) that $B_{ab}=C_B g_{ab}$. Thus we have

THEOREM 3.1. *In a Sasakian manifold (M, g, η), in order that g be a critical Riemannian metric g_B, it is necessary and sufficient that (M, g, η) be an Einstein manifold.*

Second, let us discuss the case when g is a critical Riemannian metric g_C in a Sasakian manifold (M, g, η). It follows from Theorem A, (1.1) and (1.2C) that

$$C_C g_{ab}=-\nabla_r\nabla^r R_{ab}-2R_{asrb}R^{sr}+\frac{1}{2}|R_1|^2 g_{ab}.$$

Transvecting the above equation with g^{ab}, we have

$$C_C=\frac{n-4}{2n}|R_1|^2,$$

which and (3.1C) yield

$$|R_1|^2=2nR-n(n-1)(n+1).$$

If we take account of (2.7), then we get

$$(n-1)|G|^2=-R^2+2(n^2-1)R-n(n-1)^2(n+2)\geqq 0,$$

that is,

$$n(n-1)\leqq R\leqq(n-1)(n+2). \tag{3.2}$$

Now, let us examine the following two special cases of (3.2):

(i) the scalar curvature R equals to $n(n-1)$,

(ii) the scalar curvature R equals to $(n-1)(n+2)$.

It is evident that (M, g, η) is an Einstein manifold or an η-Einstein manifold satisfying $R_{ab}=(n+1)g_{ab}-2\eta_a\eta_b$ respectively, if the case (i) or (ii) holds.

Thus we have

THEOREM 3.2. *If g is a critical Riemannian metric g_C in an n-dimensional Sasakian manifold (M, g, η), then the scalar curvature R is pinched with*

$$n(n-1)\leqq R\leqq(n-1)(n+2).$$

Especially, if the scalar curvature R equals to $n(n-1)$ or $(n-1)(n+2)$, then the manifold is Einstein satisfying $R_{ab}=(n-1)g_{ab}$ *or η-Einstein satisfying $R_{ab}=(n+1)g_{ab}-2\eta_a\eta_b$ respectively.*

In the rest of this section, we study that g is a critical Riemannian metric g_D in a Sasakian manifold (M, g, η). From Theorem A, (1.1) and (1.2D) it is clear that

$$C_D g_{ab} = -4\nabla_r\nabla^r R_{ab} + 4R_{ar}R_b{}^r - 4R_{asrb}R^{sr} - 2R_{atsr}R_b{}^{tsr} + \frac{1}{2}|R|^2 g_{ab},$$

from which by contraction this with g^{ab} we have

$$C_D = \frac{n-4}{2n}|R|^2,$$

which implies

$$|R|^2 = 6nR - 2n(n-1)(3n-1),$$

because of (3.1D). This together with (2.8) gives

$$(n^2-1)\ |U|^2 = -8R^2 + 2(n-1)(3n^2+9n-2)R - 2n(n-1)^2(n+2)(3n-1) \geqq 0,$$

that is,

$$n(n-1) \leqq R \leqq \frac{1}{4}(n-1)(n+2)(3n-1). \tag{3.3}$$

If the equalities hold respectively, then we can prove that (M, g, η) is of constant curvature 1 or of constant ϕ-holomorphic sectional curvature $H=3n-1$.

Therefore we get

THEOREM 3.3. *If g is a critical Riemannian metric g_D in an n-dimensional Sasakian manifold, then the scalar curvature R is pinched with*

$$n(n-1) \leqq R \leqq \frac{1}{4}(n-1)(n+2)(3n-1).$$

Especially, if the scalar curvature R equals to $n(n-1)$ or $\frac{1}{4}(n-1)(n+2)(3n-1)$, then the manifold is of constant curvature 1 or of constant ϕ-holomorphic sectional curvature $H=3n-1$ respectively.

4. Critical Riemannian metrics in a Sasakian manifold with vanishing C-Bochner curvature tensor.

If g is a critical Riemannian metric g_A or g_B in a Sasakian manifold with vanishing C-Bochner curvature tensor, then by virtue of Theorem 3.1 we can easily verify that it is of constant curvature 1.

Next suppose that g is a critical Riemannian metric g_C in a Sasakian manifold with vanishing C-Bochner curvature tensor, then from Theorem A, (1.1), (1.2C), (2.11) and (2.12) we find

$$2(n+1)(n-1)C_C g_{ab} = 4(n-1)\{(5n+7)-R\}R_{ab} \tag{4.1}$$

$$+\{(n+1)(n-5)|R_1|^2+4R^2-16(n+1)R-4(n-1)(n^2-3n-6)\}g_{ab}$$
$$+\{4(n+1)|R_1|^2-4R^2-4(n^2-2n-7)R+4n(n-1)(n^2-3n-6)\}\eta_a\eta_b.$$

Differentiating covariantly both side of (4.1) and using (2.10), we get

$$(n-1)(5n+7-R)(\eta_a S_{db}+\eta_b S_{da})=\{(n+1)|R_1|^2-R^2-2(n+1)(n-3)R$$
$$+(n-1)(n^3+2n^2-4n-7)\}(\eta_a\phi_{db}+\eta_b\phi_{da}).$$

If we contract this with $\eta^b\phi_c{}^d$ and make use of (1.1), then we obtain

$$(4.2)\qquad (n-1)\{R-(5n+7)\}R_{ac}$$
$$=\{-(n+1)|R_1|^2+R^2+2(n+1)(n-3)R-(n-1)(n^3+2n^2-4n-7)\}g_{ac}$$
$$+\{(n+1)|R_1|^2-R^2-(n^2-2n-7)R+n(n-1)(n^2-3n-6)\}\eta_a\eta_c,$$

from which, by contraction

$$(4.3)\qquad |R_1|^2=2nR-n(n-1)(n+1).$$

Substituting (4.3) into (4.2), we have

$$\{R-(5n+7)\}[(n-1)R_{ab}-\{R-(n-1)\}g_{ab}+\{R-n(n-1)\}\eta_a\eta_b]=0,$$

from which $R=5n+7$ or $R_{ab}=\dfrac{R-(n-1)}{n-1}g_{ab}+\dfrac{n(n-1)-R}{n-1}\eta_a\eta_b$. If the scalar curvature R equals to $5n+7$, then we can prove by Theorem 3.2 that n is necessarily 7 and the manifold is of constant curvature 1. It follows from (2.7) and (4.2) that the scalar curvature R equals to $n(n-1)$ or $(n-1)(n+2)$ if $R\neq 5n+7$. Consequently Theorem 3.2 actually shows that (M, g, η) is of constant curvature 1 or of constant ϕ-holomorphic sectional curvature $H=(n+9)/(n+1)$, respectively.

Thus we obtain

THEOREM 4.1. *If g is a critical Riemannian metric g_C in an n-dimensional Sasakian manifold (M, g, η) with vanishing C-Bochner curvature tensor, then the manifold is of constant curvature* 1 *or of constant ϕ-holomorphic sectional curvature $H=(n+9)/(n+1)$.*

Finally we assume that g is a critical Riemannian metric g_D in a Sasakian manifold with vanishing C-Bochner curvature tensor. From Theorem A, (1.1), (1.2D), (2.11), (2.12) and (2.4) we find

$$(4.4)\qquad 2(n-1)(n+1)(n+3)C_D g_{ab}=8(n-1)(9n^2+32n+31-4R)R_{ab}$$
$$+\{16(n+1)(n-5)|R_1|^2-8(n-9)R^2-4(3n^3-3n^2+37n+91)R$$
$$+2(n-1)(3n^4-27n^3-19n^2+163n+136)\}g_{ab}$$
$$+\{64(n+1)|R_1|^2-64R^2-8(3n^3+5n^2-19n-53)R$$

$$+8n(n-1)(3n^3+5n^2-27n-45)\}\,\eta_a\eta_b\,.$$

Differentiating (4.4) covariantly and making use of (2.10), we get

$$\begin{aligned}&(n-1)(9n^2+32n+31-4R)(\eta_a S_{db}+\eta_b S_{da})\\&=\{8(n+1)|R_1|^2-8R^2-(3n^3+9n^2-27n-49)R\\&\quad+(n-1)(3n^4+14n^3-4n^2-46n-31)\}(\eta_a\phi_{db}+\eta_b\phi_{da})\,,\end{aligned}$$

from which we have by contraction of this with $\eta^b\phi_e{}^d$

$$\begin{aligned}(4.5)\qquad&(n-1)\{4R-(9n^2+32n+31)\}R_{ae}\\&=\{-8(n+1)|R_1|^2+(3n^3+9n^2-27n-49)R+8R^2\\&\quad-(n-1)(3n^4+14n^3-4n^2-46n-31)\}g_{ae}\\&\quad+\{8(n+1)|R_1|^2-8R^2-(3n^3+5n^2-19n-53)R\\&\quad+n(n-1)(3n^3+5n^2-27n-45)\}\eta_a\eta_e\,,\end{aligned}$$

which implies

$$(4.6)\qquad 8(n+1)|R_1|^2=4R^2+(3n^3+18n^2+9n-22)R-n(n-1)(3n^3+14n^2+5n-14)\,.$$

Comparing this with (4.5), we have

$$\{4R-(9n^2+32n+31)\}[(n-1)R_{ab}-\{R-(n-1)\}g_{ab}+\{R-n(n-1)\}\eta_a\eta_b]=0\,.$$

First let us discuss the case of $4R\neq 9n^2+32n+31$. Then we can easily verify that the manifold is η-Einstein with $R=n(n-1)$ or $(n-1)(n+2)(3n-1)/4$. Consequently it follows by virtue of Theorem 3.3 that (M, g, η) is of constant curvature 1 or of constant ϕ-holomorphic sectional curvature $H=3n-1$. The case of $4R=9n^2+32n+31$ implies that $n\geqq 7$ by Theorem 3.3.

Gathering these, we have

THEOREM 4.2. *Let g be a critical Riemannian metric g_D in an n-dimensional Sasakian manifold (M, g, η) with vanishing C-Bochner curvature tensor. Then*

(i) *for $4R\neq 9n^2+32n+31$, the manifold is of constant curvature 1 or of constant ϕ-holomorphic sectional curvature $H=3n-1$,*

(ii) *for $4R=9n^2+32n+31$, the scalar curvature R satisfies the inequality*

$$n(n-1)\leqq R\leqq\frac{1}{4}(n-1)(n+2)(3n-1)\,,\qquad (n\geqq 7)\,.$$

5. Conditions to be critical Riemannian metrics.

In this section we shall investigate some conditions that g is a critical Riemannian metric g_C or g_D in a Sasakian manifold (M, g, η).

Let (M, g, η) be an η-Einstein manifold. Then we have

$$(n-1)R_{ab}=\{R-(n-1)\}g_{ab}+\{n(n-1)-R\}\eta_a\eta_b, \tag{5.1}$$

from which

$$(n-1)|R_1|^2=R^2-2(n-1)R+n(n-1)^2. \tag{5.2}$$

Substituting (5.1) into (2.12), we can see

$$(n-1)\nabla_r\nabla^r R_{ab}=2\{n(n-1)-R\}(g_{ab}-n\eta_a\eta_b). \tag{5.3}$$

Substituting (5.1) and (5.2) into (2.13), we have

$$(n-1)^2R_{ar}R_b{}^r=\{R-(n-1)\}^2g_{ab}+\{n(n-1)-R\}\{R+(n-1)(n-2)\}\eta_a\eta_b. \tag{5.4}$$

From (5.1), (5.4) and (2.11), we can get

$$\begin{aligned}(n-1)^2R_{asrb}R^{sr}=&\{R^2-3(n-1)R+(n-1)^2(n+1)\}g_{ab}\\&-\{R-n(n-1)\}\{R-2(n-1)\}\eta_a\eta_b.\end{aligned} \tag{5.5}$$

Consequently we get

$$\begin{aligned}2(n-1)^2C_{ab}=&\{(n-5)R^2-2(n-1)(n-9)R+(n-1)^2(n^2-9n-4)\}g_{ab}\\&+4\{R-n(n-1)\}\{R-(n-1)(n+2)\}\eta_a\eta_b\end{aligned}$$

by virtue of (1.2C), (5.2), (5.3) and (5.5). Therefore we find

THEOREM 5.1. *In an n-dimensional η-Einstein manifold, if the scalar curvature R equals to $n(n-1)$ or $(n-1)(n+2)$, then g is a critical Riemannian metric g_C.*

As for a critical Riemannian metric g_D we shall prove

THEOREM 5.2. *In an n-dimensional Sasakian manifold (M, g, η) of constant ϕ-holomorphic sectional curvature H, if H equals to 1 or $3n-1$, then g is a critical Riemannian metric g_D.*

Proof. It is well known that the C-Bochner curvature tensor coincides with $U_{abc}{}^d$ if and only if (M, g, η) is an η-Einstein manifold. Thus, (M, g, η) is considered as an η-Einstein manifold with vanishing C-Bochner curvature tensor. Thus we have the following equation by the aid of (2.8):

$$|R|^2=\frac{2}{(n-1)(n+1)}\{4R^2-2(n-1)(3n-1)R+n(n-1)^2(3n-1)\}. \tag{5.6}$$

Substituting (5.1) and (5.4) into (2.14), we have

$$\begin{aligned}(n+1)(n-1)^2R_{atsr}R_b{}^{tsr}=&2\{4R^2-2(n-1)(3n-1)R+(n-1)^3(3n+1)\}g_{ab}\\&-4\{R-n(n-1)\}\{2R-(n-1)^2\}\eta_a\eta_b.\end{aligned} \tag{5.7}$$

Compering with (1.2D), (5.3), (5.4), (5.5), (5.6) and (5.7), we find

$$\begin{aligned}(n-1)^2(n+1)D_{ab}=&\{4(n-5)R^2-2(n-1)(3n^2-22n-1)R\\&+(n-1)^2(3n^3-28n^2-3n+4)\}g_{ab}\\&+4\{R-n(n-1)\}\{4R-(n-1)(n+2)(3n-1)\}\eta_a\eta_b,\end{aligned}$$

which shows that the assertion of the theorem is true.

6. Sasakian submersion.

We consider Riemannian submersions $\Pi:(\tilde{M},\tilde{g})\to(B,{}^Bg)$ such that fibers F are complete and connected and imbedded in $(\tilde{M},\tilde{g})$ regularly as totally geodesic submanifolds. The Riemannian metrics on the total manifold $\tilde{M}$, the base manifold B and the fiber F are denoted respectively by $\tilde{g}$, ${}^Bg$ and ${}^Fg$. Let $(\tilde{M},\tilde{g},\tilde{\eta})$ be a Sasakian manifold where $\dim\tilde{M}=\tilde{n}$. Let the indices $a, b, c, d, e, \cdots$ run over the range $\{1, 2, \cdots, \tilde{n}\}$ and the indices $h, i, j, k, \cdots$ the range $\{1, 2, \cdots, n\}$ where $\tilde{n}=n+1$. A Sasakian manifold $(\tilde{M},\tilde{g},\tilde{\eta})$ admits a Riemannian submersion where the unit Killing vector $\tilde{\eta}$ is a vertical vector and the fibers are geodesics tangent to $\tilde{\eta}$. Such a Riemannian submersion is called a Sasakian submersion.

Let us recall some relations with respect to a Sasakian submersion $\Pi:(\tilde{M},\tilde{g},\tilde{\eta})\to(B,{}^Bg)$. (For details see [9], [14]). For the Riemannian metric ${}^Bg$ on the base manifold B we have

$$(6.1)\qquad {}^Bg_{ji}=\tilde{g}_{ji}-\tilde{\eta}_j\tilde{\eta}_i,\qquad {}^Bg^{ji}=\tilde{g}^{ji}.$$

Relation between the curvature tensor $\tilde{R}_{abc}{}^d$ of $(\tilde{M},\tilde{g},\tilde{\eta})$ and the curvature tensor ${}^BR_{kji}{}^h$ of $(B,{}^Bg)$ has be given as follows:

$$(6.2)\qquad (R^H)_{kji}{}^h={}^BR_{kji}{}^h-F_{ji}F_k{}^h+F_{ki}F_j{}^h+2F_{kj}F_i{}^h,\qquad (F_{ji}=F_j{}^{r}{}^Bg_{ri}),$$

where R^H denotes the horizontal part of the curvature tensor $\tilde{R}$ of $(\tilde{M},\tilde{g},\tilde{\eta})$ and $F_j{}^i$ represents a complex structure J such that $(B,{}^Bg,J)$ is a Kählerian structure on B. For the Ricci tensor and the scalar curvature we have

$$(6.3)\qquad \tilde{R}_{ji}={}^BR_{ji}-2{}^Bg_{ji}+(\tilde{n}-1)\tilde{\eta}_j\tilde{\eta}_i,$$

$$(6.4)\qquad \tilde{R}={}^BR-(\tilde{n}-1).$$

In a Riemannian manifold (M, g), since the tensors B, C and D are given by (1.2), corresponding tensors of a Sasakian manifold $(\tilde{M},\tilde{g},\tilde{\eta})$ will be denoted by $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ and their components by $\tilde{B}_{ab}$, $\tilde{C}_{ab}$ and $\tilde{D}_{ab}$, while these of the base manifold $(B,{}^Bg)$ by ${}^BB$, ${}^BC$ and ${}^BD$ and their components by ${}^BB_{ij}$, ${}^BC_{ij}$ and ${}^BD_{ij}$. If the scalar curvature $\tilde{R}$ of $(\tilde{M},\tilde{g},\tilde{\eta})$ is constant, the following identities are obtained by Y. Muto [9]:

$$(6.5\mathrm{B})\qquad \tilde{B}_{ij}={}^BB_{ij}+2(\tilde{n}-1){}^BR_{ij}+\left\{-(\tilde{n}-5){}^BR-4(\tilde{n}-1)+\frac{1}{2}(\tilde{n}-1)^2\right\}{}^Bg_{ij}+\tilde{C}_B\tilde{\eta}_i\tilde{\eta}_j,$$

(6.5C) $$\tilde{C}_{ij}={}^{B}C_{ij}+12{}^{B}R_{ij}+\left\{-2{}^{B}R+\frac{1}{2}(\tilde{n}+3)(\tilde{n}-9)\right\}{}^{B}g_{ij}+\tilde{C}_{C}\tilde{\eta}_{i}\tilde{\eta}_{j},$$

(6.5D) $$\tilde{D}_{ij}={}^{B}D_{ij}+36{}^{B}R_{ij}+\{-6{}^{B}R+(3\tilde{n}+2)(\tilde{n}-9)\}{}^{B}g_{ij}+\tilde{C}_{D}\tilde{\eta}_{i}\tilde{\eta}_{j},$$

where we put

(6.6B) $$\tilde{C}_{B}=\frac{1}{2}({}^{B}R)^{2}-3(\tilde{n}-1){}^{B}R+\frac{5}{2}(\tilde{n}-1)^{2},$$

(6.6C) $$\tilde{C}_{C}=\frac{1}{2}{}^{B}R_{ij}{}^{B}R^{ij}-6{}^{B}R+\frac{5}{2}(\tilde{n}-1)(\tilde{n}+3),$$

(6.6D) $$\tilde{C}_{D}=\frac{1}{2}{}^{B}R_{ijkl}{}^{B}R^{ijkl}-18{}^{B}R+5(3\tilde{n}+2)(\tilde{n}-1).$$

Y. Muto [9] has proved the following theorems:

THEOREM B. *Let $\Pi:(\tilde{M},\tilde{g},\tilde{\eta})\to(B,{}^{B}g)$ be a Sasakian submersion. If $\tilde{g}$ and ${}^{B}g$ are critical Riemannian metrics g_{B} on $\tilde{M}$ and B respectively, then the scalar curvature $\tilde{R}$ is constant and $(B,{}^{B}g)$ is an Einstein manifold satisfying*

$$({}^{B}R)^{2}-(\tilde{n}-1)(\tilde{n}+2){}^{B}R+(\tilde{n}-1)^{2}(\tilde{n}+1)=0.$$

Conversely consider the case where $(B,{}^{B}g)$ is an Einstein manifold where ${}^{B}R=\tilde{n}^{2}-1$ or $\tilde{n}-1$. If one of $\tilde{g}$ and ${}^{B}g$ is a critical Riemannian metric g_{B}, then the other is also a critical Riemannian metrics g_{B}.

THEOREM C. *Let $\Pi:(\tilde{M},\tilde{g},\tilde{\eta})\to(B,{}^{B}g)$ be a Sasakian submersion. If $\tilde{g}$ and ${}^{B}g$ are critical Riemannian metrics g_{C} on $\tilde{M}$ and B respectively, then the scalar curvature $\tilde{R}$ is constant and $(B,{}^{B}g)$ is an Einstein manifold satisfying (*). Conversely, consider the case where $(B,{}^{B}g)$ is an Einstein manifold satisfying (*). If one of $\tilde{g}$ and ${}^{B}g$ is a critical Riemannian metric g_{C}, then the other is also a critical Riemannian metric g_{C}. Where (*) is as follows.*

(*) $$({}^{B}R)^{2}-2(\tilde{n}-1)(\tilde{n}+2){}^{B}R+(\tilde{n}+1)(\tilde{n}+3)(\tilde{n}-1)^{2}=0.$$

THEOREM D. *Let $\Pi:(\tilde{M},\tilde{g},\tilde{\eta})\to(B,{}^{B}g)$ be a Sasakian submersion. If $\tilde{g}$ and ${}^{B}g$ are critical Riemannian metrics g_{D} on $\tilde{M}$ and B respectively, then the scalar curvature $\tilde{R}$ is constant and $(B,{}^{B}g)$ is an Einstein manifold satisfying (**). Conversely, let us consider the case where $(B,{}^{B}g)$ is an Einstein manifold satisfying (**). If one of $\tilde{g}$ and ${}^{B}g$ is a critical Riemannian metric g_{D}, then the other is also a critical Riemannian metric g_{D}. Where (**) is as follows*

(**) $${}^{B}R_{ijkl}{}^{B}R^{ijkl}=6(\tilde{n}+2){}^{B}R-2(3\tilde{n}+2)(\tilde{n}^{2}-1).$$

We shall now show the following:

THEOREM 6.1. *Let $\Pi:(\tilde{M},\tilde{g},\tilde{\eta})\to(B,{}^{B}g)$ be a Sasakian submersion. In order*

that $\tilde{g}$ be a critical Riemannian metric g_B in a Sasakian manifold $(\tilde{M}, \tilde{g}, \tilde{\eta})$, it is necessary and sufficient that $(B, {}^Bg)$ be an Einstein manifold satisfying

$$ {}^BR_{ab}=(\tilde{n}+1){}^Bg_{ab}\,, $$

and ${}^Bg$ be a critical Riemannian metric g_B.

Proof. If we assume that $\tilde{g}$ is a critical Riemannian metric g_B on $\tilde{M}$, then, it follows from Theorem 3.1 that $(\tilde{M}, \tilde{g}, \tilde{\eta})$ is an Einstein manifold satisfying

$$ \tilde{R}_{ab}=(\tilde{n}-1)\tilde{g}_{ab}\,. $$

In view of (6.1), this means that ${}^BR_{ij}=(\tilde{n}+1){}^Bg_{ij}$, namely, the base manifold $(B, {}^Bg)$ is an Einstein one with ${}^BR=(\tilde{n}-1)(\tilde{n}+1)$. Furthermore from (6.5B) we have ${}^BB_{ij}={}^BC_B{}^Bg_{ij}$, because of $\tilde{g}$ is a critical Riemannian metric g_B. The converse is trivial by Theorem B. These complete the proof.

Secondly, let us prove

THEOREM 6.2. *Let $\Pi:(\tilde{M}, \tilde{g}, \tilde{\eta})\to(B, {}^Bg)$ be a Sasakian submersion and $\tilde{g}$ be a critical Riemannian metric g_C in a Sasakian manifold $(\tilde{M}, \tilde{g}, \tilde{\eta})$. Then the scalar curvature ${}^BR$ in $(B, {}^Bg)$ is pinched with*

$$ (\tilde{n}-1)(\tilde{n}+1)\leqq {}^BR\leqq(\tilde{n}-1)(\tilde{n}+3)\,. \tag{6.7} $$

Especially, if the scalar curvature ${}^BR$ equals to $(\tilde{n}-1)(\tilde{n}+1)$ or $(\tilde{n}-1)(\tilde{n}+3)$, then the manifold is an Einstein one satisfying

$$ {}^BR_{ab}=(\tilde{n}+1){}^Bg_{ab} \quad or \quad {}^BR_{ab}=(\tilde{n}+3){}^Bg_{ab} \tag{6.8} $$

respectively, and ${}^Bg$ is a critical Riemannian metric g_C in $(B, {}^Bg)$.

Proof. Taking account of the fact that $\tilde{g}$ is a critical Riemannian metric g_C, in view of Theorem 3.2 and (6.4) it is easy to see that the scalar curvature ${}^BR$ in $(B, {}^Bg)$ is pinched with (6.7). Recalling that if the scalar curvature ${}^BR$ of $(B, {}^Bg)$ is $\tilde{n}^2-1$ or $(\tilde{n}-1)(\tilde{n}+3)$, then the scalar curvature $\tilde{R}$ of $(\tilde{M}, \tilde{g}, \tilde{\eta})$ is $\tilde{n}(\tilde{n}-1)$ or $(\tilde{n}-1)(\tilde{n}+2)$ respectively, we can easily see by virtue of Theorem 3.2 that $(\tilde{M}, \tilde{g}, \tilde{\eta})$ is Einstein or η-Einstein satisfying $\tilde{R}_{ab}=(\tilde{n}+1)\tilde{g}_{ab}-2\tilde{\eta}_a\tilde{\eta}_b$ respectively. Therefore it is evident by (6.3) that $(B, {}^Bg)$ is an Einstein manifold satisfying (6.8). Consequently, regarding to $\tilde{C}_{ab}=\tilde{C}_C\tilde{g}_{ab}$, (6.1) and the mentions above, we have ${}^BC_{ij}={}^BC_C{}^Bg_{ij}$, which means that ${}^Bg$ is a critical Riemannian metric g_C. These facts show that the assertion of the theorem is true.

Finally we shall prove the following

THEOREM 6.3. *Let $\Pi:(\tilde{M}, \tilde{g}, \tilde{\eta})\to(B, {}^Bg)$ be a Sasakian submersion and $\tilde{g}$ be a critical Riemannian metric g_D in a Sasakian manifold $(\tilde{M}, \tilde{g}, \tilde{\eta})$. Then the scalar curvature ${}^BR$ in $(B, {}^Bg)$ is pinched with*

$$ (\tilde{n}-1)(\tilde{n}+1)\leqq {}^BR\leqq\frac{1}{4}(\tilde{n}-1)(\tilde{n}+1)(3\tilde{n}+2)\,. \tag{6.9} $$

Especially, if the scalar curvature ${}^{B}R$ *equal to* $\tilde{n}^2-1$ *or* $(\tilde{n}-1)(\tilde{n}+1)(3\tilde{n}+2)/4$, *then* $(B, {}^{B}g)$ *is of constant holomorphic sectional curvature* $H=4$ *or* $H=3\tilde{n}+2$ *respectively, and* ${}^{B}g$ *is a critical Riemannian metric* g_D.

Proof. As $\tilde{g}$ is a critical Riemannian metric g_D, it follows from (6.4) and Theorem 3.3 that (6.9) holds. If ${}^{B}R$ is $\tilde{n}^2-1$ or $(\tilde{n}-1)(\tilde{n}+1)(3\tilde{n}+2)/4$, we have $\tilde{R}=\tilde{n}(\tilde{n}-1)$ or $(\tilde{n}-1)(\tilde{n}+2)(3\tilde{n}-1)/4$ respectively. So we can see from Theorem 3.3 that $(\tilde{M}, \tilde{g}, \tilde{\eta})$ is of constant curvature 1 or of constant ϕ-holomorphic sectional curvature $\tilde{H}=3\tilde{n}-1$. Making use of (6.2), we have that $(B, {}^{B}g)$ is of constant holomorphic sectional curvature $H=4$ or $3\tilde{n}+2$ respectively. These facts prove that ${}^{B}g$ is a critical Riemannian metric g_D. These complete the proof.

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