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POISSON APPROXIMATION FOR SUMS OF INDEPENDENT BIVARIATE BERNOULLI VECTORS

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1. Introduction. It is well known fact as Poisson's theorem that for a given sequence of $\{p_n, n\geqq 1\}$ such that $p_n\to 0$ $(n\to\infty)$ we have

$$P_n(m)-(\lambda_n^m/m!)e^{-\lambda_n} \longrightarrow 0 \qquad \text{as} \quad n\to\infty$$

for all non-negative integer m where

$$\lambda_n=np_n, \qquad P_n(m)=\binom{n}{m}p_n^m(1-n)^{n-m}.$$

Furthermore, if $np_n\to\lambda$ $(n\to\infty)$ then we have

$$P_n(m) \longrightarrow (\lambda^m/m!)e^{-\lambda} \qquad \text{as} \quad n\to\infty.$$

R. von Mises in the paper [3] has showed that if $\{X_{kj}, k\geqq 1, j=1, 2, \cdots, n_k\}$ is a sequence of independent random variables such that

$$P[X_{kj}=1]=1-P[X_{kj}=0]=p_{kj}, \qquad j=1, 2, \cdots, n_k$$

and

(1.1) $$\max_{1\leq j\leq n_k} p_{kj} \longrightarrow 0, \qquad \sum_{j=1}^{n_k} p_{kj} \longrightarrow \lambda>0 \quad (k\to\infty),$$

then

$$P\left[\sum_{j=1}^{n_k} X_{kj}=m\right] \longrightarrow (\lambda^m/m!)e^{-\lambda} \qquad (k\to\infty).$$

In (1977) J. Mačys (see. [2]) has proved that the conditions (1.1) are necessary as well.

Let $\{(X_k, Y_k), k\geqq 1\}$ be a sequence of random vectors bivariate Bernoulli law, i.e.

$$P[X_k=0, Y_k=0]=p_{00}, \qquad P[X_k=1, Y_k=0]=p_{10},$$

$$P[X_k=0, Y_k=1]=p_{01}, \qquad P[X_k=1, Y_k=1]=p_{11},$$

where $p_{00}+p_{10}+p_{01}+p_{11}=1$.

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K. Kawamura in [1] has proved that if $\{(X_k, Y_k),\ k\geqq 1\}$ are mutually independent and indentically distributed random vectors having bivariate Bernoulli probability, then

$$P\left[\sum_{k=1}^{n}(X_k, Y_k)=(n, m)\right]\longrightarrow \sum_{s=0}^{\min(n,m)}\frac{\lambda_{10}^{n-s}\lambda_{01}^{m-s}\lambda_{11}^{s}}{(n-s)!(m-s)!s!}e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}$$

as $n\to\infty$, where $np_{11}=\lambda_{11}$, $np_{10}=\lambda_{10}$ and $np_{01}=\lambda_{01}$ are fixed values.

The main aim of this paper is to generalize Kawamura's results [1] to nonidentically distributed random vectors $\{(X_k, Y_k),\ k\geqq 1\}$. The results presented in Section 2 extend those of Kawamura [1], and Mačys [2].

2. The result.

Let $\{(X_{kj}, Y_{kj}),\ j=1, 2, \cdots, n_k,\ k\geqq 1\}$ be a sequence of independent bivariate Bernoulli vectors with

$$P[X_{kj}=0,\ Y_{kj}=0]=p_{kj}(0, 0), \qquad P[X_{kj}=0,\ Y_{kj}=1]=p_{kj}(0, 1),$$

$$P[X_{kj}=1,\ Y_{kj}=0]=p_{kj}(1, 0), \qquad P[X_{kj}=1,\ Y_{kj}=1]=p_{kj}(1, 1),$$

where $p_{kj}(0, 0)+p_{kj}(0, 1)+p_{kj}(1, 0)+p_{kj}(1, 1)=1$.

Let

$$S_k=\sum_{j=1}^{n_k}(X_{kj}, Y_{kj}), \qquad k\geqq 1.$$

THEOREM. *In order that*

$$\lim_{k\to\infty}P[S_k=(n, m)]=\sum_{s=0}^{\min(n,m)}\frac{\lambda_{10}^{n-s}\lambda_{01}^{m-s}\lambda_{11}^{s}}{(n-s)!(m-s)!s!}e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}$$

may hold for all $n, m\geqq 0$ *it is necessary and sufficient that for* $k\to\infty$

$$\sum_{j=1}^{n_k}p_{kj}(1, 0)\longrightarrow\lambda_{10}, \tag{2.1}$$

$$\sum_{j=1}^{n_k}p_{kj}(0, 1)\longrightarrow\lambda_{01}, \tag{2.2}$$

$$\sum_{j=1}^{n_k}p_{kj}(1, 1)\longrightarrow\lambda_{11} \tag{2.3}$$

and

$$\min_{1\leq k\leq n_k}p_{kj}(0, 0)\longrightarrow 1. \tag{2.4}$$

Proof. For the sake of simplicity the index k will be omitted in the proof of Theorem, i.e. instead of p_{kj} we write p_j.

The part if. It is easy to see that

(2.5) $$P[S_k=(0, 0)]=\prod_{g=1}^{n_k} p_g(0, 0),$$

(2.6) $$P[S_k=(1, 0)]=\sum_{t=1}^{n_k}\Big\{p_t(1, 0)\prod_{\substack{g=1\\ \neq t}}^{n_k} p_g(0, 0)\Big\}$$
$$=\prod_{g=1}^{n_k} p_g(0, 0)\sum_{t=1}^{n_k} p_t(1, 0)/p_t(0, 0),$$

(2.7) $$P[S_k=(2, 0)]=\sum_{t_1<t_2}^{n_k}\Big\{p_{t_1}(1, 0)p_{t_2}(1, 0)\prod_{\substack{g=1\\ \neq t_1, t_2}}^{n_k} p_g(0, 0)\Big\}$$
$$=\prod_{g=1}^{n_k} p_g(0, 0)\sum_{t_1<t_2} p_{t_1}(1, 0)p_{t_2}(1, 0)/p_{t_1}(0, 0)p_{t_2}(0, 0).$$

In the same way, for every $n>2$, we obtain

$$P[S_k=(n, 0)]=\sum_{t_1<\cdots<t_n}\Big\{\prod_{l=1}^{n} p_{t_l}(1, 0)\prod_{\substack{g=1\\ \neq t_1,\cdots,t_n}}^{n_k} p_g(0, 0)\Big\}$$
$$=\prod_{g=1}^{n_k} p_g(0, 0)\sum_{t_1<\cdots<t_n}\Big(\prod_{l=1}^{n} p_{t_l}(1, 0)/p_{t_l}(0, 0)\Big).$$

Let $n, m\geqq 1$ be given and assume that $n=m$. If we put $\delta=\max\{2n-n_k, 0\}$, then

$$P[S_k=(n, n)]=\sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_n} P\Big[\bigcap_{p=1}^{n}[X_{j_p}=1],\ \bigcap_{\substack{l=1\\ \neq i_1,\cdots,i_n}}^{n_k}[X_l=0],$$
$$\cdot\bigcap_{r=1}^{n}[Y_{j_r}=1],\ \bigcap_{\substack{s=1\\ \neq j_1,\cdots,j_n}}^{n_k}[Y_s=0]\Big]=\sum_{i_1<\cdots<i_n}\Big\{\prod_{p=1}^{n} p_{i_p}(1, 1)\prod_{\substack{g=1\\ \neq i_1,\cdots,i_n}}^{n_k} p_g(0, 0)\Big\}$$
$$+\sum_{t_1=1}^{n_k}\sum_{\substack{i_1<\cdots<i_{n-1}\\ \neq t_1}}\sum_{\substack{j_1=1\\ \neq t_1; i_1,\cdots,i_{n-1}}}^{n_k}\Big\{p_{t_1}(1, 0)p_{j_1}(0, 1)\prod_{p=1}^{n-1} p_{i_p}(1, 1)\prod_{\substack{g=1\\ \neq t_1; i_1,\cdots,i_{n-1}; j_1}}^{n_k} p_g(0, 0)\Big\}$$
$$+\sum_{t_1<t_2}\sum_{\substack{i_1<\cdots<i_{n-2}\\ \neq t_1, t_2}}\sum_{\substack{j_1<j_2\\ \neq t_1, t_2; i_1,\cdots,i_{n-2}}}\Big\{\prod_{l=1}^{2} p_{t_l}(1, 0)\prod_{r=1}^{2} p_{j_r}(0, 1)$$
$$\cdot\prod_{p=1}^{n-2} p_{i_p}(1, 1)\prod_{\substack{g=1\\ \neq t_1 t_2; i_1,\cdots,i_{n-2}; j_1, j_2}}^{n_k} p_g(0, 0)\Big\}+\cdots+\sum_{t_1<\cdots<t_{n-\delta}}\sum_{i_1<\cdots<i_\delta}$$
$$\cdot\sum_{\substack{j_1<\cdots<j_{n-\delta}\\ \neq t_1,\cdots,t_{n-\delta}; i_1,\cdots,i_\delta}}\Big\{\prod_{l=1}^{n-\delta} p_{t_l}(1, 0)\prod_{r=1}^{n-\delta} p_{j_r}(0, 1)\prod_{p=1}^{\delta} p_{i_p}(1, 1)\Big\}.$$

Taking out the product $\prod_{g=1}^{n_k} p_g(0, 0)$ before the sign of the first sum, we may write out

(2.8) $$P[S_k=(n, n)]=\sum_{s=\max\{2n-n_k, 0\}}^{n}\prod_{g=1}^{n_k} p_g(0, 0)\Big\{\sum_{t_1<\cdots<t_{n-s}}\Big(\prod_{l=1}^{n-s} p_{t_l}(1, 0)/p_{t_l}(0, 0)$$

$$\cdot \sum_{\substack{i_1<\cdots<i_s \\ \neq t_1,\cdots,t_{n-s}}} \Big(\prod_{p=1}^{s} p_{i_p}(1,1)/p_{i_p}(0,0)\Big) \sum_{\substack{j_1<\cdots<j_{n-s} \\ \neq t_1,\cdots,t_{n-s};i_1,\cdots,i_s}} \Big(\prod_{r=1}^{n-s} p_{j_r}(0,1)/p_{j_r}(0,0)\Big)\Big\}.$$

Let now $n, m \geqq 1$ be given and suppose that $n<m$. If we put $\delta=\max\{n+m-n_k, 0\}$, then

$$P[S_k=(n,m)]=\sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_m} P\Big[\bigcap_{p=1}^{n}[X_{i_p}=1], \bigcap_{\substack{l=1 \\ \neq i_1,\cdots,i_n}}^{n_k}[X_l=0], \bigcap_{r=1}^{m}[Y_{j_r}=1],$$

$$\cdot \bigcap_{\substack{s=1 \\ \neq j_1,\cdots,j_m}}^{n_k}[Y_s=0]\Big]=\sum_{i_1<\cdots<i_n}\sum_{\substack{j_1<\cdots<j_{m-n} \\ \neq i_1,\cdots,i_n}}\Big\{\prod_{p=1}^{n} p_{i_p}(1,1)\prod_{r=1}^{m-n} p_{j_r}(0,1)$$

$$\cdot \prod_{\substack{g=1 \\ \neq i_1,\cdots,i_n;j_1,\cdots,j_m}}^{n_k} p_g(0,0)\Big\}+\sum_{t_1=1}^{n_k}\sum_{\substack{i_1<\cdots<i_{n-1} \\ \neq t_1}}\sum_{\substack{j_1<\cdots<j_{m-n+1} \\ \neq t_1;i_1,\cdots,i_{n-1}}}\Big\{p_{t_1}(1,0)$$

$$\cdot \prod_{r=1}^{m-n+1} p_{j_r}(0,1)\prod_{p=1}^{n-1} p_{i_p}(1,1)\prod_{\substack{g=1 \\ \neq t_1;i_1,\cdots,i_{n-1};j_1,\cdots,j_{m-n+1}}}^{n_k} p_g(0,0)\Big\}$$

$$+\sum_{t_1<t_2}\sum_{\substack{i_1<\cdots<i_{n-2} \\ \neq t_1,t_2}}\sum_{\substack{j_1<\cdots<j_{m-n+2} \\ \neq t_1,t_2;i_1,\cdots,i_{n-2}}}\Big\{\prod_{l=1}^{2} p_{t_l}(1,0)\prod_{r=1}^{m-n+2} p_{j_r}(0,1)\prod_{p=1}^{n-2} p_{i_p}(1,1)$$

$$\cdot \prod_{\substack{g=1 \\ \neq t_1,t_2;i_1,\cdots,i_{n-2};j_1,\cdots,j_{m-n+2}}}^{n_k} p_g(0,0)\Big\}+\cdots+\sum_{t_1<\cdots<t_{n-\delta}}\sum_{\substack{i_1<\cdots<i_\delta \\ \neq t_1,\cdots,t_{n-\delta}}}$$

$$\cdot \sum_{\substack{j_1<\cdots<j_{m-\delta} \\ \neq t_1,\cdots,t_{n-\delta};i_1,\cdots,i_\delta}}\Big\{\prod_{l=1}^{n-\delta} p_{t_l}(1,0)\prod_{r=1}^{m-\delta} p_{j_r}(0,1)\prod_{p=1}^{\delta} p_{i_p}(1,1)\Big\}.$$

Thus taking into account (2.8) and the above given equality, for all $n, m \geqq 1$, we may write

$$(2.9)\quad P[S_k=(n,m)]=\sum_{s=\max\{n+m-n_k,0\}}^{\min(n,m)}\prod_{g=1}^{n_k}\Big\{p_g(0,0)\sum_{t_1<\cdots<t_{n-s}}\Big(\prod_{l=1}^{n-s} p_{t_l}(1,0)/p_{t_l}(0,0)\Big)$$

$$\cdot \sum_{\substack{i_1<\cdots<i_s \\ \neq t_1,\cdots,t_{n-s}}}\Big(\prod_{p=1}^{s} p_{i_p}(1,1)/p_{i_p}(0,0)\Big)\sum_{\substack{j_1<\cdots<j_{m-s} \\ \neq t_1,\cdots,t_{n-s};i_1,\cdots,i_s}}\Big(\prod_{r=1}^{m-s} p_{j_r}(0,1)/p_{j_r}(0,0)\Big)\Big\}.$$

In order to prove that

$$\prod_{g=1}^{n_k} p_g(0,0) \longrightarrow e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})} \quad \text{as} \quad k\to\infty,$$

we consider the inequality $1+y \leqq e^y$, $y\in[-1,\infty)$. Putting $y=-x$, $x\in[0,1)$, we have

$$e^{-x/(1-x)} \leqq 1-x \leqq e^{-x}, \qquad x\in[0,1).$$

Now putting $\Lambda_g=p_g(1,0)+p_g(0,1)+p_g(1,1)$ and using the last inequality, we obtain

$$e^{-\frac{1}{\min p_g(0,0)}\sum_{g=1}^{n_k}\Lambda_g} \leqq \prod_{g=1}^{n_k} p_g(0,0) \leqq e^{-\Sigma\Lambda_g}.$$

Taking into account (2.1)-(2.4) one can prove that

$$\lim_{k\to\infty} \prod_{g=1}^{n_k} p_g(0, 0)=e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}\,.$$

Now we are going to prove that

$$(2.10)\quad \sum_{t_1<\cdots<t_{n-s}} \prod_{l=1}^{n-s} p_{t_l}(1, 0) \sum_{\substack{i_1<\cdots<i_s\\ \neq t_1,\cdots,t_{n-s}}} \prod_{p=1}^{s} p_{i_p}(1, 1) \sum_{\substack{j_1<\cdots<j_{m-s}\\ \neq t_1,\cdots,t_{n-s};i_1,\cdots,i_s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1)$$

$$\longrightarrow \lambda_{10}^{n-s}/(n-s)\,!\cdot\lambda_{11}^{s}/s\,!\cdot\lambda_{01}^{m-s}/(m-s)\,! \qquad \text{as} \quad k\to\infty\,.$$

In the case $m=0$ the proof is by induction with respect to n. The case $n=1$ is obvious. Let $n=2$, then by (2.1) and (2.4)

$$\sum_{t_1<t_2} p_{t_1}(1, 0)p_{t_2}(1, 0) \longrightarrow \lambda_{10}^2/2$$

as

$$2\sum_{t_1<t_2} p_{t_1}(1, 0)p_{t_2}(1, 0)=\Big(\sum_{t=1}^{n_k} p_t(1, 0)\Big)^2-\sum_{t=1}^{n_k} p_t^2(1, 0)$$

and

$$0\leqq \sum_{t=1}^{n_k} p_t^2(1, 0)\leqq(1-\min p_t(0, 0))\cdot \sum_{t=1}^{n_k} p_t(1, 0)\,.$$

Assume that

$$\sum_{t_1<\cdots<t_{n-1}} \prod_{l=1}^{n-1} p_{t_l}(1, 0) \longrightarrow \lambda_{10}^{n-1}/(n-1)\,!\,.$$

Multiplying the left hand side of the last relation by $\sum_{t_n=1}^{n_k} p_{t_n}(1, 0)$ we obtain

$$\sum_{t_1<\cdots<t_{n-1}} p_{t_1}^2(1, 0) \prod_{\substack{l=1\\ \neq t_1}}^{n-1} p_{t_l}(1, 0)+\sum_{t_1<\cdots<t_{n-1}} p_{t_2}^2(1, 0) \prod_{\substack{l=1\\ \neq t_2}}^{n-1} p_{t_l}(1, 0)+\cdots$$

$$+\sum_{t_1<\cdots<t_{n-1}} p_{t_{n-1}}^2(1, 0) \prod_{\substack{l=1\\ \neq t_{n-1}}}^{n-1} p_{t_l}(1, 0)+\sum_{t_n<t_1<\cdots<t_{n-1}} \prod_{l=1}^{n} p_{t_l}(1, 0)$$

$$+\sum_{t_1<t_n<\cdots<t_{n-1}} \prod_{l=1}^{n} p_{t_l}(1, 0)+\cdots+\sum_{t_1<\cdots<t_{n-1}<t_n} \prod_{l=1}^{n} n_{t_l}(1, 0)\,.$$

Because every sum from among the first $(n-1)$ sums we may estimate by

$$(1-\min p_t(0, 0))\sum_{t_1<\cdots<t_{n-1}} \prod_{l=1}^{n-1} p_{t_l}(1, 0)$$

so the first $(n-1)$ sums tend to 0. We have then

$$n\sum_{t_1<\cdots<t_n} \prod_{l=1}^{n} p_{t_l}(1, 0) \longrightarrow \lambda_{10}^{n}/(n-1)\,!\,.$$

The last relation proves that

$$\sum_{t_1<\cdots<t_n} \prod_{l=1}^{n} p_{t_l}(1, 0) \longrightarrow \lambda_{10}^{n}/n! . \tag{2.11}$$

In the same way one can prove that

$$\sum_{i_1<\cdots<i_s} \prod_{p=1}^{s} p_{i_p}(1, 1) \longrightarrow \lambda_{11}^{s}/s! \tag{2.12}$$

and

$$\sum_{j_1<\cdots<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1) \longrightarrow \lambda_{01}^{m-s}/(m-s)! . \tag{2.13}$$

Let us put

$$B_{n_k}(n-s, s, m-s)=\sum_{t_1<\cdots<t_{n-s}} \prod_{l=1}^{n-s} p_{t_l}(1, 0) \sum_{i_1<\cdots<i_s} \prod_{p=1}^{s} p_{i_p}(1, 1) \cdot \sum_{j_1<\cdots<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1) .$$

Then, taking into account (2.11)-(2.13), we have

$$B_{n_k}(n-s, s, m-s) \longrightarrow \lambda_{10}^{n-s}/(n-s)!\cdot\lambda_{11}^{s}/s!\cdot\lambda_{01}^{m-s}/(m-s)! \tag{2.14}$$

as $k\to\infty$.

Let us define

$$\overset{s}{\underset{t_1,\cdots,t_{n-s}}{A}}(1, 1)=\sum_{i_1<\cdots<i_s} \prod_{p=1}^{s} p_{i_p}(1, 1)-\sum_{\substack{i_1<\cdots<i_s\\ \neq t_1,\cdots,t_{n-s}}} \prod_{p=1}^{s} p_{i_p}(1, 1)$$

and

$$\overset{m-s}{\underset{t_1,\cdots,t_{n-s};i_1,\cdots,i_s}{A}}(0, 1)=\sum_{j_1<\cdots<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1)-\sum_{\substack{j_1<\cdots<j_{m-s}\\ \neq t_1,\cdots,t_{n-s};i_1,\cdots,i_s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1) .$$

Taking into account the above considerations it may be proved that $\overset{s}{\underset{t_1,\cdots,t_{n-s}}{A}}(1, 1)$ and $\overset{m-s}{\underset{t_1,\cdots t_{n-s};i_1,\cdots,i_s}{A}}(0, 1)$ tend to 0 as $k\to\infty$.

It is easy to see that

$$\sum_{t_1<\cdots<t_{n-s}} \prod_{l=1}^{n-s} p_{t_l}(1, 0) \sum_{\substack{i_1<\cdots<i_s\\ \neq t_1,\cdots,t_{n-s}}} \prod_{p=1}^{s} p_{i_p}(1, 1) \sum_{\substack{j_1<\cdots<j_{m-s}\\ \neq t_1,\cdots,t_{n-s};i_1,\cdots,i_s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1)$$

$$=B_{n_k}(n-s, s, m-s)-(Z_1+Z_2+Z_3) ,$$

where

$$Z_1=\sum_{t_1<\cdots<t_{n-s}} \prod_{l=1}^{n-s} p_{t_l}(1, 0) \overset{n-s}{\underset{t_1,\cdots,t_{n-s}}{A}}(0, 1) \sum_{j_1<\cdots<j_{m-s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1) ,$$

$$Z_2=\sum_{t_1<\cdots<t_{n-s}} \prod_{l=1}^{n-s} p_{t_l}(1, 0) \sum_{i_1<\cdots<i_s} \prod_{p=1}^{s} p_{i_p}(1, 0) \overset{m-s}{\underset{t_1,\cdots,t_{n-s};i_1,\cdots,t_s}{A}}(0, 1) ,$$

$$Z_3 = \sum_{t_1<\cdots<t_{n-s}} \prod_{l=1}^{n-s} p_{t_l}(1, 0) \underset{t_1,\cdots,t_{n-s}}{\overset{s}{A}}(1, 1) \underset{t_1,\cdots,t_{n-s};\, i_1,\cdots,i_s}{\overset{m-s}{A}}(0, 1)$$

and $Z_i \to 0$ as $k\to\infty$, $1\leqq i\leqq 3$.

The relation (2.10) finishes the proof of the part if, because

$$\sum_{t_1<\cdots<t_{n-s}} \prod_{l=1}^{n-s} p_{t_l}(1, 0) \sum_{\substack{i_1<\cdots<i_s \\ \neq t_1,\cdots,t_{n-s}}} \prod_{p=1}^{s} p_{i_p}(1, 1) \sum_{\substack{j_1<\cdots<j_{m-s} \\ \neq t_1,\cdots,t_{n-s},i_1,\cdots,i_s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1)$$

$$\leqq \sum_{t_1<\cdots<t_{n-s}} \prod_{l=1}^{n-s} p_{t_l}(1, 0)/p_{t_l}(0, 0) \sum_{\substack{i_1,\cdots,i_s \\ \neq t_1,\cdots,t_{n-s}}} \prod_{p=1}^{s} p_{i_p}(1, 1)/p_{i_p}(0, 0)/$$

$$\sum_{\substack{j_1<\cdots<j_{m-s} \\ \neq t_1,\cdots,t_{n-s};i_1,\cdots,i_s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1)/p_{j_r}(0, 0)$$

$$\leqq \left(\frac{1}{\min p_j(0, 0)}\right)^{n+m-s} \cdot \sum_{t_1<\cdots<t_{n-s}} \prod_{l=1}^{n-s} p_{t_l}(1, 0)$$

$$\cdot \sum_{\substack{i_1<\cdots<i_s \\ \neq t_1,\cdots,t_{n-s}}} \prod_{p=1}^{s} p_{i_p}(1, 1) \sum_{\substack{j_1<\cdots<j_{m-s} \\ \neq t_1,\cdots,t_{n-s};i_1,\cdots,i_s}} \prod_{r=1}^{m-s} p_{j_r}(0, 1) .$$

Proof of the part only. From (2.5)-(2.7) we have

$$\prod_{g=1}^{n_k} p_g(0, 0) \longrightarrow e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})} , \tag{2.15}$$

$$\sum_{t_1=1}^{n_k} p_{t_1}(1, 0)/p_{t_1}(0, 0) \longrightarrow \lambda_{10} , \tag{2.16}$$

$$\sum_{j_1=1}^{n_k} p_{j_1}(0, 1)/p_{j_1}(0, 0) \longrightarrow \lambda_{01}$$

and

$$\sum_{t_1<t_2} p_{t_1}(1, 0)p_{t_2}(1, 0)/p_{t_1}(0, 0)p_{t_2}(0, 0) \longrightarrow \lambda_{10}^2/2 . \tag{2.17}$$

$$\sum_{j_1<j_2} p_{j_1}(0, 1)p_{j_2}(0, 1)/p_{j_1}(0, 0)p_{j_2}(0, 0) \longrightarrow \lambda_{01}^2/2 .$$

From (2.15)-(2.17) we get

$$\sum_{t_1=1}^{n_k} (p_{t_1}(1, 0)/p_{t_1}(0, 0))^2 \longrightarrow 0 , \qquad \sum_{j_1=1}^{n_k} (p_{j_1}(0, 1)/p_{j_1}(0, 0))^2 \longrightarrow 0 ,$$

which implies that

$$\max (p_t(1, 0)/p_t(0, 0)) \longrightarrow 0 , \qquad \max (p_j(0, 1)/p_j(0, 0)) \longrightarrow 0 .$$

Now we will prove that $\max (p_i(1, 1)/p_i(0, 0))\to 0$. Indeed, from (2.8) for $n=m=1$ and (2.15) we have

$$\sum_{i_1=1}^{n_k} p_{i_1}(1, 0)/p_{i_1}(0, 0) \sum_{j_1=1}^{n_k} p_{j_1}(0, 1)/p_{j_1}(0, 0)+ \sum_{i_1=1}^{n_k} p_{i_1}(1, 1)/p_{i_1}(0, 0)$$

$$\longrightarrow \lambda_{10}\cdot\lambda_{01}+\lambda^{11}.$$

On the other hand, taking into account the inequality

$$\sum_{i_1=1}^{n_k} p_{i_1}(1, 0)p_{i_1}(0, 1)/p_{i_1}^2(0, 0) \leqq \max(p_j(0, 1)/p_j(0, 0)) \sum_{i_1=1}^{n_k} p_{i_1}(1, 0)/p_{i_1}(0, 0),$$

(2.15) and (2.16), we get

$$\sum_{i_1=1}^{n_k} p_{i_1}(1, 1)/p_{i_1}(0, 0) \longrightarrow \lambda_{11}. \tag{2.18}$$

In the same way, putting in (2.8) $n=m=2$, we obtain

$$\sum_{i_1<i_2} p_{i_1}(1, 1)p_{i_2}(1, 1)/p_{i_1}(0, 0)p_{i_2}(0, 0) \longrightarrow \lambda_{11}^2/2. \tag{2.19}$$

From (2.18) and (2.19) it follows that $\max(p_i(1, 1)/p_i(0, 0)) \to 0$, and therefore $\min p_g(0, 0) \to 1$ as $k\to\infty$.

The last relation and (2.16) imply that

$$\sum_{i=1}^{n_k} p_i(1, 0) \longrightarrow \lambda_{10},$$

because

$$\min p_i(0, 0) \sum_{i_1=1}^{n_k} p_{i_1}(1, 0)/p_{i_1}(0, 0) \leqq \sum_{i_1=1}^{n_k} p_{i_1}(1, 0) \leqq \sum_{i_1=1}^{n_k} p_{i_1}(1, 0)/p_{i_1}(0, 0).$$

In the same way one can prove that (2.2) and (2.3) are satisfied. Thus the proof of Theorem is completed.

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