K. YAMAGUCHI
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ON THE HOMOTOPY OF TYPE *CW* COMPLEXES WITH THE FORM $S^2 \cup e^4 \cup e^6$

BY KOHHEI YAMAGUCHI

§1. Introduction.

The purpose of this paper is to classify the homotopy type of *CW* complexes with the form $S^2 \cup e^4 \cup e^6$. For example, the total space of a sphere boundle over a sphere (or of a spherical fibration over a sphere) is a *CW* complex with the form $S^p \cup e^q \cup e^{p+q}$ up to homotopy. The homotopy type classification of such a complex was partially given by James and Whitehead [8] and Sasao [6], and for more general cases Toda considered. [7]

In general, it is not easy to find the complete invariants which determine the homotopy type of it. But we can find them in the case of *CW* complexes with the form $S^2 \cup e^4 \cup e^6$.

Let X be a *CW* complex with the form $S^2 \cup e^4 \cup e^6$, and $x_j \in H^{2j}(X, Z)$ be the generator for $j=1$, 2 or 3 such that,

$$(x_1)^2 = m \cdot x_2 \quad \text{and} \quad x_1 \cdot x_2 = n \cdot x_3 . \qquad (m, n \geqq 0)$$

Then we have

THEOREM 4.5. (a) *If m is odd, then*

$$Sq^2 : H^4(X, Z_2) \longrightarrow H^6(X, Z_2)$$

is trivial and the homotopy type of X is uniquely determined by the pair of integers (m, n).

(b) *If m is even and*

$$Sq^2 : H^4(X, Z_2) \longrightarrow H^6(X, Z_2)$$

is trivial, then the homotopy type of X is uniquely determined by the pair of integers (m, n).

(c) *If m is even and*

$$Sq^2 : H^4(X, Z_2) \longrightarrow H^6(X, Z_2)$$

is non-trivial, then X has precisely two homotopy types which can be distingushed

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by some element of order two in $\pi_5(L_m)$.

In particular, in the case of manifolds we also have

COROLLARY 4.6. *Let M be a closed* 6-*dimensional smooth manifold with the form* $S^2\cup e^4\cup e^6$ *such that*

$$(x_1)^2=m\cdot x_2,$$

where $x_k\in H^{2k}(M, Z)$ *is a generator for* $k=1$, 2 *or* 3.

(a) *If m is odd, then*

$$Sq^2: H^4(M, Z_2)\longrightarrow H^6(M, Z_2)$$

is trivial and the homotopy type of M is uniquely determined by the integer m.

(b) *If m is even and*

$$Sq^2: H^4(M, Z_2)\longrightarrow H^6(M, Z_2)$$

is trivial, then the homotopy type of M is uniquely determined by the integer m.

(c) *If m is even and*

$$Sq^2: H^4(M, Z_2)\longrightarrow H^6(M, Z_2)$$

is non-trivial, then M has precisely two homotopy types which can be distinguished by the element of order two in $\pi_5(L_m)$.

The plan of this paper is as follows: In § 2, we calculate homotopy groups of a *CW* complex L with the form $S^2\cup e^4$. In § 3, at first, we calculate $\varepsilon(L)$ which is the group of self-homotopy equivalences over L. Secondly, we determine the actions of $\varepsilon(L)$ on $\pi_5(L)$. In § 4, we give the proof of the main results.

The problem of this paper was suggested by Professor S. Sasao and the author would like to take this opportunity to thank him for his many valuable suggestions and encouragement.

§ 2. Homotopy groups of L_m.

Let $\eta_2: S^3\to S^2$ be the Hopf map. It is well-known that the homotopy group $\pi_3(S^2)$ is isomorphic to $Z\{\eta_2\}$. For each integer m, let L_m denote the *CW* complex formed by attaching the 4-cell e^4 to S^2 with the map $m\eta_2: S^3\to S^2$, and the map $a_m: (E^4, S^3)\to(L_m, S^2)$ denote the characteristic map of 4-cell e^4 of L_m. For example, L_0 and L_1 are homotopy equivalent to the wedge of spheres $S^2\vee S^4$ and the 2-dimensional projective space CP^2, respectively. Let $SO(n)$ be the n-th rotation group, and $p: SO(3)\to SO(3)/SO(2)\cong S^2$ be the canonical fibration with its fibre S^1. Let X_m be the S^2 bundle over S^4 with the characteritic element $c_m\in\pi_3(SO(3))$ satisfying $p_*(c_m)=m\eta_2$, where p_* is the induced homomorphism $p_*: \pi_3(SO(3))\to\pi_3(S^2)=Z\{\eta_2\}$. It is easy to see that X_m is homotopy equivalent to the *CW* complex $L_m \cup_{b_m} e^6$ formed by attaching 6-cell e^6 to L_m with $b_m \in \pi_5(L_m)$, which is a generator of order infinity because X_m is a closed manifold. (See in detail [8]) Denote by ι_n the generator of $\pi_n(S^n)$ and by η_m the map $E^{m-2}\eta_2$ for integer $m \geqq 2$. It is also well-known that

$$\pi_4(S^2) = Z_2\{\eta_2^2\}$$

and

$$\pi_5(S^2) = Z_2\{\eta_2^3\},$$

where we denote by η_n^2 the composition map $\eta_n \circ \eta_{n+1}$ and by η_n^3 the composition map $\eta_n \circ \eta_{n+1} \circ \eta_{n+2}$.

LEMMA 2.1. (a) *$L_m = S^2 \cup_{m\eta_2} e^4$ is simply-connected,*
(b) *$\pi_2(L_m) = \pi_2(S^2) = Z\{\iota_2\}$, and*
(c) *$\pi_3(L_m) \cong Z/mZ = Z_m$.*

Proof. Statements (a) and (b) are clear. Consider the exact sequence

$$\pi_4(L_m, S^2) \xrightarrow{\partial_4} \pi_3(S^2) = Z\{\eta_2\} \longrightarrow \pi_3(L_m) \longrightarrow 0\,. \tag{2.2}$$

Since $\pi_4(L_m, S^2) = Z\{a_m\}$ and $\partial_4(a_m) = m\eta_2$, the statement (c) is also obtained.
Q. E. D.

LEMMA 2.3. (a) *If m is odd, then $\pi_4(L_m) = 0$.*
(b) *If m is even and $m \neq 0$, then*

$$\pi_4(L_m) = \pi_4(S^2) = Z_2\{\eta_2^2\},$$

and in particular,

(c) $$\pi_4(L_0) = \pi_4(S^2 \vee S^4) \cong Z\{\iota_4\} \oplus Z_2\{\eta_2^2\}.$$

Proof. Since $\pi_4(L_m, S^2) = Z\{a_m\}$, it is easy to see that $\pi_5(L_m, S^2) = a_{m*}\pi_5(E^4, S^3) \oplus Z\{[a_m, \iota_2]_r\}$, where

$$a_{m*}: \pi_5(E^4, S^3) \longrightarrow \pi_5(L_m, S^2)$$

is the homomorphism induced by a_m, and $[\ ,\ ]_r$ denotes a relative Whitehead product.

Now consider the exact sequence

$$\pi_5(L_m, S^2) \xrightarrow{\partial_5} \pi_4(S^2) = Z\{\eta_2^2\} \longrightarrow \pi_4(L_m) \longrightarrow \pi_4(L_m, S^2) \xrightarrow{\partial_4} \pi_3(S^2)\,. \tag{2.4}$$

Since $a_m | S^2 = m\eta_2$, we have $\partial_5 a_{m*}\pi_5(E^4, S^3) = Z_2\{m\eta_2^2\}$. On the other hand, taking account of $[\eta_2, \iota_2] = 0$, we obtain

$$\partial_5[a_m, \iota_2]_r = 0\,.$$

Hence we also have

(2.5) $$\mathrm{Im}\,[\partial_5 : \pi_5(L_m, S^2) \longrightarrow \pi_4(S^2)] = Z_2\{m\eta_2^2\} = \begin{cases} Z_2\{\eta_2^2\} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even}. \end{cases}$$

Proof of Lemma 2.1 also shows that ∂_4 is a monomorphism if $m \neq 0$. Then we have statements (a) and (b). The statement (c) is obvious. Q. E. D.

LEMMA 2.6. (a) *If m is odd, then*

$$\pi_5(L_m) \cong Z\{[a_m, \iota_2]_r\} \cong Z\{b_m\},$$

(b) *if m is even and $m \neq 0$, we have the exact sequence*

$$0 \longrightarrow \pi_5(S^2) \longrightarrow \pi_5(L_m) \longrightarrow Z\{[a_m, \iota_2]_r\} \oplus a_{m*}\pi_5(E^4, S^3) \longrightarrow 0,$$

and in particular, for $m=0$

(c) $$\begin{aligned} \pi_5(L_0) &= \pi_5(S^2 \vee S^4) \\ &= \pi_5(S^2) \oplus \pi_5(S^4) \oplus [\pi_2(S^2), \pi_4(S^4)] \\ &= Z_2\{\eta_2^3\} \oplus Z_2\{\eta_4\} \oplus Z\{[\iota_2, \iota_4]\}. \end{aligned}$$

Here we can identify $[a_m, \iota_2]_r = \pm b_m \in \pi_5(L_m)$.

Proof. The statement (c) is obvious. From (2.5) we have the exact sequence

(2.7) $$\pi_6(L_m, S^2) \xrightarrow{\partial_6} \pi_5(S^2) \longrightarrow \pi_5(L_m) \longrightarrow \pi_5(L_m, S^2) \xrightarrow{\partial_5} Z_2\{m\eta_2^2\} \longrightarrow 0.$$

Since $\pi_6(L_m, S^2) = a_{m*}\pi_6(E^4, S^3) \oplus [\pi_5(L_m, S^2), \pi_2(S^2)]_r$ and $[\eta_2 \circ \eta_3, \iota_2] = 0$, we have

$$\partial_6[\pi_5(L_m, S^2), \pi_2(S^2)]_r = 0.$$

It follows from $a_m | S^3 = m\eta_2$ and $\pi_5(S^3) = Z_2\{\eta_4^2\}$ that we obtain

(2.8) $$\begin{aligned} \mathrm{Im}\,[\partial_6 : \pi_6(L_m, S^2) \longrightarrow \pi_5(S^2)] &= Z_2\{m\eta_2^3\} \\ &= \begin{cases} Z_2\{\eta_2^3\} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

Therefore we have statements (a) and (b). The rest of the proof is easy. Q. E. D.

LEMMA 2.9. $$\pi_5(L_m) \cong Z\{b_m\} \oplus \pi_5(X_m).$$

Proof. Let $b'_m : (E^6, S^5) \to (X_m, L_m)$ be the characteristic map of 6-cell of X_m. Consider the exact sequence

$$\pi_6(X_m, L_m) \xrightarrow{\partial'_6} \pi_5(L_m) \longrightarrow \pi_5(X_m) \longrightarrow \pi_5(X_m, L_m) = 0.$$

Since $\partial'_6(b'_m)=b_m\neq 0$ and $\pi_6(X_m, L_m)=Z\{b'_m\}$, we have the exact sequence

$$0 \longrightarrow Z\{b_m\} \xrightarrow{i_*} \pi_5(L_m) \longrightarrow \pi_5(X_m) \longrightarrow 0 . \tag{2.10}$$

Here we recall that $b_m\in\pi_5(L_m)$ is a generator. Then is follows by using the functional cup-product that the exact sequence (2.10) is split. Q. E. D.

The preceding argument also shows

COROLLARY 2.11. (a) *If m is odd, then*

$$\pi_5(X_m)=0 ,$$

and

(b) *if m is even and $m\neq 0$, then the sequence*

$$0 \longrightarrow \pi_5(S^2) \longrightarrow \pi_5(X_m) \longrightarrow \pi_5(S^4) \longrightarrow 0$$

is exact.

§ 3. Actions of $\varepsilon(L_m)$.

We denote by $\varepsilon(X)$ the group of self-homotopy equivalences over X with multiplication induced from composition. If $i: S^2\to L_m$ is the inclusion map, the induced homomorphism

$$i_*: \pi_2(S^2) \xrightarrow{\cong} \pi_2(L_m)$$

is an isomorphism. Since $H(\eta_2)=\iota_3$ and $[\iota_2, \iota_2]=2\eta_2$, we have

$$\begin{aligned} i_*((-\iota_2)\circ\eta_2) &= -i_*(\eta_2)+i_*([\iota_2, \iota_2]\circ H(\eta_2)) \\ &= -i_*(\eta_2)+2i_*(\eta_2) \\ &= i_*(\eta_2) . \end{aligned}$$

Thus there is a map

$$f: L_m \longrightarrow L_m$$

such that f has a degree $(-1)^j$ on each cell e^{2j} of L_m for $j=1$ or 2, and we denote by (-1) one of such maps. Let $u: L_m\to L_m\vee S^4$ be the co-action map and $\nabla: L_m\vee L_m\to L_m$ be a folding map. For $h=id$ or (-1), we denote by $h\vee\eta_2\eta_3$ the composite

$$L_m \xrightarrow{u} L_m\vee S^4 \xrightarrow{h\vee\eta_2\eta_3} L_m\vee L_m \xrightarrow{\nabla} L_m .$$

LEMMA 3.1. (a) *If m is odd, then $\varepsilon(L_m)=\{id, (-1)\}$.*

(b) *If m is even and $m\neq 0$, then*

$$\varepsilon(L_m)=\{id, (-1), id\vee\eta_2\eta_3, (-1)\vee\eta_2\eta_3\} .$$

(c) *In particular, for $m=0$, we have the split extension*

$$0 \longrightarrow \pi_4(S^2\vee S^4) \longrightarrow \varepsilon(S^2\vee S^4)=\varepsilon(L_0) \longrightarrow Z_2\times Z_2 \longrightarrow 0\,,$$

where $Z_2\times Z_2$ operates on the homotopy group $\pi_4(S^2\vee S^4)$ by

$$(a,\, b)\circ c=a\circ c\circ b \qquad \text{for } (a,\, b)\in Z_2\times Z_2 \text{ and } c\in\pi_4(S^2\vee S^4)\,.$$

Proof. The statement (c) is clear. (See in detail [3]) Now suppose $m\neq 0$. It follows from (6.1) of [1] that we have the exact sequence

$$\text{(3.2)} \qquad \operatorname{Im}[i_*: \pi_4(S^2) \longrightarrow \pi_4(L_m)] \xrightarrow{d_*} \varepsilon(L_m) \xrightarrow{r} \varepsilon(S^2) \longrightarrow 0\,.$$

At first, suppose m is odd. It follows from Lemma 2.3 the statement (a) is clear. Therefore we may assume m is even and that $m\neq 0$. It follows from (3.2) and Lemma 2.3 we also have the exact sequence

$$\text{(3.3)} \qquad \pi_4(S^2)=Z_2\{\eta_2^2\} \xrightarrow{d_*} \varepsilon(L_m) \xrightarrow{r} \varepsilon(S^2) \longrightarrow 0\,.$$

Hence it suffices to prove $h\neq h\vee\eta_2\eta_3$ for $h=id$ or (-1). Now consider the isomorphism $\pi_5(E^4,\, S^3) \xrightarrow[\cong]{\partial} \pi_4(S^3)=Z_2\{\eta_3\}$. If $j: L_m\to(L_m,\, S^2)$ is the inclusion map, it follows from (2.6) that the induced homomorphism

$$j_*: \pi_5(L_m) \longrightarrow \pi_5(L_m,\, S^2)=Z\{b_m\}\oplus a_{m*}\pi_5(E^4,\, S^3)$$

is an epimorphism. Thus there is an element $\gamma_0\in\pi_5(L_m)$ such that $j_*(\gamma_0)=a_{m*}(\partial^{-1}\eta_3)$. Then we have

$$\text{(3.4)} \qquad \pi_5(L_m,\, S^2)=Z\{b_m\}\oplus Z_2\{j_*(\gamma_0)\}\,.$$

On the other hand,

$$\begin{aligned}\pi_5(L_m\vee S^4)&=\pi_5(L_m)\oplus\pi_5(S^4)\oplus[\pi_2(L_m),\, \pi_4(S^4)]\\ &=\pi_5(L_m)\oplus Z_2\{\eta_4\}\oplus Z\{[\iota_2,\, \iota_4]\}\,.\end{aligned}$$

Therefore by using $[\iota_2,\, \eta_2\eta_3]=0$, we have

$$\text{(3.5)} \qquad \begin{aligned}(h\vee\eta_2\eta_3)\circ\gamma_0&=h\circ\gamma_0+\eta_2\eta_3\eta_4+[h\,|\,S^2,\, \eta_2\eta_3]\\ &=h\circ\gamma_0+\eta_2\eta_3\eta_4\pm[\eta_2\eta_3,\, \iota_2]\\ &=h\circ\gamma_0+\eta_2\eta_3\eta_4\,.\end{aligned}$$

Hence we have $h\vee\eta_2\eta_3\neq h$ for $h=id$ or (-1). Q.E.D.

Remark 3.6. Suppose m is even and $m\neq 0$. Since $[\iota_2,\, \iota_2]=2\eta_2$, we have

(3.7) $$m\eta_2+(-\iota_2)\circ m\eta_2=[\iota_2, \iota_2]\circ H_0(m\eta_2) = 2m\eta_2 .$$

Furthermore, it follows from $[\eta_2, \iota_2]=0$ that we also have

(3.8) $$\eta_2\circ E(m\eta_2)+[\iota_2, \eta_2]\circ EH_0(m\eta_2)=0 .$$

Hence taking account of Theorem 3.15 in [3], we have the exact sequence

(3.9) $$1 \longrightarrow \pi_4(S^2) \longrightarrow \varepsilon(L_m) \longrightarrow Z_2 \longrightarrow 1 .$$

Remark 3.10. Since X_m is the total space of S^2-bundle over S^4 with its characteristic element $c_m\in\pi_3(SO(3))$, we may also regard X_m as the space

(3.11) $$S^2\times E^4\cup S^2\times E^4/\sim , \qquad \text{where } (x, y)\sim(c_m(y)x, y) \quad \text{for } (x, y)\in S^2\times S^3 .$$

Then we define a map $f_m: X_m\to X_m$ by

(3.12) $$f_m(x, y)=(-x, y) \qquad \text{for } (x, y)\in S^2\times E^4 .$$

Then the map f_m has a degree $(-1)^j$ on each cell e^{2j} of X_m for $j=1$, 2 or 3. Hence without loss of generalities, we may set $(-1)=f_m|L_m$. Therefore $(-1)\circ(-1)=id$.

LEMMA 3.13. (a) $(-1)\circ b_m=-b_m$.
(b) *If m is even and $b\in\pi_5(L_m)$, then*

$$(h\vee\eta_2\eta_3)b=\begin{cases} h\circ b & \text{if } j_*(b)\in Z\{b_m\} , \\ h\circ b+\eta_2\eta_3\eta_4 & \text{if } j_*(b)\notin Z\{b_m\} , \end{cases}$$

where $j_*: \pi_5(L_m)\to\pi_5(L_m, S^2)=Z\{b_m\}\oplus Z_2\{j_*(\gamma_0)\}$.

Proof. It follows from Remark 3.10 the statement (a) is clear. The preceding proof of Lemma 3.1 also shows the assertion (b). Q. E. D.

§ 4. Proof of the main results.

Throughout this section we assume X is a *CW* complex with the form $S^2\cup e^4\cup e^6$ such that,

(4.1) $$(x_1)^2=m\cdot x_2 \quad \text{and} \quad x_1\cdot x_2=n\cdot x_3 , \qquad (m, n\geqq 0)$$

where $x_j\in H^{2j}(X, Z)$ is a generator for $j=1$, 2 or 3. Furthermore, taking account of the Hopf invariant, we may also suppose that the attaching map of 4-cell e^4 of X is $m\eta_2$. Hence we have

(4.2) $$X=L_m\cup_b e^6 \quad \text{for some } b\in\pi_5(L_m),$$

up to homotopy.

At first we recall

LEMMA 4.3. *Let $j: L_m\to(L_m, S^2)$ be the inclusion map. Then the following two conditions are equivalent*:

(a) $j_*(b)=nb_m+a$ *for some* $a\in Z_2$.

(b) $(x_1)^2=mx_2$, $x_1\cdot x_2=n\cdot x_3$ *and the second Steenrod square*

$$Sq^2: H^4(X, Z_2)\longrightarrow H^6(X, Z_2)$$

satisfies $Sq^2(x_2)=a\cdot x_3$.

Proof. See (2) in detail. Q. E. D.

Remark 4.4. Taking account of Lemma 2.6, it is easy to see that

$$Sq^2: H^4(X, Z_2)\longrightarrow H^6(X, Z_2)$$

is trivial if m is odd.

Then we have

THEOREM 4.5. (a) *If m is odd, then*

$$Sq^2: H^4(X, Z_2)\longrightarrow H^6(X, Z_2)$$

is trivial and the homotopy type of X is uniquely determined by the pair of integers (m, n).

(b) *If m is even and*

$$Sq^2: H^4(X, Z_2)\longrightarrow H^6(X, Z_2)$$

is trivial, then the homotopy type of X is uniquely determined by the pair of integers (m, n).

(c) *If m is even and*

$$Sq^2: H^4(X, Z_2)\longrightarrow H^6(X, Z_2)$$

is non-trivial, then X has precisely two homotopy types which can be distinguished by some element of order two in $\pi_5(L_m)$.

In particular, in the case of manifolds, we also have

COROLLARY 4.6. *Let M be a closed 6-dimensional smooth manifold with the form $S^2\cup e^4\cup e^6$ such that*

$$(x_1)^2=mx_2,$$

where $x_k\in H^{2k}(M, Z)$ is a generator for $k=1$, 2 or 3.

(a) *If m is odd, then*

$$Sq^2 : H^4(M, Z_2) \longrightarrow H^6(M, Z_2)$$

is trivial and the homotopy type of M is uniquely determined by the integer m.

(b) *If m is even and*

$$Sq^2 : H^4(M, Z_2) \longrightarrow H^6(M, Z_2)$$

is trivial, then the homotopy type of M is uniquely determined by the integer m.

(c) *If m is even and*

$$Sq^2 : H^4(M, Z_2) \longrightarrow H^6(M, Z_2)$$

is non-trivial, then M has precisely two homotopy types which can be distinguished by element of order two in $\pi_5(L_m)$.

Proof of Theorem 4.5. Without loss of generalities, we may assume $X = L_m \cup_b e^6$ for some $b \in \pi_5(L_m)$. At first, consider the case that $Sq^2 : H^4(M, Z_2) \to H^6(M, Z_2)$ is trivial. It follows from (3.1), (3.13) and (4.1) that we have $b = n \cdot b_m$. Therefore taking account of (4.4), the assertion (a) and (b) can be obtained. Secondly consider the case (c). Let X' be a CW complex with the form $L_m \cup_{b'} e^6$ satisfying the same assumptions as X. If follows from (4.3) that we have

$$j_*(b) = j_*(b),$$

where $j_* : \pi_5(L_m) \to \pi_5(L_m, S^2) = Z\{b_m\} \oplus Z_2\{\eta_2\eta_3\eta_4\}$. Thus it follows from (2.6) that we have

$$b = b' \quad \text{or} \quad b = b' + \eta_2\eta_3\eta_4 .$$

Hence taking account of (3.1) and (3.13), the assertion (c) is also obtained.

Q. E. D.

Remark 4.7. It is well-known that for each pair of integers (m, n), there is a simply connected CW complex X with the form $S^2 \cup e^4 \cup e^6$ such that,

$$(x_1)^2 = m \cdot x_2 \quad \text{and} \quad x_1 \cdot x_2 = n \cdot x_3$$

for each generator $x_j \in H^{2j}(X, Z)$. (See [6] in detail.)

Remark 4.8. Let M be a closed six dimensional smooth manifold with the CW decomposition $S^2 \cup e^4 \cup e^6$. Then M has the same homotopy type as a S^2 bundle over S^4 if and only if m is odd, or m is even and one of the following conditions is satisfied:

(a) $Sq^2 : H^4(M, Z_2) \longrightarrow H^6(M, Z_2)$ is trivial,

or

(b) $P_1(M) + 4m \equiv 0 \pmod{48}$, where we denote by $P_1(M)$ the first Pontrjagin class of M. (See (4) in detail.)

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Tokyo Institute of Technology
Oh-Okayama, Meguro, Tokyo
Japan