H. UEDA
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AN ENTIRE FUNCTION RELATED TO THEOREMS OF BARRY

BY HIDEHARU UEDA

0. Introduction.

Let $f(z)$ be an entire function of order ρ and lower order μ, where $0 \leqq \mu \leqq \rho \leqq 1$. The classical $\cos \pi\rho$ theorem of Wiman and Valiron states that, given $\varepsilon > 0$, the inequality

$$\log m^*(r, f) > (\cos \pi\rho - \varepsilon) \log M(r, f) \tag{1}$$

holds for a sequence $r = r_n \to \infty$, where

$$m^*(r, f) = \min_{|z|=r} |f(z)|, \qquad M(r, f) = \max_{|z|=r} |f(z)|.$$

This was sharpened by Kjellberg [5], who showed that (1) holds with ρ replaced by μ (<1), independently of the value of ρ.

Much work including the above has been performed related to the $\cos \pi\rho$ theorem. The starting point of the considerations presented here is the following results due to Barry.

THEOREM A. ([1]) *If $\rho < \alpha < 1$, and if*

$$E = \{r ; \log m^*(r, f) > \cos \pi\alpha \log M(r, f)\}, \tag{2}$$

then

$$\underline{\log \operatorname{dens}}\, E \geqq 1 - \rho/\alpha. \tag{3}$$

THEOREM B. ([2]) *If $\mu < \alpha < 1$, and if E is defined by* (2), *then*

$$\overline{\log \operatorname{dens}}\, E \geqq 1 - \mu\backslash\alpha. \tag{4}$$

The estimates (3) and (4) are both sharp in the sense that the sign $\geqq$ cannot be replaced by $>$. In fact, the following theorem was proved by Hayman.

THEOREM C. ([4, Theorem 1.]) *Given any numbers ρ, α, such that $0 < \rho < \alpha < 1$, there exists an entire function $f(z)$ of order ρ and regular growth such that*

$$\underline{\log \operatorname{dens}}\, E = \overline{\log \operatorname{dens}}\, E = 1 - \rho/\alpha,$$

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where E is the set defined by (2).

The function $f(z)$ in Theorem C satisfies both (3) and (4) with the sign of equality. Motivated by this fact, the following problem is naturally raised.

Problem. Let μ, ρ, α be any numbers such that $0 \leqq \mu \leqq \rho < \alpha < 1$. Then is it possible to construct an entire function $f(z)$ of order ρ and lower order μ such that

$$1-\rho/\alpha = \underline{\log \text{dens}}\, E \leqq \overline{\log \text{dens}}\, E = 1-\mu/\alpha\,,$$

where E is the set defined by (2)?

Observe first that for entire functions $f(z)$ of order 0, the Barry's estimates (3) and (4) imply $\log \text{dens}\, E=1$, so that our problem is solved affirmatively for $\mu=\rho=0$. And since, for $0<\mu=\rho<\alpha<1$, Hayman has given examples satisfying the conclusion of our problem, we may consider the case $0 \leqq \mu < \rho < \alpha < 1$.

In this paper we prove the following

THEOREM. *Given any numbers μ, ρ, α, such that $0 \leqq \mu < \rho < \alpha < 1$, there exists an entire function $f(z)$ of order ρ and lower order μ and such that*

$$1-\rho/\alpha = \underline{\log \text{dens}}\, E < \overline{\log \text{dens}}\, E = 1-\mu/\alpha\,,$$

where E is defined by (2).

All the above results combine to show that our problem is solved affirmatively in all cases.

In §§ 1-4, we suppose $\mu>0$; a special argument when $\mu=0$ is in § 5.

1. Construction of a continuous increasing function $\nu(t)$.

Let l be the positive number satisfying

$$\rho = \frac{\mu\alpha(l+1)}{\alpha+\mu l}\,. \tag{1.1}$$

Define a sequence $\{r_m\}_0^\infty$ by

$$r_0=1\,, \quad r_m = 3^{(l+1)^{m-1}} \qquad (m \geqq 1)\,. \tag{1.2}$$

Further let $\{\alpha_m\}_0^\infty$ be a decreasing sequence tending to α such that $\alpha_0<1$, and let $\{r'_m\}_0^\infty$ be an increasing sequence defined by

$$\left(\frac{r'_m}{r_m}\right)^{\alpha_m} = \left(\frac{r_{m+1}}{r_m}\right)^{\mu}\,. \tag{1.3}$$

Then, since $\mu<\alpha\leqq\alpha_m$, we deduce from (1.2) and (1.3) that

$$r_m < r'_m < r_{m+1} \qquad (m=0,\ 1,\ 2,\ \cdots)\,.$$

Now, we define a nonnegative function $\lambda(t)$ $(t \geqq 1)$ as follows:

$$\lambda(t)=\begin{cases}\alpha_m & (r_m\leqq t\leqq r'_m\,;\ m=0,\ 1,\ 2,\ \cdots)\\ 0 & (r'_m<t<r_{m+1}\,;\ m=0,\ 1,\ 2,\ \cdots)\,.\end{cases}\tag{1.4}$$

Then corresponding to $\lambda(t)$, we take a continuous increasing function $\nu(r)$ $(r\geqq 1)$ with

$$\nu(r)=\exp\left(\int_1^r \lambda(t)t^{-1}dt\right).\tag{1.5}$$

Here we show the following

LEMMA 1. *The order and lower order of $\nu(r)$ are equal to ρ and μ, respectively.*

Proof. Consider the interval $r_m\leqq r<r_{m+1}$ $(m=0,\ 1,\ 2,\ \cdots)$. By (1.5)

$$\log\nu(r)=\int_1^r \lambda(t)t^{-1}dt\,.$$

Hence, if $r_m\leqq r\leqq r'_m$, we deduce from (1.4), (1.3) and (1.2) that

$$\begin{aligned}\log\nu(r)&=\sum_{s=0}^{m-1}\int_{r_s}^{r'_s}\frac{\alpha_s}{t}dt+\int_{r_m}^{r}\frac{\alpha_m}{t}dt\\ &=\sum_{s=0}^{m-1}\alpha_s\log\left(\frac{r'_s}{r_s}\right)+\alpha_m\log\left(\frac{r}{r_m}\right)\\ &=\sum_{s=0}^{m-1}\mu\log\left(\frac{r_{s+1}}{r_s}\right)+\alpha_m\log\left(\frac{r}{r_m}\right)\\ &=\mu\log r_m+\alpha_m\log\left(\frac{r}{r_m}\right).\end{aligned}$$

Similarly, if $r'_m\leqq r<r_{m+1}$, we deduce that

$$\begin{aligned}\log\nu(r)&=\sum_{s=0}^{m}\int_{r_s}^{r'_s}\frac{\alpha_s}{t}dt\\ &=\mu\log r_{m+1}\,.\end{aligned}$$

Thus

$$\frac{\log\nu(r)}{\log r}=\begin{cases}\alpha_m-(\alpha_m-\mu)\dfrac{\log r_m}{\log r} & (r_m\leqq r\leqq r'_m\,;\ m=0,\ 1,\ 2,\ \cdots)\\ \mu\dfrac{\log r_{m+1}}{\log r} & (r'_m\leqq r<r_{m+1}\,;\ m=0,\ 1,\ 2,\ \cdots)\,.\end{cases}\tag{1.6}$$

From this, we see that

$$\varliminf_{r\to\infty}\frac{\log\nu(r)}{\log r}=\varliminf_{m\to\infty}\frac{\log\nu(r_m)}{\log r_m}=\mu\,,$$

and

$$\overline{\lim_{r\to\infty}}\frac{\log\nu(r)}{\log r}=\overline{\lim_{m\to\infty}}\frac{\log\nu(r'_m)}{\log r'_m}=\overline{\lim_{m\to\infty}}\,\mu\frac{\log r_{m+1}}{\log r'_m}. \tag{1.7}$$

It remains to compute $\log r_{m+1}/\log r'_m$. We have

$$\frac{\log r_{m+1}}{\log r'_m}=\frac{\log r_{m+1}}{\log r_m+\log(r'_m/r_m)}.$$

Using (1.2) and (1.3), we obtain

$$\begin{aligned}\frac{\log r_{m+1}}{\log r'_m}&=\frac{(l+1)^m\log 3}{(l+1)^{m-1}\log 3+\frac{\mu}{\alpha_m}l(l+1)^{m-1}\log 3}\\ &=\frac{l+1}{1+\frac{\mu}{\alpha_m}l} \qquad (1.8)\\ &=\frac{\alpha_m(l+1)}{\alpha_m+\mu l}.\end{aligned}$$

Since $\alpha_m\to\alpha$ as $m\to\infty$, it follows from (1.8), (1.7) and (1.1) that

$$\overline{\lim_{r\to\infty}}\frac{\log\nu(r)}{\log r}=\frac{\mu\alpha(l+1)}{\alpha+\mu l}=\rho.$$

This proves Lemma 1.

2. A set F on the positive real axis.

Set

$$K'_m=\frac{r_{m+1}}{r_m}=3^{l(l+1)^{m-1}} \qquad (m\geqq 1), \tag{2.1}$$

and define

$$K_m=(\log K'_m)^{2/\mu}. \tag{2.2}$$

In view of (1.3), (2.1) and (2.2), we have $r'_m/K_m>K_mr_m$ $(m\geqq m_0)$

Now let

$$F=\bigcup_{m=m_0}^{\infty}[K_mr_m,\, r'_m/K_m]. \tag{2.3}$$

Then we have the following

LEMMA 2. $\overline{\log\mathrm{dens}}\,F\geqq\rho/\alpha$, $\underline{\log\mathrm{dens}}\,F\geqq\mu/\alpha$.

Proof. Let R be a large positive number and let m_1 be the integer such that $r'_{m_1}/K_{m_1}\leqq R<r'_{m_1+1}/K_{m_1+1}$. Suppose first that $r'_{m_1}/K_{m_1}\leqq R<K_{m_1+1}r_{m_1+1}$ and $m_1\geqq m_0$. Then we have from (2.3), (1.3) and (2.1) that

$$\int_{F\cap[1,R]}\frac{dt}{t}=\sum_{m=m_0}^{m_1}\int_{K_m r_m}^{r'_m/K_m}\frac{dt}{t}=\sum_{m=m_0}^{m_1}\left\{\log\left(\frac{r'_m}{r_m}\right)-2\log K_m\right\}$$

$$=\sum_{m=m_0}^{m_1}\left\{\frac{\mu}{\alpha_m}\log K'_m-2\log K_m\right\}.$$

In view of (2.2)

(2.4) $$\log K_m=o(\log K'_m)\qquad (m\to\infty).$$

Also $\alpha_m\to\alpha$ as $m\to\infty$. Hence given $\varepsilon>0$, we can choose $N=N(\varepsilon)$, so that for $m_1\geqq N$

(2.5) $$\int_{F\cap[1,R]}\frac{dt}{t}>\frac{\mu}{\alpha}(1-\varepsilon)\sum_{m=N}^{m_1}\log K'_m=\frac{\mu}{\alpha}(1-\varepsilon)\log\frac{r_{m_1+1}}{r_N}.$$

Since $R<K_{m_1+1}r_{m_1+1}$, it follows from (1.2), (2.1) and (2.4) that

$$\frac{\log r_{m_1+1}}{\log R}>\frac{\log r_{m_1+1}}{\log(K_{m_1+1}r_{m_1+1})}=\frac{\log r_{m_1+1}}{\log K_{m_1+1}+\log r_{m_1+1}}$$

$$=\frac{\log r_{m_1+1}}{(1+o(1))\log r_{m_1+1}}\qquad (m_1\to\infty)$$

$$>1-\varepsilon\qquad (m_1\geqq N_1(\varepsilon)).$$

Thus for all sufficiently large $R\in\bigcup_{m=0}^{\infty}[r'_m/K_m,\ K_{m+1}r_{m+1}]$

(2.6) $$\frac{1}{\log R}\int_{F\cap[1,R]}\frac{dt}{t}>\frac{\mu}{\alpha}(1-\varepsilon)^3.$$

Next suppose that $K_{m_1+1}r_{m_1+1}\leqq R<r'_{m_1+1}/K_{m_1+1}$ and $m_1\geqq m_0$. In this case, we have from (2.3), (1.3), (2.1) and (2.5) that

$$\int_{F\cap[1,R]}\frac{dt}{t}=\sum_{m=m_0}^{m_1}\left\{\frac{\mu}{\alpha_m}\log K'_m-2\log K_m\right\}+\log\frac{R}{K_{m_1+1}r_{m_1+1}}$$

$$>\frac{\mu}{\alpha}(1-\varepsilon)\log\frac{r_{m_1+1}}{r_N}+\log\frac{R}{K_{m_1+1}r_{m_1+1}}\qquad (m_1\geqq N).$$

Since $K_{m_1+1}r_{m_1+1}\leqq R$, it follows from (1.2), (2.1) and (2.4) that

$$\frac{\log R-\log K_{m_1+1}-(1-(\mu/\alpha)(1-\varepsilon))\log r_{m_1+1}-(\mu/\alpha)(1-\varepsilon)\log r_N}{\log R}$$

$$>1-o(1)-(1-(\mu/\alpha)(1-\varepsilon))\frac{\log r_{m_1+1}}{\log R}\qquad (m_1\to\infty)$$

$$\geqq 1-o(1)-(1-(\mu/\alpha)(1-\varepsilon))\frac{\log r_{m_1+1}}{\log K_{m_1+1}r_{m_1+1}}$$

$$>1-\varepsilon-(1-(\mu/\alpha)(1-\varepsilon))(1-\varepsilon)\qquad (m_1\geqq N_2(\varepsilon))$$

$$=(\mu/\alpha)(1-\varepsilon)^2 .$$

Thus for all sufficiently large $R\in\bigcup_{m=0}^{\infty}[K_{m+1}r_{m+1},\ r'_{m+1}/K_{m+1}]$

$$\frac{1}{\log R}\int_{F\cap[1,R]}\frac{dt}{t}>\frac{\mu}{\alpha}(1-\varepsilon)^2 . \tag{2.7}$$

Together, (2.6) and (2.7) give

$$\frac{1}{\log R}\int_{F\cap[1,R]}\frac{dt}{t}>\frac{\mu}{\alpha}(1-\varepsilon)^3$$

for all sufficiently large R, i.e.

$$\underline{\log\text{dens}}\,F\geqq\frac{\mu}{\alpha}(1-\varepsilon)^3 .$$

Since ε is an arbitrary positive number independent of F, we have

$$\underline{\log\text{dens}}\,F\geqq\frac{\mu}{\alpha} .$$

In order to show that $\overline{\log\text{dens}}\,F\geqq\rho/\alpha$, we put $R=r'_{m_1}/K_{m_1}\equiv R_{m_1}$ in (2.5). Then from (2.1), (1.3), (2.4) and (1.1) it follows that

$$\frac{1}{\log R_{m_1}}\int_{F\cap[1,R_{m_1}]}\frac{dt}{t}>\frac{\mu}{\alpha}(1-\varepsilon)\frac{\log r_{m_1+1}-\log r_N}{\log r'_{m_1}-\log K_{m_1}}$$

$$=\frac{\mu}{\alpha}(1-\varepsilon)\frac{\log r_{m_1}+\log K'_{m_1}-\log r_N}{\log r_{m_1}+\dfrac{\mu}{\alpha_{m_1}}\log K'_{m_1}-\log K_{m_1}}$$

$$=\frac{\mu}{\alpha}(1-\varepsilon)\frac{(l+1)^{m_1-1}\log 3+l(l+1)^{m_1-1}\log 3-O(1)}{(l+1)^{m_1-1}\log 3+\left(\dfrac{\mu}{\alpha}-o(1)\right)l(l+1)^{m_1-1}\log 3}\qquad(m_1\to\infty)$$

$$>\frac{\mu}{\alpha}(1-\varepsilon)^2\frac{1+l}{1+\dfrac{\mu}{\alpha}l}\qquad(m_1>N_3(\varepsilon))$$

$$=\frac{\mu(1+l)}{\alpha+\mu l}(1-\varepsilon)^2=\frac{\rho}{\alpha}(1-\varepsilon)^2 .$$

Thus

$$\overline{\log\text{dens}}\,F\geqq\varlimsup_{m_1\to\infty}\frac{1}{\log R_{m_1}}\int_{F\cap[1,R_{m_1}]}\frac{\alpha t}{t}\geqq\frac{\rho}{\alpha}(1-\varepsilon)^2 .$$

Agoin, since ε is an arbitrary positive number independent of F, we obtain

$$\overline{\log\text{dens}}\,F\geqq\frac{\rho}{\alpha} .$$

This completes the proof of Lemma 2.

3. An entire function $f(z)$ of genus zero associated with $[\nu(t)]$.

Let $f(z)$ be a canonical product all of whose zeros $\{a_n\}_1^\infty$ are real and negative and such that

$$n(t) \equiv n(t, 0) = \begin{cases} 0 & (t<1) \\ [\nu(t)] & (t \geqq 1)\,. \end{cases} \tag{3.1}$$

It follows from Lemma 1.4 in [3], Lemma 1 and (3.1) that

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|} = \int_1^\infty \frac{n(t)}{t^2}\,dt < \int_1^\infty \frac{\nu(t)}{t^2}\,dt < \infty\,.$$

This implies that $f(z)$ has genus zero, and so for $|\arg z| < \pi$ we have [3, p 21]

$$\log f(z) = z\int_1^\infty \frac{n(t)}{t(t+z)}\,dt\,. \tag{3.2}$$

First we prove the following

LEMMA 3. *$f(z)$ has order ρ and lower order μ.*

Proof. We denote the order and lower order of $f(z)$ by ρ_f and μ_f, respectively. Take $\varepsilon>0$ small so that $0<\mu-\varepsilon<\rho+\varepsilon<1$. By Lemma 1

$$t^{\mu-\varepsilon} \leqq n(t) \leqq t^{\rho+\varepsilon} \qquad (t \geqq t_0 \equiv t_0(\varepsilon) \geqq 1)\,. \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$\begin{aligned} \log M(r, f) &= r\int_1^\infty \frac{n(t)}{t(t+r)}\,dt \\ &\leqq r\int_1^{t_0} \frac{n(t_0)}{t(t+r)}\,dt + r\int_0^\infty \frac{t^{\rho+\varepsilon}}{t(t+r)}\,dt \\ &= n(t_0)\log\frac{t_0(1+r)}{t_0+r} + r^{\rho+\varepsilon}\int_0^\infty \frac{u^{\rho+\varepsilon-1}}{u+1}\,du \\ &= O(r^{\rho+\varepsilon}) \qquad (r\to\infty)\,. \end{aligned}$$

Similarly

$$\begin{aligned} \log M(r, f) &\geqq r\int_{t_0}^\infty \frac{t^{\mu-\varepsilon}}{t(t+r)}\,dt \\ &= r^{\mu-\varepsilon}\int_{t_0/r}^\infty \frac{u^{\mu-\varepsilon-1}}{u+1}\,du \\ &\geqq r^{\mu-\varepsilon}\int_1^\infty \frac{u^{\mu-\varepsilon-1}}{2u}\,du \qquad (r \geqq t_0) \\ &= \frac{1}{2}\,\frac{r^{\mu-\varepsilon}}{1+\varepsilon-\mu} = O(r^{\mu-\varepsilon}) \qquad (r\to\infty)\,. \end{aligned}$$

Since we can choose $\varepsilon(>0)$ arbitrarily small, we deduce that

$$\mu \leqq \mu_f \leqq \rho_f \leqq \rho .$$

Next, we proceed to show that $\rho_f \geqq \rho$. For this purpose, note that $N(t, 0)$ has the same order as $n(t)$, and so by Lemma 1 it has order ρ. Further the first fundamental theorem gives $T(t, f) \geqq N(t, 0)$. Thus we have $\rho_f \geqq \rho$.

It remains to prove that $\mu_f \leqq \mu$. The proof is a little more complicated. Set

$$R = r_m / K_{m-1} \equiv R_m ,$$

and write $\log M(R, f)$ as follows:

$$\log M(R, f) = R\Big(\int_1^{r'_{m-1}} + \int_{r'_{m-1}}^{r_m} + \int_{r_m}^{\infty}\Big)\frac{n(t)}{t(t+R)}\,dt \tag{3.4}$$
$$\equiv I_1 + I_2 + I_3 .$$

Using (1.3) and (2.4), we have $R_m \geqq r'_{m-1}$ $(m \geqq m_2)$. It is clear that

$$I_3 \leqq R\int_{r_m}^{\infty} \frac{\nu(t)}{t^2}\,dt .$$

It is a consequence of (1.6) that $\nu(t)/t^{\alpha_0}$ decreases for all $t \geqq 1$. Thus

$$I_3 \leqq R\int_{r_m}^{\infty} \frac{\nu(R)}{R^{\alpha_0}} t^{\alpha_0} \frac{dt}{t^2} = R^{1-\alpha_0}\nu(R)\frac{1}{1-\alpha_0}\Big(\frac{1}{r_m}\Big)^{1-\alpha_0} \tag{3.5}$$
$$= \frac{1}{1-\alpha_0}\Big(\frac{1}{K_{m-1}}\Big)^{1-\alpha_0} r_m^{\mu} \qquad (m \geqq m_2) .$$

Now, by (1.6) and (3.1), $n(t) = [r_m^{\mu}]$ for $r'_{m-1} \leqq t \leqq r_m$. Thus, from (2.1), (1.3) and (2.4) it follows that

$$I_2 = [r_m^{\mu}] \log \frac{r_m}{r'_{m-1}} \frac{r'_{m-1}+R}{r_m+R}$$
$$= [r_m^{\mu}] \log \frac{1+\dfrac{1}{K_{m-1}}\dfrac{r_m}{r'_{m-1}}}{1+\dfrac{1}{K_{m-1}}} \tag{3.6}$$
$$= [r_m^{\mu}] \log \frac{1+\dfrac{1}{K_{m-1}}(K'_{m-1})^{1-(\mu/\alpha_{m-1})}}{1+\dfrac{1}{K_{m-1}}}$$
$$\sim \Big(1-\frac{\mu}{\alpha}\Big)(\log K'_{m-1}) r_m^{\mu} \qquad (m \to \infty) .$$

Finally, by (1.6), (2.1) and (1.3)

$$I_1 \leqq \int_1^{r'_{m-1}} \frac{\nu(t)}{t} dt \leqq \sum_{s=1}^{m-1} \nu(r_s) \log\left(\frac{r_s}{r_{s-1}}\right) + \int_{r_{m-1}}^{r'_{m-1}} \frac{r_{m-1}^{\mu}(t/r_{m-1})^{\alpha_{m-1}}}{t} dt$$

$$< \frac{r_{m-1}^{\mu}}{\alpha_{m-1}}\left\{\left(\frac{r'_{m-1}}{r_{m-1}}\right)^{\alpha_{m-1}} - 1\right\} + (\log K'_{m-2}) \sum_{s=1}^{m-1} r_s^{\mu}$$

$$< \frac{r_m^{\mu}}{\alpha_{m-1}} + (\log K'_{m-2}) r_m^{\mu} \sum_{s=1}^{m-1}\left(\frac{r_s}{r_m}\right)^{\mu}.$$

Also in view of (2.4)

$$\frac{r_j}{r_{j+1}} = \frac{1}{K'_j} = \frac{1}{3^{l(l+1)^{j-1}}} < \frac{1}{3} \qquad (j > j_0(l)).$$

Further, it is easy to see that

$$\sum_{s=1}^{\infty}\left(\frac{1}{3}\right)^{\mu s} = \frac{1}{3^{\mu}-1} < \frac{1}{\mu}.$$

Thus

$$I_1 < \frac{r_m^{\mu}}{\alpha_{m-1}} + (\log K'_{m-2})\left\{\sum_{s>j_0}^{m-1}\left(\frac{1}{3}\right)^{\mu(m-s)} + j_0\right\} r_m^{\mu}$$

(3.7)

$$< \frac{r_m^{\mu}}{\alpha_{m-1}} + \left(\frac{1}{\mu} + j_0\right)(\log K'_{m-2}) r_m^{\mu}.$$

Substituting (3.5)-(3.7) into (3.4), we obtain

$$\log M(R, f) < \{o(1) + O(\log K'_{m-1}) + O(\log K'_{m-2})\} r_m^{\mu}$$

$$= r_m^{\mu} O(\log K'_{m-1})$$

$$= r_m^{\mu} O(\log r_{m-1}) = r_m^{\mu} O(\log r_m) \qquad (m \to \infty).$$

Therefore by (1.2), (2.1) and (2.4)

$$\frac{\log\log M(R, f)}{\log R} < \frac{(1+o(1))\mu \log r_m}{\log r_m - \log K_{m-1}} = (1+o(1))\mu \qquad (m \to \infty).$$

This shows that $\mu_f \leqq \mu$. This completes the proof of Lemma 3.

Now, we choose $\{\alpha_m\}_0^{\infty}$ as follows:

(3.8)
$$\alpha_m = \alpha + \frac{1-\alpha}{2} \frac{1}{(l+1)^{\sqrt{m+1}}}.$$

Here we show the following

LEMMA 4. *For all sufficiently large* $r \in F$

$$\log m^*(r, f) < \cos \pi\alpha \log M(r, f).$$

Proof. We make use of Lemma 3 in [4]. Because of (1.2), (3.8), (1.3), (1.4), (1.5) and (2.2), this lemma is applicable to our $f(z)$. Hence we deduce that for

$r\in[K_m r_m, r'_m/K_m]$

$$\log M(r, f)\geqq\nu(r)\left\{\frac{\pi}{\sin \pi\alpha_m}+O\left(\frac{1}{K_m^\delta}\right)\right\}+O(\log r),$$

$$\log m^*(r, f)\leqq\nu(r)\left\{\pi \cot \pi\alpha_m+O\left(\frac{1}{K_m^\delta}\right)\right\}+O(\log r),$$

where $\delta\in(0, 1]$ is a constant depending only on α_0 and μ.

Therefore by (2.1), (2.2), Lemma 1 and (3.8)

$$\begin{aligned}\frac{\log m^*(r, f)}{\log M(r, f)}&\leqq\cos \pi\alpha_m+O\left(\frac{1}{K_m^\delta}\right)+O\left(\frac{\log r}{\nu(r)}\right)\\&=\cos \pi\alpha_m+O\left\{\frac{1}{(l+1)^{(m-1)2\delta/\mu}}\right\}\\&=\cos \pi\alpha_m+o(\alpha_m-\alpha) \qquad (m\to\infty)\\&<\cos \pi\alpha \qquad (m\geqq m_3).\end{aligned}$$

This gives the desired result.

4. Proof of Theorem; the case $\mu>0$.

Define E by (2). Then by Lemma 4 $E\cap F\cap(R, \infty)=\phi$ for all sufficiently large R. Hence

$$\overline{\log \text{dens}}\,(E+F)\leqq 1,$$

so that

(4.1) $\qquad \underline{\log \text{dens}}\, E+\overline{\log \text{dens}}\, F\leqq 1, \qquad \overline{\log \text{dens}}\, E+\underline{\log \text{dens}}\, F\leqq 1.$

It follows from (4.1) and Lemma 2 that

(4.2) $\qquad \underline{\log \text{dens}}\, E\leqq 1-\rho/\alpha, \qquad \overline{\log \text{dens}}\, E\leqq 1-\mu/\alpha.$

Now, we use Lemma 3 and Theorem A or Theorem B to obtain

(4.3) $\qquad \underline{\log \text{dens}}\, E\geqq 1-\rho/\alpha, \qquad \overline{\log \text{dens}}\, E\geqq 1-\mu/\alpha.$

Combining (4.2) and (4.3), we have

$$1-\rho/\alpha=\underline{\log \text{dens}}\, E<\overline{\log \text{dens}}\, E=1-\mu/\alpha.$$

This is the desired result.

5. The case $\mu=0$.

For given ρ and α, we put

$$L=\alpha\rho/(\alpha-\rho),$$

and define three sequences $\{r_m\}_1^\infty$, $\{\alpha_m\}_1^\infty$, $\{\mu_m\}_1^\infty$ by

$$r_m=3^{m!},\quad \alpha_m=\alpha+\frac{1-\alpha}{2}\frac{1}{m+1},\quad \mu_m=L/m.$$

Further let $\{r'_m\}_1^\infty$ be a sequence defined by

$$\left(\frac{r'_m}{r_m}\right)^{\alpha_m}=\left(\frac{r_{m+1}}{r_m}\right)^{\mu_m}.$$

Since $\alpha_m\to\alpha$ and $\mu_m\to 0$, there exists a positive integer m_3 such that $m\geqq m_3$ implies $\mu_m<\alpha_m$, so we deduce that $r_m<r'_m<r_{m+1}$ $(m\geqq m_3)$. Now, we define a nonnegative function $\lambda(t)$ $(t\geqq r_{m_3})$ by (1.4) and set

$$\nu(r)=\exp\left(\int_{r_{m_3}}^{r}\lambda(t)t^{-1}dt\right)\qquad(r\geqq r_{m_3}).$$

LEMMA 5. *The order and lower order of $\nu(r)$ are equal to ρ and 0, respectively.*

Proof. Consider the interval $r_m\leqq r<r_{m+1}$ $(m\geqq m_3)$. As in the proof of Lemma 1, we have for $r_m\leqq r\leqq r'_m$

$$\log\nu(r)=\sum_{s=m_3}^{m-1}\mu_s\log\left(\frac{r_{s+1}}{r_s}\right)+\alpha_m\log\left(\frac{r}{r_m}\right)$$

$$=L\log 3\sum_{s=m_3}^{m-1}s!+\alpha_m\log\left(\frac{r}{r_m}\right)$$

$$=\alpha_m\log r-\left\{\alpha_m-L\frac{\sum_{s=m_3}^{m-1}s!}{m!}\right\}\log r_m,$$

and for $r'_m\leqq r<r_{m+1}$

$$\log\nu(r)=\sum_{s=m_3}^{m}\mu_s\log\left(\frac{r_{s+1}}{r_s}\right)=L\log 3\sum_{s=m_3}^{m}s!=L\frac{\sum_{s=m_3}^{m}s!}{m!}\log r_m.$$

Hence

$$\varliminf_{r\to\infty}\frac{\log\nu(r)}{\log r}=\varliminf_{m\to\infty}L\frac{\sum_{s=m_3}^{m-1}s!}{m!}=0,$$

$$\varlimsup_{r\to\infty}\frac{\log\nu(r)}{\log r}=\varlimsup_{m\to\infty}L\frac{\sum_{s=m_3}^{m}s!}{m!}\frac{\log r_m}{\log r'_m}$$

$$=L\varlimsup_{m\to\infty}\frac{\log r_m}{\log r'_m}$$

$$=L\,\overline{\lim_{m\to\infty}}\,\frac{\log r_m}{\log r_m+(\mu_m/\alpha_m)\log(r_{m+1}/r_m)}$$

$$=L\,\overline{\lim_{m\to\infty}}\,\frac{1}{1+L/\alpha_m}=\frac{\alpha L}{\alpha+L}=\rho\,.$$

Next, we set

$$K_m=r_{m+1}/r_m=3^{m\cdot m!}$$

and define

$$K_m=(\log K'_m)^{2/\mu_m}\,.$$

It is easy to see that $r'_m/K_m>K_m r_m$ $(m\geqq m_4)$. Here we estimate the size of the set

$$F=\bigcup_{m=m_4}^{\infty}[K_m r_m,\ r'_m/K_m]\,.$$

LEMMA 6. $\overline{\text{log dens}}\,F\geqq\rho/\alpha$.

Proof. Put $R=r'_m/K_m$ $(m\geqq m_4)$. Then

$$\int_{F\cap[1,R]}\frac{dt}{t}=\sum_{s=m_4}^{m}\int_{K_s r_s}^{r'_s/K_s}\frac{dt}{t}$$

$$=\sum_{s=m_4}^{m}\left\{\frac{\mu_s}{\alpha_s}\log K'_s-2\log K_s\right\}>\frac{1-\varepsilon}{\alpha}\sum_{s=m_5}^{m}\mu_s\log K'_s\qquad(s\geqq m_5(\varepsilon))$$

$$=\frac{L(1-\varepsilon)}{\alpha}\log 3\sum_{s=m_5}^{m}s!\,,$$

so that

$$\frac{1}{\log R}\int_{F\cap[1,R]}\frac{dt}{t}>\frac{L(1-\varepsilon)}{\alpha}\log 3\,\frac{\sum_{s=m_5}^{m}s!}{\log r'_m-\log K_m}$$

$$>\frac{L(1-\varepsilon)}{\alpha}\,\frac{\sum_{s=m_5}^{m}s!}{m!+(L/\alpha_m)m!}=\frac{L(1-\varepsilon)\sum_{s=m_5}^{m}s!}{\{\alpha+L(\alpha/\alpha_m)\}\,m!}$$

Thus

$$\overline{\text{log dens}}\,F\geqq(1-\varepsilon)\rho/\alpha\,.$$

Since ε is an arbitrary positive number independent of F, we have

$$\overline{\text{log dens}}\,F\geqq\rho/\alpha\,.$$

Now, set $n(t)=0$ $(t<r_{m_3})$, $=[\nu(t)]$ $(t\geqq r_{m_3})$, and define $f(z)$ as in § 3. In this case, $f(z)$ satisfies

$$\log f(z)=z\int_{r_{m_3}}^{\infty}\frac{n(t)}{t(t+z)}\,dt\qquad(|\arg z|<\pi)\,.$$

LEMMA 7. *$f(z)$ has order ρ and lower order* 0.

Proof. As in the proof of Lemma 3, we can easily see that $f(z)$ has order ρ. We prove that the lower order of $f(z)$ is equal to 0. Set $R=r_m/K_{m-1}\equiv R_m$, and write $\log M(R, f)$ as (3.4). Then

$$I_3 \leqq \frac{1}{1-\alpha_1}\left(\frac{1}{K_{m-1}}\right)^{1-\alpha_1}\nu(r_m) \qquad (m\geqq m_6),$$

$$I_2 \sim (\log K'_{m-1})\nu(r_m) \qquad (m\to\infty),$$

$$\begin{aligned} I_1 &< \frac{\nu(r_m)}{\alpha_{m-1}}+(\log K'_{m-2})\sum_{s=m_3+1}^{m-1}\nu(r_s) \\ &=\nu(r_m)\left[\frac{1}{\alpha_{m-1}}+(\log K'_{m-2})\sum_{s=m_3+1}^{m-1}\left\{\frac{\nu(r_s)}{\nu(r_m)}\right\}\right] \\ &=\nu(r_m)\left[\frac{1}{\alpha_{m-1}}+(\log K'_{m-2})\sum_{s=m_3+1}^{m-1}e^{-L(\log 3)\sum_{l=s}^{m-1}l!}\right] \\ &<\nu(r_m)\left[\frac{1}{\alpha_{m-1}}+(m-m_3-1)e^{-L(\log 3)(m-1)!}(\log K'_{m-2})\right]. \end{aligned}$$

Thus

$$\log M(R, f)<\nu(r_m)O(\log K'_{m-1}) \qquad (m\to\infty),$$

$$\begin{aligned} \frac{\log\log M(R, f)}{\log R} &< \frac{\log\nu(r_m)+O(\log\log K'_{m-1})}{\log r_m-\log K_{m-1}} \\ &\leqq(1+o(1))\frac{\log\nu(r_m)}{\log r_m}=(1+o(1))\frac{L\sum_{s=m_3}^{m-1}s!}{m!}\longrightarrow 0 \\ &\qquad\qquad (m\to\infty). \end{aligned}$$

Finally we modify the argument of the proof of Lemma 3 in [4] to obtain for $K_m r_m \leqq r \leqq r'_m/K_m$ $(m\geqq m_3)$

$$\log M(r, f)\geqq\nu(r)\left\{\frac{\pi}{\sin\pi\alpha_m}+O\left(\frac{m}{\log K'_m}\right)\right\}+O(\log r),$$

$$\log m^*(r, f)\leqq\nu(r)\left\{\pi\cot\pi\alpha_m+O\left(\frac{m}{\log K'_m}\right)\right\}+O(\log r).$$

Thus

$$\begin{aligned} \frac{\log m^*(r, f)}{\log M(r, f)} &\leqq \cos\pi\alpha_m+O\left(\frac{m}{\log K'_m}\right)+O\left(\frac{\log r}{\nu(r)}\right) \\ &=\cos\pi\alpha_m+O\left(\frac{1}{m!}\right) \\ &=\cos\pi\alpha_m+o(\alpha_m-\alpha)<\cos\pi\alpha \qquad (r\in F\cap[r_{m_7}, \infty)). \end{aligned}$$

From this and Lemma 6 we deduce that

$$\underline{\log \text{dens}}\, E \leqq 1-\rho/\alpha .$$

On the other hand, from Lemma 7 and Theorem A or Theorem B it follows that

$$1-\rho/\alpha \leqq \underline{\log \text{dens}}\, E < \overline{\log \text{dens}}\, E = 1 .$$

Hence

$$1-\rho/\alpha = \underline{\log \text{dens}}\, E < \overline{\log \text{dens}}\, E = 1 .$$

This completes the proof.

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DEPARTMENT OF MATHEMATICS
DAIDO INSTITUTE OF TECHNOLOGY
DAIDO-CHO, MINAMI-KU, NAGOYA, JAPAN