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CANONICAL DECOMPOSITION OF HARMONIZABLE ISOTROPIC RANDOM CURRENTS

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Abstracts. First we show that the difference between harmonizable isotropic random currents and homogeneous isotropic ones is small in some sense. Next we obtain the quasi canonical decomposition of harmonizable isotropic random currents, and discuss the possibility of the canonical decomposition of them. Then we show that a certain kind of harmonizable isotropic random currents which is not homogeneous has the canonical decomposition.

§ 1. Introduction

Let $\mathfrak{L}_p$ be a class of random p-currents. We say that a random current U_p has the quasi canonical decomposition in $\mathfrak{L}_p$ if it has the unique decomposition $U_p=U_p^h+U_p^i+U_p^s$ in $\mathfrak{L}_p$ such that $dU_p^i=0$, $\delta U_p^s=0$ and $\Delta U_p^h=0$. The random currents U_p^i, U_p^s and U_p^h are called the irrotational, the solenoidal and the harmonic components of U_p respectively. If the covariances between any two components are zero in the quasi canonical decomposition, we call it the canonical decomposition. Physically the quasi canonical decomposition means the decomposition of a wave into the longitudinal one and the transversal one, and the canonical decomposition corresponds to the case where these two kinds of waves are stochastically independent.

Let $\mathfrak{U}_p$ be the class of homogeneous isotropic random p-currents, and $\mathfrak{W}_p$ be the class of harmonizable isotropic random p-currents. K. Ito (1956) has shown that every random current in $\mathfrak{U}_p$ has the canonical decomposition in it. In this paper we investigate the possibility of the canonical decomposition in $\mathfrak{W}_p$.

In Theorem 1 of § 2 we have two characterizations of the class $\mathfrak{U}_p$ in the broader class $\mathfrak{W}_p$. The results may be understood as those stating that the difference between two classes is not large. Then we introduce a class $\mathfrak{V}_p$ of isotropic random currents which are superpositions of independent plane waves. This class stands between $\mathfrak{U}_p$ and $\mathfrak{W}_p$.

In § 3 we first show that every random current in $\mathfrak{W}_p$ has the quasi canonical decomposition, but it is not necessarily the canonical one. Next we show

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that the canonical decomposition is possible in $\mathfrak{V}_p$. This means that we can not characterize $\mathfrak{U}_p$ by using the possibility of the canonical decomposition.

§ 2. Relations among classes $\mathfrak{U}_p$, $\mathfrak{V}_p$ and $\mathfrak{W}_p$

Let L be the Hilbert space of random variables with mean 0 and finite variance. In L the inner product $\langle , \rangle$ of two elements is defined by their covariance. Let $\mathfrak{D}_p$ be the linear space of smooth p-vector fields with compact support, where we introduce the Schwartz topology. A random p-current U_p is defined as a continuous linear map

$$U_p : \mathfrak{D}_{n-p} \ni \phi_{n-p} \to U_p(\phi_{n-p}) \in L .$$

For a random p-current U_p and a constant p-vector a_p, we define (U_p, a_p) as a random distribution, $(U_p, a_p)(\phi)=U_p(\phi^* a_p)$. The function defined by

$$\rho(\phi, \psi ; a_p, b_p)=\langle (U_p, a_p)(\phi), (U_p, b_p)(\psi)\rangle$$

is called the covariance bilinear form of U_p. For a p-vector field $\phi_p \in \mathfrak{D}_p$ and a random p-current U_p, a motion $g : R^n \ni x \to gx \in R^n$ induces a new p-vector field $g\phi_p$ and a new random p-current gU_p as

$$(g\phi_p)(x)=g\phi_p(g^{-1}x) \quad \text{and} \quad (gU_p)(\phi_{n-p})=U_p(g\phi_{n-p}) .$$

If the covariance bilinear form of gU_p does not depend on all translations g or all rotations g, then U_p is said to be homogeneous or isotropic respectively. A random current U_p is said to be isotropic about the point h if the translated random current hU_p is isotropic. We say that a random current is harmonizable and its spectral measures are $m(d\lambda, d\mu ; a_p, b_p)$ if its its covariance bilinear form is written as

$$\rho(\phi, \psi ; a_p, b_p)=\int_{R^n \times R^n} \mathcal{F}\phi(\lambda)\overline{\mathcal{F}\phi(\mu)} m(d\lambda, d\mu ; a_p, b_p) ,$$

where $m(d\lambda, d\mu ; a_p, b_p)$ is a complex-valued tempered measure for each pair of constant p-vectors a_p, b_p, and $\mathcal{F}\phi$ denotes the Fourier transform of a function ϕ,

$$\mathcal{F}\phi(\lambda)=\int_{R^n} \exp(i(x, \lambda))\phi(x)dx .$$

A random measure is defined as a random current whose covariance bilinear form is of the form

$$\rho(\phi, \psi ; a_p, b_p)=\int_{R^n \times R^n} \phi(\lambda)\overline{\psi(\mu)} m(d\lambda, d\mu ; a_p, b_p) ,$$

where m is a complex-valued tempered measure. In particular, if the support of m is contained in the diagonal set $\{(\lambda, \mu) \in R^n \times R^n ; \lambda=\mu\}$ for every a_p, b_p, it is called an orthogonal random measure.

In the below we frequently use the following lemma, which can be proved as in the proof of Proposition 5.1 of M. M. Rao (1969).

LEMMA 1. *A harmonizable random current with spectral measures* $m(d\lambda, d\mu; a_p, b_p)$ *is isotropic if and only if* $m(d(g\lambda), d(g\mu); ga_p, gb_p)=m(d\lambda, d\mu; a_p, b_p)$ *hold for all rotations* g *and all* a_p *and* b_p.

K. Ito has shown that a homogeneous random current is the Fourier transform of an orthogonal random measure. Applying Theorem 3.1 of M. M. Rao, we have similar representations for harmonizable random currents. We denote the Fourier transform of $\phi_{n-p}\in\mathfrak{D}_{n-p}$ by $\mathcal{F}\phi_{n-p}$.

PROPOSITION 1. *A harmonizable random current* U_p *can be represented by a random measure* M_p *as its Fourier transform,* $U_p(\phi_{n-p})=M_p(\mathcal{F}\phi_{n-p})$. *Conversely the random current defined by the Fourier transform of a random measure is harmonizable.*

It is obvious from Proposition 1 that $\mathfrak{U}_p$ is contained in $\mathfrak{W}_p$. The following theorem states that the difference between these two classes is not so large as was expected by the comparison of their spectral measures.

THEOREM 1. *A harmonizable random current in* $\mathfrak{W}_p$ *which is isotropic about the point* $h\neq 0$ *is homogeneous. Similarly, a random current* U_p *in* $\mathfrak{W}_p$ *whose covariance bilinear form coincides with that of* hU_p *for a translation* h *is homogeneous.*

Proof. We only prove the first assertion as the proof of the second is similar. If U_p has the spectral measures $m(d\lambda, d\mu; a_p, b_p)$, hU_p has the spectral measure $\exp(i(\lambda-\mu, h))m(d\lambda, d\mu; a_p, b_p)$. Using Lemma 1, we have

$$\{1-\exp(i(\lambda-\mu, h-g^{-1}h))\}\, m(d\lambda, d\mu; a_p, b_p)=0$$

for all a_p, b_p, λ, μ and g. Throughout the argument we fix a_p and b_p. For any g, there is a null set N_g with respect to the measure $m(d\lambda, d\mu; a_p, b_p)$ such that

$$\exp(i((\lambda-\mu)-g(\lambda-\mu), h))=1 \tag{1}$$

for all $(\lambda, \mu)\notin N_g$. Take a countable dense subset C of the orthogonal group $O(n)$, and put $N=\bigcup_{g\in C} N_g$. Choose any element $(\lambda, \mu)\notin N$, and fix it. Then, noting the continuity, we see that (1) hold for all $g\in O(n)$. We define $F=\{(\lambda-\mu)-g(\lambda-\mu);\, g\in O(n)\}$ and $P=\{x\in R^n;\ x$ is orthogonal to $h\}$. Since F is connected, F is contained in P. If $\lambda-\mu\neq 0$, we have $n=\dim F\leqq\dim P=n-1$. This is a contradiction. Accordingly the complement of the diagonal set is contained in N. Since N is a null set, the measure $m(d\lambda, d\mu; a_p, b_p)$ is concentrated on the diagonal set. Therefore U_p is homogeneous.

We introduce a class of random p-currents which may be understood as the superpositions of independent plane waves. Let h be a complex-valued function on R^1 which is expressed as

$$h(y)=\int_{R^1}\exp(i\alpha y)H(d\alpha)$$

by a complex-valued measure H such that the integrals

$$\int_{|\alpha|\leq 1}|\alpha|^{-N}|H(d\alpha)|,\ \int_{|\alpha|>1}|\alpha|^{N}|H(d\alpha)|$$

are finite for all $N\geqq 0$. Let M_p be an orthogonal isotropic random measure on R^n by which complex-valued tempered measures m are defined as

$$m(dt;\ a_p,\ b_p)=\langle(M_p(dt),\ a_p),\ (M_p(dt),\ b_p)\rangle\,.$$

We can prove easily that the map $\mathfrak{D}_{n-p}\ni\phi_{n-p}\to\mathfrak{H}\phi_{n-p}\in\mathfrak{D}_{n-p}$ is continuous, where

$$(\mathfrak{H}\phi_{n-p})(t)=\int_{R^n}h((x,\ t))\phi_{n-p}(x)dx\,.$$

Accordingly we can define a random current U_p as the $\mathfrak{H}$-transform of M_p, $U_p(\phi_{n-p})=M_p(\mathfrak{H}\phi_{n-p})$. Now we define a class $\mathfrak{V}_p$ as the totality of the $\mathfrak{H}$-transform of orthogonal isotropic random p-measures for any h.

PROPOSITION 2. $\mathfrak{U}_p\subset\mathfrak{V}_p\subset\mathfrak{W}_p$

Proof. It is obvious that $\mathfrak{V}_p$ contains $\mathfrak{U}_p$. Consider a random current U_p in $\mathfrak{V}_p$. From the assumption on h it follows that $\mathfrak{H}\phi$ is a rapidly decreasing infinitely differentiable function for any $\phi\in\mathfrak{D}$. Thus we can write the covariance bilinear form of U_p as

$$\rho(\phi,\ \psi;\ a_p,\ b_p)=\int_{R^n}(\mathfrak{H}\phi)(t)\overline{(\mathfrak{H}\psi)(t)}m(dt;\ a_p,\ b_p)\,.$$

From this expression we can see easily that U_p is isotropic. Now we define $\Lambda(t)=\{\alpha\in R^1;\ \alpha t\in\Lambda\}$ for any bounded Borel set Λ in R^n, and

$$z(\Lambda_1,\ \Lambda_2;\ a_p,\ b_p)=\int_{t\neq 0}H(\Lambda_1(t))\overline{H(\Lambda_2(t))}m(dt;\ a_p,\ b_p)$$

for any bounded Borel set Λ_1 and Λ_2. Noting that

$$(\mathfrak{H}\phi)(t)=\int_{R^1}(\mathfrak{F}\phi)(\alpha t)H(d\alpha)\,,$$

we can rewrite the covariance bilinear form as

$$z(\phi,\ \psi;\ a_p,\ b_p)=\int_{R^n\times R^n}(\mathfrak{F}\phi)(\lambda)\overline{(\mathfrak{F}\psi)(\mu)}z(d\lambda,\ d\mu;\ a_p,\ b_p)\,.$$

Moreover we can prove that z is a tempered measure. Therefore U_p belongs to $\mathfrak{W}_p$.

§ 3. Canonical decomposition in classes $\mathfrak{U}_p$, $\mathfrak{V}_p$ and $\mathfrak{W}_p$

First we consider the quasi canonical decomposition of harmonizable random currents.

PROPOSITION 3. *Every harmonizable random current has the quasi canonical decomposition in which all components are harmonizable.*

Proof. Let U_p be a harmonizable random current which is the Fourier transform of M_p. We define four random measures M_p^h, M_p^u, M_p^i and M_p^s as

$$M_p^h(d\lambda)=M_p(d\lambda\cap\{0\}), \quad M_p^u(d\lambda)=M_p(d\lambda-\{0\}),$$
$$M_p^i(d\lambda)=|\lambda|^{-2}\lambda\wedge(\lambda\vee M_p^u(d\lambda)), \quad M_p^s(d\lambda)=|\lambda|^{-2}\lambda\vee(\lambda\wedge M_p^u(d\lambda)).$$

Moreover we define three random currents U_p^h, U_p^i and U_p^s as the Fourier transform of M_p^h, M_p^i and M_p^s respectively. Then, as in the proof of Theorem 5.2 of K. Ito, we can show that U_p has the quasi canonical decomposition.

Now we investigate the possibility of the canonical decomposition in $\mathfrak{W}_p$. We put $D=\{(\lambda, \mu)\in R^n\times R^n;\ \lambda$ and μ are linearly independent$\}$, and for each $(\lambda, \mu)\in D$, we choose a system of vectors $\{\xi=|\lambda|^{-1}\lambda,\ \eta=|\mu|^{-1}\mu,\ \zeta_j\ (1\leqq j\leqq n-2)\}$ such that ζ_j's are orthonormal and orthogonal to both ξ and η. Then, defining a measure $\tilde{m}$ in the domain D as

$$\tilde{m}(d\lambda, d\mu)=m\Big(d\lambda, d\mu;\xi\wedge\zeta_1\wedge\zeta_2\wedge\cdots\wedge\zeta_{p-1}, \frac{\xi-(\xi, \eta)\eta}{(1-(\xi, \eta)^2)^{1/2}}\wedge\zeta_1\wedge\cdots\wedge\zeta_{p-1}\Big),$$

we have the following theorem.

THEOREM 2. *Every random current U_p in $\mathfrak{W}_p$ has the quasi canonical decomposition in it. Its irrotational component and solenoidal component are mutually orthogonal if and only if the measure $\tilde{m}$ is identically zero in the domain D.*

Proof. From Proposition 3, U_p has the quasi canonical decomposition $U_p=U_p^h+U_p^i+U_p^s$ where all components are harmonizable. We have to show that they are also isotropic. We only prove for U_p^i. Noting the identity $(\lambda\wedge(\lambda\vee a_p), b_p)=(a_p, \lambda\wedge(\lambda\vee b_p))$, we have the following expression

$$m^i(d\lambda, d\mu; a_p, b_p)=m^u(d\lambda, d\mu;\xi\wedge(\xi\vee a_p), \eta\wedge(\eta\vee b_p))$$

for spectral measures of U_p^i. Then, using Lemma 1, we can see that U_p^i is isotropic.

Similarly cross-spectral measures of U_p^i and U_p^s can be written as

$$m^{is}(d\lambda, d\mu; a_p, b_p)=m^u(d\lambda, d\mu;\xi\wedge(\xi\vee a_p), \eta\vee(\eta\wedge b_p)).$$

By the same reasoning as in the proof of Theorem 7.5 of I. Kubo (1967), we can rewrite them as

$$m^{is}(d\lambda,\ d\mu;\ a_p,\ b_p)=\frac{1}{(1-(\xi,\ \eta)^2)^{1/2}}(\xi\wedge(\xi\vee a_p),\ \eta\vee(\eta\wedge b_p))\tilde{m}(d\lambda,\ d\mu)$$

in the domain D.

From this expression we derive heuristically that m^{is} is identically zero in the complement of D although we can prove it rigorously by the same reasoning as in the above. In fact, we see easily that the coefficient of $\tilde{m}$ in the above expression is bounded in ξ and η, and moreover, when η tends to ξ or $(-\xi)$, $\xi-(\xi,\ \eta)\eta$ is nearly perpendicular to ξ and then $\tilde{m}(d\lambda,\ d\mu)$ is nearly equal to zero. Thus the proof is completed.

Every random current in $\mathfrak{W}_p$ does not have the canonical decomposition, while every random current in $\mathfrak{U}_p$ has it. On the other hand, Theorem 1 states that the difference between $\mathfrak{U}_p$ and $\mathfrak{W}_p$ is not large. Thus one may ask whether a random current in $\mathfrak{W}_p$ which has the canonical decomposition is homogeneous or not. The following theorem answers this question in the negative way.

THEOREM 3. *Every random current in $\mathfrak{V}_p$ has the canonical decomposition in it.*

Proof. The proof of the quasi canonical decomposition of random currents in $\mathfrak{V}_p$ is similar to that of harmonizable random currents in Proposition 3. We can see easily the orthogonality of the harmonic component and the other two ones. In the below we show that the irrotational component and the solenoidal component are mutually orthogonal. For any bounded Borel set Λ not containing 0, we put $c(\Lambda)=\{t\in R^n;$ there is $\alpha\in R^1$ and $\lambda\in R^n$ such that $t=\alpha\lambda\}$. Using the notation in the proof of Proposition 2, we see that $z(\Lambda_1,\ \Lambda_2;\ a_p,\ b_p)$ is zero for any bounded Borel sets Λ_1 and Λ_2 such that $c(\Lambda_1)\cap c(\Lambda_2)=\{0\}$. Accordingly, by Theorem 2, we have the conclusion.

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