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CONTACT *CR* SUBMANIFOLDS

Dedicated to Professor Shigeru Ishihara
on his sixtieth birthday

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Introduction.

The *CR* submanifolds of a Kaehlerian manifold have been defined and studied by A. Bejancu [1] and are now being studied by many authors [3, 4, 5, 10, 11, 13, 14].

The main purpose of the present paper is to define what we call contact *CR* submanifolds of a Sasakian manifold and to study their properties [2, 13].

In § 1, we first of all state some known results on submanifolds of a Sasakian manifold and define the contact *CR* submanifolds of a Sasakian manifold. We then prove a theorem which gives a necessary and sufficient condition in order for a submanifold tangent to the structure vector field ξ of a Sasakian manifold to be a contact *CR* submanifold.

§ 2 is devoted to the study of integrability conditions of the distributions defining contact *CR* structure of the contact *CR* submanifolds.

In § 3, we deal with contact *CR* submanifolds of a Sasakian manifold whose normal connection is flat and in § 4 we study minimal contact *CR* submanifolds of a Sasakian manifold.

§ 1. Submanifolds of Sasakian manifolds.

Let $\bar{M}$ be a $(2m+1)$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g). The structure tensors of $\bar{M}$ satisfy

$$\phi^2X=-X+\eta(X)\xi, \qquad \phi\xi=0, \qquad \eta(\xi)=1, \qquad \eta(\phi X)=0,$$

$$g(\phi X, \phi Y)=g(X, Y)-\eta(X)\eta(Y), \qquad \eta(X)=g(X, \xi)$$

for any vector fields X and Y on $\bar{M}$. We denote by $\bar{\nabla}$ the operator of covariant differentiation with respect to the metric g on $\bar{M}$. We then have

$$\bar{\nabla}_X\xi=\phi X, \qquad (\bar{\nabla}_X\phi)Y=\bar{R}(X, \xi)Y=-g(X, Y)\xi+\eta(Y)X,$$

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where $\bar{R}$ denotes the Riemannian curvature tensor of $\bar{M}$.

Let M be an $(n+1)$-dimensional submanifold isometrically immersed in $\bar{M}$. *Throughout this paper, we assume that the submanifold M of $\bar{M}$ is tangent to the structure vector field ξ.*

We denote by the same g the Riemannian metric tensor field induced on M from that of $\bar{M}$. The operator of covariant differentiation with respect to the induced connection on M will be denoted by ∇. Then the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_X Y=\nabla_X Y+B(X, Y) \quad \text{and} \quad \bar{\nabla}_X V=-A_V X+D_X V$$

for any vector fields X, Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. A and B appearing here are both called the second fundamental forms of M and are related by

$$g(B(X, Y), V)=g(A_V X, Y)\,.$$

The second fundamental form A can be considered as a symmetric $(n+1, n+1)$-matrix. The mean curvature vector μ of M is defined to be $\mu=(\operatorname{Tr} B)/(n+1)$, $\operatorname{Tr} B$ denoting the trace of B. If $\mu=0$, then M is said to be minimal. If the second fundamental form B vanishes identically, then M is said to be totally geodesic. A vector field V normal to M is said to be parallel if $D_X V=0$ for any vector field X tangent to M. The covariant derivative $\nabla_X B$ of B is defined to be

$$(\nabla_X B)(Y, Z)=D_X(B(Y, Z))-B(\nabla_X Y, Z)-B(Y, \nabla_X Z)$$

and the covariant derivative $\nabla_X A$ of A is defined to be

$$(\nabla_X A)_V Y=\nabla_X(A_V Y)-A_{D_X V}Y-A_V\nabla_X Y\,.$$

If $\nabla_X B=0$ for any vector field X tangent to M, then the second fundamental form of M is said to be parallel, which is equivalent to $\nabla_X A=0$. Let R be the Riemannian curvature tensor field of M. Then we have

$$\bar{R}(X, Y)Z=R(X, Y)Z-A_{B(Y,Z)}X+A_{B(X,Z)}Y+(\nabla_X B)(Y, Z)-(\nabla_Y B)(X, Z)$$

for any vector fields X, Y and Z tangent to M. Then we have equations of Gauss and Codazzi respectively

$$g(\bar{R}(X, Y)Z, W)=g(R(X, Y)Z, W)-g(B(X, W), B(Y, Z))+g(B(Y, W), B(X, Z))\,,$$

$$(\bar{R}(X, Y)Z)^{\perp}=(\nabla_X B)(Y, Z)-(\nabla_Y B)(X, Z)\,,$$

$(\bar{R}(X, Y)Z)^{\perp}$ denoting the normal component of $\bar{R}(X, Y)Z$. We now define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X, Y)V=D_X D_Y V-D_Y D_X V-D_{[X,Y]}V\,.$$

Then we have equation of Ricci

$$g(\bar{R}(X, Y)U, V)=g(R^{\perp}(X, Y)U, V)+g([A_V, A_U]X, Y).$$

If $R^{\perp}=0$, then the normal connection of M is said to be flat.

For any vector field X tangent to M, we put

$$\phi X=PX+FX, \tag{1.1}$$

where PX is the tangential part and FX the normal part of ϕX. Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. Similarly, for any vector field V normal to M, we put

$$\phi V=tV+fV, \tag{1.2}$$

where tV is the tangential part and fV the normal part of ϕV. For any vector field Y tangent to M, we have, from (1.1), $g(\phi X, Y)=g(PX, Y)$, which shows that $g(PX, Y)$ is skew-symmetric. Similarly, for any vector field U normal to M, we have, from (1.2), $g(\phi V, U)=g(fV, U)$, which shows that $g(fV, U)$ is skew-symmetric. We also have, from (1.1) and (1.2),

$$g(FX, V)+g(X, tV)=0, \tag{1.3}$$

which gives the relation between F and t.

If we put $X=\xi$ in (1.1), we have

$$\phi\xi=P\xi+F\xi=0,$$

from which

$$P\xi=0, \qquad F\xi=0. \tag{1.4}$$

Now, applying ϕ to (1.1) and using (1.1) and (1.2), we find

$$P^2=-I-tF+\eta\otimes\xi, \qquad FP+fF=0. \tag{1.5}$$

Applying ϕ to (1.2) and using (1.1) and (1.2), we find

$$Pt+tf=0, \qquad f^2=-I-Ft. \tag{1.6}$$

DEFINITION. Let M be a submanifold isometrically immersed in a Sasakian manifold $\bar{M}$ tangent to the structure vector field ξ. Then M is called a contact CR submanifold of $\bar{M}$ if there exists a differentiable distribution $\mathcal{D}$; $x\to\mathcal{D}_x\subset T_x(M)$ on M satisfying the following conditions:

(i) $\mathcal{D}$ is invariant with respect to ϕ, i.e., $\phi\mathcal{D}_x\subset\mathcal{D}_x$ for each $x\in M$, and

(ii) the complementary orthogonal distribution $\mathcal{D}^{\perp}$: $x\to\mathcal{D}_x^{\perp}\subset T_x(M)$ is anti-invariant with respect to ϕ, i.e., $\phi\mathcal{D}_x^{\perp}\subset T_x(M)^{\perp}$ for each $x\in M$.

Remark. For a contact CR submanifold M, the structure vector field ξ

satisfies $\xi\in\mathcal{D}$ or $\xi\in\mathcal{D}^{\perp}$. Indeed, from $\phi^2X=-X+\eta(X)\xi$ for any $X\in\mathcal{D}$, we see that $\eta(X)\xi\in\mathcal{D}$. Thus we have $\xi\in\mathcal{D}$ or $\eta(X)=0$ and hence $\xi\in\mathcal{D}^{\perp}$.

Let M be a contact CR submanifold of a Sasakian manifold $\overline{M}$. We denote by l and $l^{\perp}$ the projection operators on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. Then we have

$$l+l^{\perp}=I,\quad l^2=l,\quad l^{\perp 2}=l^{\perp},\quad ll^{\perp}=l^{\perp}l=0. \tag{1.7}$$

From (1.1), we have

$$\phi lX=PlX+FlX\,,$$

from which, the distribution $\mathcal{D}$ being invariant, we have

$$l^{\perp}Pl=0\,,\qquad Fl=0\,. \tag{1.8}$$

From (1.1), we also have

$$\phi l^{\perp}X=Pl^{\perp}X+Fl^{\perp}X\,,$$

from which, the distribution $\mathcal{D}^{\perp}$ being anti-invariant, we have $Pl^{\perp}=0$, and consequently

$$Pl=P\,, \tag{1.9}$$

since $l^{\perp}=I-l$.

Now applying l from the right to the second equation of (1.5) and using the second equation of (1.8) and (1.9), we find

$$FP=0 \tag{1.10}$$

and consequently

$$fF=0\,. \tag{1.11}$$

Thus, remembering the skew-symmetry of f and the relation (1.3), we have

$$tf=0 \tag{1.12}$$

and consequently, from the first equation of (1.6),

$$Pt=0\,. \tag{1.13}$$

Thus, from the first equation of (1.5) we have

$$P^3+P=0\,, \tag{1.14}$$

which shows that P is an f-structure in M and from the second equation of (1.6), we have

$$f^3+f=0\,, \tag{1.15}$$

which shows that f is an f-structure in the normal bundle $T(M)^{\perp}$ (see [8]).

Conversely, for a submanifold M of a Sasakian manifold $\overline{M}$, assume that we have (1.10). Then we have (1.11), (1.12), (1.13) and consequently (1.14) and (1.15). We now put

$$l=-P^2+\eta\otimes\xi\,, \qquad l^{\perp}=I-l\,. \tag{1.16}$$

Then we can easily verify that

$$l+l^{\perp}=I\,, \qquad l^2=l\,, \qquad l^{\perp 2}=l^{\perp}, \qquad ll^{\perp}=l^{\perp}l=0\,,$$

which means that l and $l^{\perp}$ are complementary projection operators and consequently define complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.

From the first equation of (1.16), we have

$$Pl=P$$

since $P^3=-P$ and $P\xi=0$. This equation can be written as

$$Pl^{\perp}=0\,.$$

But $g(PX, Y)$ is skew-symmetric and $g(l^{\perp}X, Y)$ is symmetric and consequently the above equation gives

$$l^{\perp}P=0$$

and hence

$$l^{\perp}Pl=0\,.$$

From the first equation of (1.16), we have

$$Fl=0\,,$$

since $FP=0$ and $F\xi=0$.

The above equations show that the distribution $\mathcal{D}$ is invariant and $\mathcal{D}^{\perp}$ is anti-invariant with respect to ϕ. Moreover, we have

$$l\xi=\xi\,, \qquad l^{\perp}\xi=0$$

and consequently $\mathcal{D}$ contains ξ.

On the other hand, putting

$$l=-P^2\,, \qquad l^{\perp}=I+P^2\,, \tag{1.17}$$

we still see that l and $l^{\perp}$ define complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively since P is an f-structure. We also have

$$Pl=P\,, \qquad l^{\perp}P=0\,, \qquad Fl=0\,, \qquad Pl^{\perp}=0$$

and see that $\mathcal{D}$ is invariant and $\mathcal{D}^{\perp}$ is anti-invariant with respect to ϕ and that

$$l\xi=0\,, \qquad l^{\perp}\xi=\xi\,,$$

which means that $\mathcal{D}^{\perp}$ contains ξ.

Thus we have

THEOREM 1.1. *In order for a submanifold M of a Sasakian manifold $\bar{M}$ to be a contact CR submanifold, it is necessary and sufficient that $FP=0$.*

THEOREM 1.2. *Let M be a contact CR submanifold of a Sasakian manifold $\bar{M}$. Then P is an f-structure in M and f is an f-structure in the normal bundle.*

Let M be a contact CR submanifold of a Sasakian manifold $\bar{M}$. If $\dim \mathcal{D}=0$, then M is an anti-invariant submanifold of $\bar{M}$, and if $\dim \mathcal{D}^{\perp}=0$, then M is an invariant submanifold of $\bar{M}$. If $\phi\mathcal{D}^{\perp}=T(M)^{\perp}$, then M is a generic submanifold of $\bar{M}$ (see [10], [12]).

In the following, we state certain properties of the second fundamental form of a submanifold M of a Sasakian manifold $\bar{M}$. Since ξ is tangent to M, for any vector field X tangent to M, we have

$$\bar{\nabla}_X\xi=\phi X=\nabla_X\xi+B(X, \xi),$$

from which

$$\nabla_X\xi=PX, \qquad FX=B(X, \xi), \qquad A_V\xi=-tV, \tag{1.18}$$

where V is a vector field normal to M. Especially, we have

$$B(\xi, \xi)=0. \tag{1.19}$$

Let X and Y be vector fields tangent to M. Then we obtain

$$(\nabla_X P)Y=-g(X, Y)\xi+\eta(Y)X+A_{FY}X+tB(X, Y) \tag{1.20}$$

and

$$(\nabla_X F)Y=fB(X, Y)-B(X, PY), \tag{1.21}$$

where we have defined $(\nabla_X P)Y$ and $(\nabla_X F)Y$ respectively by

$$(\nabla_X P)Y=\nabla_X(PY)-P\nabla_X Y \quad \text{and} \quad (\nabla_X F)Y=D_X(FY)-F\nabla_X Y.$$

For any vector field X tangent to M and any vector field V normal to M, we have

$$(\nabla_X t)V=A_{fV}X-PA_V X \tag{1.22}$$

and

$$(\nabla_X f)V=-FA_V X-B(X, tV), \tag{1.23}$$

where we have defined $(\nabla_X t)V$ and $(\nabla_X f)V$ respectively by

$$(\nabla_X t)V=\nabla_X(tV)-tD_X V \quad \text{and} \quad (\nabla_X f)V=D_X(fV)-fD_X V.$$

If M is a contact CR submanifold of $\bar{M}$, then $PX=PY=0$ for any $X, Y\in\mathcal{D}^{\perp}$, and then we have $g((\nabla_Z P)X, Y)=g(\nabla_Z(PX), Y)-g(P\nabla_Z X, Y)=0$ for any vector field Z tangent to M. Therefore, (1.20) implies

$$0=g((\nabla_Z P)X, Y)=-\eta(Y)g(Z, X)+\eta(X)g(Z, Y) +g(A_{FX}Z, Y)+g(tB(Z, X), Y),$$

from which

$$g(A_{FX}Y, Z)-g(A_{FY}X, Z)=\eta(Y)g(Z, X)-\eta(X)g(Z, Y).$$

Thus we have

$$A_{FX}Y-A_{FY}X=\eta(Y)X-\eta(X)Y \quad \text{for} \quad X, Y\in\mathcal{D}^{\perp}. \tag{1.24}$$

For a contact CR submanifold M we have the following decomposition of the tangent space $T_x(M)$ at each $x\in M$:

$$T_x(M)=H_x(M)+\{\xi\}+N_x(M),$$

where $H_x(M)=\phi H_x(M)$ and $N_x(M)$ is the orthogonal complement of $H_x(M)+\{\xi\}$ in $T_x(M)$. Then $\phi N_x(M)=FN_x(M)\subset T_x(M)^{\perp}$. Similarly, we have

$$T_x(M)^{\perp}=FN_x(M)+N_x(M)^{\perp},$$

where $N_x(M)^{\perp}$ is the orthogonal complement of $FN_x(M)$ in $T_x(M)^{\perp}$. Then $\phi N_x(M)^{\perp}=fN_x(M)^{\perp}=N_x(M)^{\perp}$.

We take an orthonormal frame $e_1, \cdots, e_{2m+1}$ of $\overline{M}$ such that, restricted to M, $e_1, \cdots, e_{n+1}$ are tangent to M. Then $e_1, \cdots, e_{n+1}$ form an orthonormal frame of M. We can take $e_1, \cdots, e_{n+1}$ such that $e_1, \cdots, e_p$ form an orthonormal frame of $N_x(M)$ and $e_{p+1}, \cdots, e_n$ form an orthonormal frame of $H_x(M)$ and $e_{n+1}=\xi$, where $\dim N_x(M)=p$. Moreover, we can take $e_{n+2}, \cdots, e_{2m+1}$ of an orthonormal frame of $T_x(M)^{\perp}$ such that $e_{n+2}, \cdots, e_{n+1+p}$ form an orthonormal frame of $FN_x(M)$ and $e_{n+2+p}, \cdots, e_{2m+1}$ form an orthonormal frame of $N_x(M)^{\perp}$. In case of need, we can take $e_{n+2}, \cdots, e_{n+1+p}$ such that $e_{n+2}=Fe_1, \cdots, e_{n+1+p}=Fe_p$. Unless otherwise stated, we use the conventions that the ranges of indices are respectively:

$$i, j, k=1, \cdots, n+1; \quad x, y, z=1, \cdots, p; \quad a, b, c=p+1, \cdots, n;$$
$$\alpha, \beta, \gamma=n+2, \cdots, n+1+p.$$

§ 2. Integrability of distributions

We consider the integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ of a contact CR submanifold M of a Sasakian manifold $\overline{M}$.

Let $Y, Y\in\mathcal{D}^{\perp}$. Then we have

$$\phi[X, Y]=P[X, Y]+F[X, Y]=-(\nabla_X P)Y+(\nabla_Y P)X+F[X, Y]$$
$$=A_{FX}Y-A_{FY}X-\eta(Y)X+\eta(X)Y+F[X, Y]=F[X, Y],$$

from which $\phi[X, Y] \in T(M)^{\perp}$. Thus we have $[X, Y] \in \mathcal{D}^{\perp}$.

PROPOSITION 2.1. *Let M be an $(n+1)$-dimensional contact CR submanifold of a $(2m+1)$-dimensional Sasakian manifold $\overline{M}$. Then the distribution $\mathcal{D}^{\perp}$ is completely integrable and its maximal integral submanifold is a p-dimensional anti-invariant submanifold of $\overline{M}$ normal to ξ or a $(p+1)$-dimensional anti-invariant submanifold of $\overline{M}$ tangent to ξ.*

Let $X, Y \in \mathcal{D}$. Then we have

$$\phi[X, Y] = P[X, Y] + F[X, Y] = P[X, Y] + (\nabla_Y F)X - (\nabla_X F)Y$$
$$= P[X, Y] + B(X, PY) - B(Y, PX).$$

Thus we see that $[X, Y] \in \mathcal{D}$ if and only if $B(X, PY) = B(Y, PX)$ for any $X, Y \in \mathcal{D}$. If $\mathcal{D}$ is normal to the structure vector field ξ, then we have

$$g([X, Y], \xi) = 2g(X, PY)$$

for any $X, Y \in \mathcal{D}$. Therefore, if $\mathcal{D}$ is completely integrable and is normal to the structure vector field ξ, then we have $g(X, PY) = 0$, which shows that $\dim \mathcal{D} = 0$. Therefore we have

PROPOSITION 2.2. *Let M be an $(n+1)$-dimensional contact CR submanifold of a $(2m+1)$-dimensional Sasakian manifold $\overline{M}$. Then the distribution $\mathcal{D}$ is completely integrable if and only if*

$$B(X, PY) = B(Y, PX)$$

for any vector fields $X, Y \in \mathcal{D}$, and then $\xi \in \mathcal{D}$. Moreover, the maximal integral submanifold of $\mathcal{D}$ is an $(n+1-p)$-dimensional invariant submanifold of $\overline{M}$.

§ 3. Flat normal connection

Let S^{2m+1} be a $(2m+1)$-dimensional unit sphere. We know that S^{2m+1} admits a standard Sasakian structure. Let M be an $(n+1)$-dimensional contact CR submanifold of S^{2m+1}.

LEMMA 3.1. *If the normal connection of M is flat, then*

$$A_{fV} = 0$$

for any vector field V normal to M.

Proof. Let V and U be vector fields normal to M. Since $R^{\perp} = 0$, equation of Ricci implies that $A_V A_U = A_U A_V$. Thus, from (1.18), we find

$$A_V tU = A_U tV. \tag{3.1}$$

Since $tf=0$, using (3.1), we see that $A_{fV}tU=0$ and $A_{fV}\xi=0$. Moreover, from (1.23), we have

$$g((\nabla_X f)fV,\ U)=-g(FA_{fV}X,\ U)-g(B(X,\ tfV),\ U)=g(A_{fV}tU,\ X)=0\,,$$

from which

$$(\nabla_X f)fV=0\,.$$

Thus, from (1.15) and (1.21), we have

$$g((\nabla_X f)fV,\ FY)=-g(f^2V,\ (\nabla_X F)Y)=-g(A_{fV}X,\ Y)+g(A_{f^2V}X,\ PY)=0\,.$$

From this and the fact that $A_{fV}A_{f^2V}=A_{f^2V}A_{fV}$, we have

$$\begin{aligned}\mathrm{Tr}\ A_{fV}^2&=\mathrm{Tr}\ A_{f^2V}PA_{fV}=-\mathrm{Tr}\ A_{fV}PA_{f^2V}=-\mathrm{Tr}\ A_{f^2V}A_{fV}P\\ &=-\mathrm{Tr}\ A_{fV}A_{f^2V}P=-\mathrm{Tr}\ A_{f^2V}PA_{fV}=-\mathrm{Tr}\ A_{fV}^2\,.\end{aligned}$$

Consequently, we have $\mathrm{Tr}\ A_{fV}^2=0$ and hence $A_{fV}=0$.

LEMMA 3.2. *Let M be an $(n+1)$-dimensional contact CR submanifold of S^{2m+1} with flat normal connection. If $PA_V=A_VP$ for any vector field V normal to M, then*

$$g(A_UX,\ A_VY)=g(X,\ Y)g(tU,\ tV)-\sum_i g(A_UtV,\ e_i)g(A_{Fe_i}X,\ Y)\,. \tag{3.2}$$

Proof. From the assumption we see that

$$g(A_UPX,\ tV)=0\,,$$

from which

$$g((\nabla_Y A)_UPX,\ tV)+g(A_U(\nabla_Y P)X,\ tV)+g(A_UPX,\ (\nabla_Y t)V)=0\,.$$

Thus, from (1.20) and (1.22), we have

$$\begin{aligned}&g((\nabla_Y A)_UPX,\ tV)-g(X,\ Y)g(A_U\xi,\ tV)+\eta(X)g(A_UY,\ tV)+g(A_UA_{FX}Y,\ tV)\\ &\quad+g(A_UtB(Y,\ X),\ tV)+g(A_UPX,\ A_{fV}Y)-g(A_UPX,\ PA_VY)=0\,,\end{aligned}$$

from which and Lemma 3.1, we find

$$\begin{aligned}&g((\nabla_{PY} A)_UPX,\ tV)+g(X,\ PY)g(tU,\ tV)\\ &\quad+g(A_UtV,\ tB(PY,\ X))-g(A_UPX,\ PA_VPY)=0\,.\end{aligned}$$

On the other hand, we have

$$g(A_UtV,\ tB(PY,\ X))=-\sum_i g(A_UtV,\ e_i)g(A_{Fe_i}X,\ PY)\,,$$

$$-g(A_UPX,\ PA_VPY)=g(A_UPX,\ A_VY)\,.$$

From these equations we have

$$g((\nabla_{PY}A)_U PX,\ tV)+g(X,\ PY)g(tU,\ tV)$$
$$-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}X,\ PY)+g(A_U PX,\ A_V Y)=0\,.$$

Therefore, the Codazzi equation implies

$$g(X,\ PY)g(tU,\ tV)-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}X,\ PY)+g(A_U PX,\ A_V Y)=0\,,$$

from which

$$g(PX,\ PY)g(tU,\ tV)-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}PX,\ PY)+g(A_U P^2X,\ A_V Y)=0\,. \tag{3.3}$$

On the other hand, we have

$$g(PX,\ PY)g(tU,\ tV)$$
$$=g(X,\ Y)g(tU,\ tV)-\eta(X)\eta(Y)g(tU,\ tV)-g(FX,\ FY)g(tU,\ tV)\,,$$
$$-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}PX,\ PY)=-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}X,\ Y)$$
$$+\eta(Y)g(A_U tV,\ X)+\eta(X)\eta(Y)g(tU,\ tV)-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}X,\ tFY)\,,$$
$$g(A_U P^2X,\ A_V Y)=-g(A_U X,\ A_V Y)-\eta(Y)g(A_U tV,\ X)-g(A_U X,\ A_V tFY)\,.$$

Substituting these equations into (3.3), we find

$$g(X,\ Y)g(tU,\ tV)-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}X,\ Y)-g(A_U X,\ A_V Y)$$
$$-g(FX,\ FY)g(tU,\ tV)-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}X,\ tFY)-g(A_U X,\ A_V tFY)=0\,.$$

Moreover, we obtain

$$-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}X,\ tFY)=g(A_U tV,\ A_{FY}X)+g(FX,\ FY)g(tU,\ tV)\,,$$
$$-g(A_U X,\ A_V tFY)=-g(A_U tV,\ A_{FY}X)\,.$$

From these equations we have

$$g(X,\ Y)g(tU,\ tV)-\sum_i g(A_U tV,\ e_i)g(A_{Fe_i}X,\ Y)-g(A_U X,\ A_V Y)=0\,,$$

which proves (3.2).

LEMMA 3.3. *Let M be an $(n+1)$-dimensional contact CR submanifold of S^{2m+1} with flat normal connection. If the mean curvature vector of M is parallel, and if $PA_V=A_VP$ for any vector field V normal to M, then the square of the length of the second fundamental form of M is constant.*

Proof. From Lemma 3.1 the square of the length of the second fundamental form of M is given by $\sum_\alpha \operatorname{Tr} A_\alpha^2$, where $A_\alpha = A_{e_\alpha}$. Using (3.2), we have

$$\sum_\alpha \operatorname{Tr} A_\alpha^2 = (n+1)p + \sum_{\alpha,\beta} g(A_\alpha te_\alpha, te_\beta) \operatorname{Tr} A_\beta .$$

Since the normal connection of M is flat, we can take $\{e_\alpha\}$ such that $D_X e_\alpha = 0$ for each α, because, for any $V \in FN(M)$ we have $D_X V \in FN(M)$ by (1.23) and (3.1). Then we have

$$\begin{aligned}\nabla_X(\sum_\alpha \operatorname{Tr} A_\alpha^2) &= \sum_{\alpha,\beta} g((\nabla_X A)_\alpha te_\alpha, te_\beta) \operatorname{Tr} A_\beta \\ &= \sum_{\alpha,\beta} g((\nabla_{te_\alpha} A)_\beta te_\alpha, X) \operatorname{Tr} A_\beta\end{aligned}$$

by using $\nabla_X(te_\alpha) = (\nabla_X t)e_\alpha = A_{fe_\alpha} X - PA_\alpha X$ and $Pt = 0$.

On the other hand, using $PA_V = A_V P$, we have, for any $X \in T_x(M)$,

$$\begin{aligned}&\sum_i g((\nabla_{Pe_i} A)_\alpha Pe_i, X) \\ &\qquad = \sum_i [g((\nabla_{Pe_i} P)A_\alpha e_i, X) + g(P(\nabla_{Pe_i} A)_\alpha e_i, X) - g(A_\alpha(\nabla_{Pe_i} P)e_i, X)] .\end{aligned}$$

Since A_α is symmetric and P is skew-symmetric, using (1.4), (1.10), (1.13) and (1.20), we see that

$$\sum_i g((\nabla_{Pe_i} P)A_\alpha e_i, X) = 0 \quad \text{and} \quad \sum_i g(A_\alpha(\nabla_{Pe_i} P)e_i, X) = 0 .$$

Therefore, we have

$$\begin{aligned}\sum_i g((\nabla_{Pe_i} A)_\alpha Pe_i, X) &= \sum_i g(P(\nabla_{Pe_i} A)_\alpha e_i, X) \\ &= -\sum_i g((\nabla_{Pe_i} A)_\alpha e_i, PX) = -\sum_i g((\nabla_{PX} A)_\alpha Pe_i, e_i) = 0 ,\end{aligned}$$

where we have used the Codazzi equation and the fact that $(\nabla_{PX} A)_\alpha$ is symmetric and P is skew-symmetric.

Since we have $\sum_a (\nabla_{e_a} A)_\alpha e_a = \sum_i (\nabla_{Pe_i} A)_\alpha Pe_i$, the above equation implies

$$\sum_a (\nabla_{e_a} A)_\alpha e_a = 0 . \tag{3.4}$$

Moreover, we see that

$$(\nabla_\xi A)_\alpha \xi = 0 . \tag{3.5}$$

From the assumption the mean curvature vector of M is parallel, and hence

$$\begin{aligned}0 &= \sum_i (\nabla_{e_i} A)_\alpha e_i = \sum_a (\nabla_{e_a} A)_\alpha e_a + (\nabla_\xi A)_\alpha \xi + \sum_x (\nabla_{e_x} A)_\alpha e_x \\ &= \sum_x (\nabla_{e_x} A)_\alpha e_x = \sum_\beta (\nabla_{te_\beta} A)_\alpha te_\beta .\end{aligned}$$

Therefore the square of the length of the second fundamental form of M is constant.

From Lemmas 3.1 and 3.3, using a theorem of [9], we have (see [6])

LEMMA 3.4. *Let M be an $(n+1)$-dimensional contact CR submanifold of S^{2m+1} with flat normal connection. If the mean curvature vector of M is parallel, and if $PA_V=A_VP$ for any vector field V normal to M, then*

$$g(\nabla A, \nabla A)=-(n+1)\sum_{\alpha}\operatorname{Tr} A_\alpha^2+\sum_{\alpha}(\operatorname{Tr} A_\alpha)^2 + \sum_{\alpha,\beta}[\operatorname{Tr}(A_\alpha A_\beta)]^2-\sum_{\alpha,\beta}\operatorname{Tr} A_\beta \operatorname{Tr} A_\alpha^2 A_\beta . \tag{3.6}$$

LEMMA 3.5. *Under the same assumptions as those of Lemma 3.4, the second fundamental form of M is parallel.*

Proof. From (3.2) we have

$$\operatorname{Tr} A_\alpha^2 A_\beta=\operatorname{Tr} A_\alpha g(e_\alpha, e_\beta)+\sum_{\gamma}\operatorname{Tr}(A_\gamma A_\alpha)g(A_\gamma te_\alpha, te_\beta),$$

$$\operatorname{Tr}(A_\alpha A_\beta)=(n+1)g(e_\alpha, e_\beta)+\sum_{\gamma}\operatorname{Tr} A_\gamma g(A_\gamma te_\alpha, te_\beta).$$

Thus we have

$$\sum_{\alpha,\beta}[\operatorname{Tr}(A_\alpha A_\beta)]^2=(n+1)\sum_{\alpha}\operatorname{Tr} A_\alpha^2+\sum_{\alpha,\beta,\gamma}\operatorname{Tr}(A_\alpha A_\beta)\operatorname{Tr} A_\gamma g(A_\gamma te_\alpha, te_\beta),$$

$$-\sum_{\alpha,\beta}\operatorname{Tr} A_\beta \operatorname{Tr} A_\alpha^2 A_\beta=-\sum_{\alpha}(\operatorname{Tr} A_\alpha)^2-\sum_{\alpha,\beta,\gamma}\operatorname{Tr}(A_\alpha A_\beta)\operatorname{Tr} A_\gamma g(A_\gamma te_\alpha, te_\beta).$$

Substituting these equations into (3.6), we find $g(\nabla A, \nabla A)=0$, that is, the second fundamental form of M is parallel.

THEOREM 3.1. *Let M be an $(n+1)$-dimensional complete contact CR submanifold of S^{2m+1} with flat normal connection. If the mean curvature vector of M is parallel, and if $PA_V=A_VP$ for any vector field V normal to M, then M is an S^{n+1} or*

$$S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k), \quad n+1=\sum_{i=1}^{k} m_i, \quad 2\leqq k\leqq n+1, \quad \sum_{i=1}^{k} r_i^2=1$$

in some S^{n+1+p}, where $m_1, \cdots, m_k$ are odd numbers.

Proof. We first assume that $F=0$, that is, M is an invariant submanifold of S^{2m+1}. Then the second fundamental form of M satisfies $PA_V+A_VP=0$ (cf. [10]). Thus we have $PA_V=0$, which implies that $A_V=0$ and hence M is totally geodesic in S^{2m+1}. Therefore M is an S^{n+1} and $n+1$ is odd.

We next assume that $F\neq 0$. Since the second fundamental form of M is parallel and $R^\perp=0$, by Lemma 1.2 of [11], the sectional curvature of M is non-negative. On the other hand, from (3.2), we see that $A_V\neq 0$ for any $V\in FN_x(M)$.

Thus Lemma 3.1 shows that the first normal space is of dimension p. Therefore, by a theorem of [9] and a result of Example 3 of [11] (see also [14]), we have our assertion.

COROLLARY 3.1. *Let M be an $(n+1)$-dimensional complete generic submanifold of S^{2m+1} with flat normal connection. If the mean curvature vector of M is parallel, and if $PA_V=A_VP$ for any vector field V normal to M, then M is*

$$S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k), \quad n+1=\sum_{i=1}^{k} m_i, \quad 2\leqq k\leqq n+1, \quad \sum_{i=1}^{k} r_i^2=1,$$

where $m_1, \cdots, m_k$ are odd numbers.

§ 4. Minimal contact CR submanifolds

Let M be an $(n+1)$-dimensional contact CR submanifold of S^{2m+1} with flat normal connection. We denote by S the Ricci tensor of M. For any vector field X of M, we have generally (see [7])

$$\mathrm{div}(\nabla_X X)-\mathrm{div}((\mathrm{div}\, X)X)=S(X, X)+\frac{1}{2}|L(X)g|^2-|\nabla X|^2-(\mathrm{div}\, X)^2,$$

where $L(X)g$ denotes the Lie derivative of the Riemannian metric g with respect to a vector field X and $|Y|$ denotes the length with respect to the Riemannian metric of a vector field Y on M.

Let V be a parallel vector field normal to M. Then, by Lemma 3.1, $A_{fV}=0$. Thus (1.22) implies

$$\nabla_X tV=-PA_V X.$$

Hence we have

$$\mathrm{div}\, tV=-\mathrm{Tr}\, PA_V=0, \qquad \mathrm{div}((\mathrm{div}\, tV)tV)=0.$$

Consequently, we obtain

$$\mathrm{div}(\nabla_{tV}tV)=S(tV, tV)+\frac{1}{2}|L(tV)g|^2-|\nabla tV|^2. \tag{4.1}$$

In the sequel, we assume that M is minimal. Then the Ricci tensor S of M is given by

$$S(X, Y)=ng(X, Y)-\sum_\alpha g(A_\alpha^2 X, Y)$$

because of $A_{fV}=0$.

On the other hand, we have

$$|\nabla tV|^2=\mathrm{Tr}\, A_V^2-g(tV, tV)-\sum_i g(FA_Ve_i, FA_Ve_i)$$

$$=\mathrm{Tr}\, A_V^2-g(tV, tV)-\sum_\alpha g(A_\alpha tV, A_\alpha tV).$$

Therefore, equation (4.1) reduces to

(4.2) $$\mathrm{div}(\nabla_{tV}tV)=(n+1)g(tV,\ tV)-\mathrm{Tr}\ A_V^2+\frac{1}{2}|L(tV)g|^2\ .$$

PROPOSITION 4.1. *Let M be a compact orientable $(n+1)$-dimensional contact CR submanifold of S^{2m+1} with flat normal connection and with parallel section V in the normal bundle. If M is minimal and*

$$\int_M[\mathrm{Tr}\ A_V^2-(n+1)g(tV,\ tV)]*1\leqq 0\ ,$$

then tV is an infinitesimal isometry of M and $PA_V=A_VP$.

Proof. For any vector fields X, Y tangent to M, we have

$$\begin{aligned}(L(tV)g)(X,\ Y)&=g(\nabla_X tV,\ Y)+g(\nabla_Y tV,\ X)\\&=g((A_VP-PA_V)X,\ Y)\ ,\end{aligned}$$

from which we have our assertion.

Since the normal connection of M is flat, we can take a frame $\{e_\alpha\}$ of $FN(M)$ such that $De_\alpha=0$ for each α. Thus we find

$$\mathrm{div}(\sum_\alpha \nabla_{te_\alpha}te_\alpha)=(n+1)p-\sum_\alpha \mathrm{Tr}\ A_\alpha^2+\frac{1}{2}\sum_\alpha|L(te_\alpha)g|^2\ .$$

From this we have

THEOREM 4.1. *Let M be a compact orientable $(n+1)$-dimensional minimal contact CR submanifold of S^{2m+1} with flat normal connection. Then*

$$0\leqq\frac{1}{2}\int_M\sum_\alpha|L(te_\alpha)g|^2*1=\int_M[\sum_\alpha \mathrm{Tr}\ A_\alpha^2-(n+1)p]*1\ .$$

As an application of Theorem 4.1, we have

THEOREM 4.2. *Let M be a compact orientable $(n+1)$-dimensional minimal contact CR submanifold of S^{2m+1} with flat normal connection. If the square of the length of the second fundamental form of M is $(n+1)p$, then M is*

$$S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k)\ ,\quad r_t=(m_t/(n+1))^{1/2}\ (t=1,\ \cdots,\ k)\ ,$$

$$n+1=\sum_{t=1}^k m_t\ ,\quad 2\leqq k\leqq n+1\ ,\quad \sum_{t=1}^k r_t^2=1$$

in some S^{n+1+p}, where $m_1,\ \cdots,\ m_k$ are odd numbers.

Proof. Since $A_{fV}=0$, the square of the length of the second fundamental form of M is given by $\sum_\alpha \mathrm{Tr}\ A_\alpha^2$. Thus, from Theorem 4.1, we have $|L(te_\alpha)g|=0$ for each α and hence $PA_\alpha=A_\alpha P$. On the other hand, from the assumption, M is not totally geodesic. Therefore, our assertion follows from Theorem 3.1.

If M is minimal, the scalar curvature r of M is given by

$$r=n(n+1)-\sum_{\alpha} \operatorname{Tr} A_{\alpha}^{2}.$$

From this and Theorem 4.2 we have

THEOREM 4.3. *Let M be a compact orientable $(n+1)$-dimensional minimal contact CR submanifold of S^{2m+1} with flat normal connection. If $r=(n+1)(n-p)$, then M is*

$$S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k), \quad r_t=(m_t/(n+1))^{1/2} \ (t=1, \cdots, k),$$

$$n+1=\sum_{t=1}^{k} m_t, \quad 2\leqq k\leqq n+1, \quad \sum_{t=1}^{k} r_t^2=1$$

in some S^{n+1+p}, where $m_1, \cdots, m_k$ are odd numbers.

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