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A CHARACTERIZATION OF A COMPLEX PROJECTIVE SPACE BY THE SPECTRUM

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0. Introduction

Let (M, g) be a compact m-dimensional Riemannian manifold and (S^m, g_0) be an m-sphere of constant curvature 1. We denote the spectrum of the Laplacian acting on functions on (M, g) by Spec (M, g). If $m \leqq 6$, Spec $(M, g)=$Spec (S^m, g_0) implies that (M, g) is isometric to (S^m, g_0) (Berger [1], Tanno [8]). For $m \geqq 7$ it is an open question if (S^m, g_0) is characterized by the spectrum. In [9] we proved the following.

THEOREM A (Tanno [9]). *Assume that* Spec $(M, g)=$Spec (S^m, g_0). *If g is sufficiently close to constant curvature metric, then (M, g) is isometric to (S^m, g_0).*

In this paper we give the Kähler version of Theorem A. Let (M, J, g) be a compact Kählerian manifold of dimension $m=2n$, and (CP^n, J_0, g_0) be a complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4. Then, we get the following. (Cf. PROPOSITION 2.1.)

THEOREM B. *Assume that* Spec $(M, J, g)=$Spec (CP^n, J_0, g_0). *If g is sufficiently close to constant holomorphic sectional curvature metric, then (M, J, g) is holomorphically isometric to (CP^n, J_0, g_0).*

As for isospectral deformations of flat metrics, see [4], [5], and for inverse spectral results for negatively curved manifolds, see [2], [3].

1. Preliminaries

Let (M, J, g) be a Kählerian manifold, where $J=(J_j^i)$ denotes the complex structure tensor and $g=(g_{ij})$ a Kähler metric. By $R=(R^i_{jkl})$, $\rho=(R_{jl})$ and S we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature of (M, J, g), respectively. We put

$$R^*_{ij}=R_{ir}J^r_j\,.$$

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Then we have the following classical relations:

$$2R^*_{ij}=R_{ijrs}J^{rs}=2R_{irjs}J^{rs},$$

$$R_{ijkl}=J^r_iJ^s_jR_{rskl}, \quad J^i_rR^r_{jkl}=J^r_jR^i_{rkl},$$

$$R_{rs}J^r_iJ^s_j=R_{ij}, \qquad R^*_{rs}J^r_iJ^s_j=R^*_{ij},$$

$$R^*_{ij}J^{ij}=S, \qquad R^*_{ij}R^{*ij}=R_{ij}R^{ij}.$$

The Bochner curvature tensor $B=(B^i_{jkl})$ is by definition

$$\begin{aligned}B_{ijkl}=R_{ijkl}-(&g_{ik}R_{jl}-g_{il}R_{jk}+R_{ik}g_{jl}-R_{il}g_{jk}+J_{ik}R^*_{jl}-J_{il}R^*_{jk}\\&+R^*_{ik}J_{jl}-R^*_{il}J_{jk}+2R^*_{ij}J_{kl}+2J_{ij}R^*_{kl})/(m+4)\\&+S(g_{ik}g_{jl}-g_{il}g_{jk}+J_{ik}J_{jl}-J_{il}J_{jk}+2J_{ij}J_{kl})/(m+2)(m+4),\end{aligned}$$

where $m=2n=\dim M$. Easily we get

$$g^{ik}B_{ijkl}=0, \qquad J^{kl}B_{ijkl}=0,$$

$$J^{ik}B_{ijkl}=0.$$

We use the following notations:

$$(P,\ Q)=P_{ijkl}Q^{ijkl}, \qquad |P|^2=(P,\ P),$$

$$(P,\ Q,\ T)=P^{ij}{}_{kl}Q^{kl}{}_{rs}T^{rs}{}_{ij},$$

$$(U;\ Q,\ T)=U^{rs}Q_{rjkl}T_s{}^{jkl},$$

$$(U;\ V;\ T)=U^{ik}V^{jl}T_{ijkl},$$

$$(UVW)=U^i{}_jV^j{}_kW^k{}_i,$$

where P, Q, and T are tensor fields of type (1, 3); and U, V, and W are tensor fields of type (1, 1).

In the following calculations the methods are similar to ones in [9]. We obtain

$$\begin{aligned}(1.1)\qquad (R,\ R,\ R)=&(R,\ B,\ B)+8(\rho;\ R,\ B)/(n+2)+8(\rho;\ \rho;\ B)/(n+2)\\&-2S(R,\ B)/(n+1)(n+2)+24(\rho\rho\rho)/(n+2)^2\\&+4(n+6)(\rho;\ \rho;\ R)/(n+2)^2+2(n-9)S(\rho,\ \rho)/(n+1)(n+2)^2\\&-(n-1)S^3/(n+1)^2(n+2)^2,\end{aligned}$$

$$(1.2)\qquad (\rho;\ R,\ B)=(\rho;\ B,\ B)+4(\rho;\ \rho;\ B)/(n+2),$$

(1.3) $$(\rho\,;\,R,\,R)=(\rho\,;\,R,\,B)+4[(\rho\rho\rho)+(\rho\,;\,\rho\,;\,R)]/(n+2)$$
$$-2S(\rho,\,\rho)/(n+1)(n+2)\,,$$

(1.4) $$(R,\,B)=(B,\,B)\,,$$

(1.5) $$(\rho\,;\,\rho\,;\,R)=(\rho\,;\,\rho\,;\,B)+(2n+1)S(\rho,\,\rho)/2(n+1)(n+2)$$
$$+2(\rho\rho\rho)/(n+2)-S^3/4(n+1)(n+2)\,,$$

where we have used the following

$$B_{ijkl}R^{*ij}R^{*kl}=2(\rho\,;\,\rho\,;\,B)$$

etc.

Next we assume that M is compact. Let λ_i be the i-th eigenvalue of the Laplacian acting on functions on $(M,\,g)$ and let Spec $(M,\,g)$ denote the spectrum of $(M,\,g)$. Then the asymptotic expansion by Minakshisundaram-Pleijel is

$$\sum e^{-\lambda_i t}\sim(4\pi t)^{-n}\sum_{\beta=0} a_\beta t^\beta \qquad (t\downarrow 0)\,,$$

where $a_\beta=a_\beta(M,\,g)$ and

(1.6) $$a_0=\mathrm{Vol}\,(M,\,g)\,,$$

(1.7) $$a_1=(1/6)\int_M S\,,$$

(1.8) $$a_2=(1/360)\int_M[2|R|^2-2|\rho|^2+5S^2]\,,$$

(1.9) $$a_3=(1/6\,!)\int_M[-Z+2S|R|^2/3-2S|\rho|^2/3+5S^3/9+A]\,,$$

where we have put

$$Z=|\nabla R|^2/9+26|\nabla\rho|^2/63+142|\nabla S|^2/63\,,$$

$$A=8(R,\,R,\,R)/21-8(\rho\,;\,R,\,R)/63+20(\rho\,;\,\rho\,;\,R)/63-4(\rho\rho\rho)/7\,.$$

We put

$$G_{jl}=R_{jl}-Sg_{jl}/2n\,.$$

Then, $(g,\,G)=0$ and

(1.10) $$|G|^2=|\rho|^2-S^2/2n\,,$$

(1.11) $$(\rho\,;\,\rho\,;\,B)=(G\,;\,G\,;\,B)\,,$$

(1.12) $$(\rho\rho\rho)=(\rho GG)+S|G|^2/n+S^3/4n^2\,.$$

Since

$$|B|^2=|R|^2-8|\rho|^2/(n+2)+2S^2/(n+1)(n+2)\,,$$

we obtain (cf. [8])

$$2|R|^2-2|\rho|^2+5S^2=2|B|^2+2(6-n)|G|^2/(n+2) \tag{1.13}$$
$$+(5n^2+4n+3)S^2/n(n+1).$$

By (1.1)~(1.5), we obtain

$$A=8(R,\ B,\ B)/21+8(22-n)(\rho\,;\ B,\ B)/63(n+2) \tag{1.14}$$
$$-16S|B|^2/21(n+1)(n+2)+p\cdot(\rho\,;\ \rho\,;\ B)$$
$$+q\cdot(\rho\rho\rho)+uS|\rho|^2+vS^3,$$

where

$$p=4(5n^2+76n+420)/63(n+2)^2, \tag{1.15}$$

$$q=8(n^2+92n+276)/63(n+2)^3-4/7 \tag{1.16}$$
$$=-4(9n^3+52n^2-76n-480)/63(n+2)^3,$$

$$u=2(10n^3+109n^2+196n-252)/63(n+1)(n+2)^3, \tag{1.17}$$

$$v=-(5n^3+65n^2+208n+100)/63(n+1)^2(n+2)^3. \tag{1.18}$$

LEMMA 1.1. *a_3 and A are expressed as follows*:

$$a_3=(1/6\,!)\int_M[-Z+2S|B|^2/3+2(6-n)S|G|^2/3(n+2) \tag{1.19}$$
$$+(5/9-(n-3)/3n(n+1))S^3+A],$$

$$A=8(R,\ B,\ B)/21+8(22-n)(\rho\,;\ B,\ B)/63(n+2) \tag{1.20}$$
$$-16S|B|^2/21(n+1)(n+2)+p\cdot(G\,;\ G\,;\ B)$$
$$+q\cdot(\rho GG)+(q/n+u)S|G|^2+(q/4n^2+u/2n+v)S^3.$$

Proof. Eliminating $S(|R|^2-|\rho|^2)$ from (1.9) by (1.13) we get (1.19). By (1.10)~(1.12) and (1.14) we obtain (1.20). Q. E. D.

A Kählerian manifold $(M,\ J,\ g)$ is of constant holomorphic sectional curvature 4, if and only if

$$R_{ijkl}=g_{ik}g_{jl}-g_{il}g_{jk}+J_{ik}J_{jl}-J_{il}J_{jk}+2J_{ij}J_{kl}. \tag{1.21}$$

We define E as the space of all (0, 4)-tensor fields T satisfying the following conditions;

$$T_{ijkl}=T_{klij}, \qquad T_{ijkl}=-T_{jikl},$$
$$T_{ijkl}J_r^iJ_s^j=T_{rskl}, \qquad J^{ij}T_{ijkl}=0.$$

The Bochner curvature tensor B belongs to E.

If (M, J, g) is of constant holomorphic sectional curvature 4, then we obtain

$$(R, T, T)=4(T, T)$$

for any T in E. This follows from (1.21) and the definition of E.

A Kählerian manifold (M, J, g) is of constant holomorphic sectional curvature, if and only if $B=G=0$.

2. Kähler metrics close to constant holomorphic sectional curvature metric

Let (CP^n, J_0, g_0) be a complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4. Then the scalar curvature S_0 is equal to $4n(n+1)$. We put $a_\beta^0=a_\beta(CP^n, J_0, g_0)$

PROPOSITION 2.1. *There exists a positive number $\delta=\delta(n)<1$ with the following property: Assume that a compact Kählerian manifold (M, J, g),* dim $M=2n$, *satisfies the following conditions;*

(i) $(R, T, T)\leqq 4(1+\delta)(T, T)$ *for* $T\in E$,

(ii) $2(n+1)(1-\delta)<$ *Ricci curvature* $<2(n+1)(1+\delta)$,

(iii) $|B|<\delta$,

(iv) $a_\beta=a_\beta^0$, $\beta=0, 1, 2, 3$.

Then (M, J, g) is holomorphically isometric to (CP^n, J_0, g_0).

If $n\leqq 6$, Spec $(M, J, g)=$Spec (CP^n, J_0, g_0) implies that (M, J, g) is holomorphically isometric to (CP^n, J_0, g_0) (cf. [8]). Therefore in the proof of the Proposition 2.1 we can assume that $n\geqq 7$. We assume that (M, J, g) satisfies (i)$\sim$(iv) and show that δ can be determined so that (M, J, g) is holomorphically isometric to (CP^n, J_0, g_0). $a_0=a_0^0$ means that Vol $(M, J, g)=$Vol (CP^n, J_0, g_0). $a_1=a_1^0$ implies that $\int S=4n(n+1)$ Vol (CP^n, J_0, g_0).

LEMMA 2.2. *We have $S>0$ and*

$$|S-4n(n+1)|<4n(n+1)\delta, \tag{2.1}$$

$$\int_M[S^2-S_0^2]\geqq 0, \tag{2.2}$$

$$0\leqq\int_M[S^3-S_0^3]\leqq(3+\delta)4n(n+1)\int_M[S^2-S_0^2]. \tag{2.3}$$

Proof. $S>0$ and (2.1) follow from (ii). As for (2.2) and (2.3) see Lemma 3 and Lemma 6 in [9]. Q.E.D.

By (1.13) and $a_2=a_2^0$ we get

$$2(n-6)\int_M |G|^2=2(n+2)\int_M |B|^2 \tag{2.4}$$

$$+[(n+2)(5n^2+4n+3)/n(n+1)]\int_M [S^2-S_0^2]\,.$$

By Lemma 1.1 and $a_3=a_3^0$ we obtain

$$\begin{aligned} D:=\int_M [&-Z+8(R,\ B,\ B)/21+8(22-n)(\rho\,;\ B,\ B)/63(n+2)\\ &+[2/3-16/21(n+1)(n+2)]S|B|^2+p\cdot(G\,;\,G\,;\,B)\\ &+q\cdot(\rho GG)+[q/n+u-2(n-6)/3(n+2)]S|G|^2\\ &+[q/4n^2+u/2n+v+5/9-(n-3)/3n(n+1)](S^3-S_0^3)\\ =0\,. \end{aligned}$$

LEMMA 2.3. *We put* $\mu=1$ *if* $22>n$ *and* $\mu=-1$ *if* $22<n$. *Then*

$$(22-n)(\rho\,;\,B,\ B)\leqq(22-n)2(n+1)(1+\mu\delta)|B|^2\,, \tag{2.5}$$

$$2/3-16/21(n+1)(n+2)>0\,,\quad p>0\,,\quad v<0\,, \tag{2.6}$$

$$q/n+u-2(n-6)/3(n+2)<0\,, \tag{2.7}$$

$$q/4n^2+u/2n+v+5/9-(n-3)/3n(n+1)>0\,, \tag{2.8}$$

$$q\cdot(\rho GG)\leqq 2q(n+1)(1-\delta)|G|^2\,, \tag{2.9}$$

$$(G\,;\,G\,;\,B)\leqq\delta|G|^2\,. \tag{2.10}$$

Proof. (2.5) follows from (ii). (2.6) is trivial. To prove (2.8) first we get

$$q/4n^2+u/2n+v=-4(n^2-2n-15)/63n^2(n+1)^2\,. \tag{2.11}$$

Then (2.8) is clear. (2.9) follows from (ii). (2.10) follows from (iii) (cf. Lemma 7 in [9]). Q. E. D.

Applying (i), (ii), Lemma 2.2 and Lemma 2.3 to D we get

$$\begin{aligned} D\leqq\int_M [&-Z+32(1+\delta)|B|^2/21\\ &+16(22-n)(n+1)(1+\mu\delta)|B|^2/63(n+2)\\ &+[2/3-16/21(n+1)(n+2)]4n(n+1)(1+\delta)|B|^2\\ &+p\delta|G|^2+2(n+1)(1-\delta)q|G|^2\\ &+[q/n+u-2(n-6)/3(n+2)]4n(n+1)(1-\delta)|G|^2\\ &+[q/4n^2+u/2n+v+5/9-(n-3)/3n(n+1)](3+\delta)4n(n+1)(S^2-S_0^2)]. \end{aligned} \tag{2.12}$$

Next applying (2.4) to $|G|^2$ in (2.12) we obtain

$$D \leqq \int_M [-Z-(U-P\delta)|B|^2-(V-Q\delta)(S^2-S_0^2)] , \tag{2.13}$$

where we have put

$$U = -32/21-16(22-n)(n+1)/63(n+2)+64n/21(n+2)$$
$$-2(n+1)(n+2)(3q+2nu)/(n-6) ,$$
$$P = 32/21+16(22-n)(n+1)\mu/63(n+2)$$
$$+[4/3-16/21(n+1)(n+2)]4n(n+1)$$
$$+(n+1)(n+2)(p/(n+1)-6q-4nu)/(n-6) ,$$
$$V = -12n(n+1)[q/4n^2+u/2n+v+5/9-(n-3)/3n(n+1)]$$
$$-(n+2)(5n^2+4n+3)[3q+2nu-4n(n-6)/3(n+2)]/n(n-6) ,$$
$$Q = 4n(n+1)[q/4n^2+u/2n+v+5/9-(n-3)/3n(n+1)]$$
$$+(n+2)(5n^2+4n+3)[p/(n+1)-6q-4nu+8n(n-6)/3(n+2)]/2n(n-6) .$$

By calculations we get

$$3q+2nu = -(68n^2+24n-1440)/63(n+1)(n+2) , \tag{2.14}$$

$$U = 2(76n^4+144n^3-960n^2-3136n-2496)/63(n+2)^2(n-6) . \tag{2.15}$$

Therefore we see that U is positive. Since $q/4n^2+u/2n+v$ is negative by (2.11) and $3q+2nu$ is negative by (2.14), we get

$$V > -12n(n+1)[5/9-(n-3)/3n(n+1)]+4(5n^2+4n+3)/3$$
$$= 8(n-3)/3 .$$

Therefore V is also positive. Since P and Q are positive, we can define δ by

$$\delta = \min (U/P, V/Q, 99/100) . \tag{2.16}$$

Proof of Proposition 2.1. By the definition of δ, (2.13) shows that $Z=B=0$ and $S=S_0$. Furthermore (2.4) shows that $G=0$, and hence (M, J, g) is of constant holomorphic sectional curvature 4. So (M, J, g) is holomorphically isometric to (CP^n, J_0, g_0). Q.E.D.

REFERENCES

[1] M. BERGER, Le spectre des variétés riemanniennes, Rev. Roum. Math. Pure Appl., 13 (1969), 915-931.

[2] V. GUILLEMIN AND D. KAZHDAN, Some inverse spectral results for negatively curved 2-manifolds, Topology, **19** (1980), 301-312.
[3] V. GUILLEMIN AND D. KAZHDAN, Some inverse spectral results for negatively curved n-manifolds, Proc. Symp. Pure Math. **36** (1980), 135-180.
[4] R. KUWABARA, On isospectral deformations of Riemannian metrics, Compositio Math., **40** (1980), 319-324.
[5] R. KUWABARA, On the characterization of flat metrics by the spectrum, to appear.
[6] H. F. MCKEAN AND I. M. SINGER, Curvature and the eigenvalues of the Laplacian, Journ. Diff. Geom., **1** (1967), 43-69.
[7] T. SAKAI, On eigenvalues of Laplacian and curvature of Riemannian manifold, Tôhoku Math. Journ., **23** (1971), 589-603.
[8] S. TANNO, Eigenvalues of the Laplacian of Riemannian manifolds, Tôhoku Math. Journ., **25** (1973), 391-403.
[9] S. TANNO, A characterization of the canonical spheres by the spectrum, Math. Zeit., **175** (1980), 267-274.

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