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ON L-EQUIVALENCE CLASSES OF SUBMANIFOLDS AND FORMALITY

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§1. Introduction

Let N be a compact differentiable manifold. Two submanifolds of N are called L-equivalent if they are cobordant in $N\times I$. L-equivalence classes of oriented submanifolds of N of codimension d are represented, via Pontrjagin-Thom construction, by elements of $[N, MSO(d)]$: the set of homotopy classes of maps from N to $MSO(d)$. If we consider those submanifolds with the complex normal bundle, their L-equivalence classes are represented by elements of $[N, MU(d)]$.

In general, L-equivalence classes of submanifolds are not derermined by homological conditions in the unstable range. The purpose of the present paper is to show that the L-equivalence classes of submanifolds of certain type can be determined by homological informations.

We work in the rational homotopy category. Spaces and maps are called formal if they are determined, up to rational homotopy, at the cohomology level.

In §2, we characterize formal spaces and formal maps using automorphisms of minimal models. In §3, as an application, we show that the Thom space $D(E)/S(E)$ is formal, where $D(E)$ and $S(E)$ are a disk and sphere bundle associated with a vector bundle E over a formal space. Moreover, we give a comment about the formality of Thom maps. Throughout this paper we assume that spaces are simply connected, cohomology is one with rational-coefficient, $D.G.A.$ means a differential graded commutative algebra over Q and $|\ |$ means the degree.

§2. Preliminaries

We recall some facts from Sullivan's theory of minimal model (see [2], [6]). Let X be a triangulated space. Then there is a $D.G.A.$ $A^*(X)$ which is constructed from Q-polynomial forms on X. From $A^*(X)$ we can construct a paticular $D.G.A$, $m^*(X)$ called the minimal model of X and a morphism ρ_X: $m^*(X)\rightarrow A^*(X)$. It is known that $m^*(X)$ contains all the rational homotopy informations about X.

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DEFINITION 2.1. X is called a formal space if there is a $D.G.A.$ map

$$\varphi_X : m^*(X) \longrightarrow H^*(X)$$

inducing isomorphism on cohomology.

Then, formality is characterized as follows ([4], [6])

THEOREM 2.2. *X is formal if and only if, for any rational number r, there is a $D.G.A.$ automorphism*

$$F_r : m^*(X) \longrightarrow m^*(X)$$

such that $F_r^* = r^i \cdot Id : H^i(X) \to H^i(X)$ for each degree i, where Id denotes the identity.

Remark 2.3. By the proof of Lemma 3.3 in [4], F_r gives a direct sum decomposition

$$m^i(X) = \bigoplus_{k \geqq 0} W_k^i \qquad i=0, 1, 2, \cdots .$$

So that $F_r(x) = r^{i+k}x$ for $x \in W_k^i$. Then it is easy to see

$$dW_k^i \subset W_{k-1}^{i+1}, \qquad W_k^i \cdot W_m^j \subset W_{k+m}^{i+j} .$$

All elements in W_0^i are d-closed and any closed element in W_k^i ($k \geqq 1$) is exact. Thus $\varphi_X(x)=0$ for $x \in W_k^i$ $k \geqq 1$.

DEFINITION 2.4 ([1]) Let X and Y be formal spaces. A map $f: X \to Y$ is called formal map if the following diagram is commutative

$$\begin{array}{ccc} m^*(Y) & \xrightarrow{\hat{f}} & m^*(X) \\ \downarrow{\scriptstyle \varphi_Y} & & \downarrow{\scriptstyle \varphi_X} \\ H^*(Y) & \xrightarrow{f^*} & H^*(X) \end{array} \tag{2.1}$$

Formal maps are characterized as follows

THEOREM 2.5. *A map $f: X \to Y$ between formal spaces is formal if and only if the following diagram is homotopy commutative.*

$$\begin{array}{ccc} m^*(Y) & \xrightarrow{\hat{f}} & m^*(X) \\ \downarrow{\scriptstyle F_r} & & \downarrow{\scriptstyle F_r} \\ m^*(Y) & \xrightarrow{\hat{f}} & m^*(X) \end{array} \tag{2.2}$$

Proof. Suppose that the Diagram 2.2 is homotopy-commutative. Then f preserves the decomposition of Remark 2.3. Therefore $\varphi_X f(y)=0$, $f^*\varphi_Y=0$ for $y \in W_k^i$ with $k \geqq 1$ and $\varphi_X f(y)=[f(y)]=f^*([y])=f^*\varphi_Y(y)$ for $y \in W_0^i$, where [ ] denotes the cohomology class. Hence f is formal.

Conversely suppose that the Diagram 2.1 is commutative. Since F_r is formal, $F_r f$ and fF_r are formal and $(fF_r)^*=(F_r f)^*=r^*f^*$. Therefore $F_r f$ and fF_r are both lifting of the diagram:

$$\begin{array}{ccc} & \dashrightarrow & m^*(X) \\ & & \downarrow \varphi_X \\ m^*(Y) & \xrightarrow[r^*f^*\varphi_Y]{} & H^*(X) \end{array}$$

Then, by Theorem 1.2 of [1], they are homotopic and the proof is completed.

In the Theorem 2.2 if we admit the exponential of r to be arbitrary integer greater than the degree, we have the following definition introduced by Morgan and Sullivan,

DEFINITION 2.6. The rational homotopy type of X is called having positive weights if for any rational number r there is a $D.G.A.$ automorphism

$$P_r : m^*(X) \longrightarrow m^*(X)$$

such that there is a direct sum decomposition $H^i(X)=\bigoplus_{k\geqq 0} V_k^i$ $i=0, 1, 2, \cdots$. So that $P_r^*(x)=r^{i+k}(x)$ for $x \in V_k^i$.

Let X be a formal space and let Y be of positive weights, Suppose the following diagram is homotopy-commutative

$$\begin{array}{ccc} Y_{(0)} & \xrightarrow{h} & X_{(0)} \\ {\scriptstyle P_r}\downarrow & & \downarrow{\scriptstyle F_r} \\ Y_{(0)} & \xrightarrow[h]{} & X_{(0)} \end{array}$$

Then for the cofibration

$$Y \xrightarrow[h]{} X \xrightarrow[c]{} X \cup_h cY$$

we have an induced map G_r and a homotopy-commutative diagram

$$\begin{array}{ccccc} Y_{(0)} & \xrightarrow[h]{} & X_{(0)} & \xrightarrow[c]{} & X \cup_h cY_{(0)} \\ {\scriptstyle P_r}\downarrow & & \downarrow{\scriptstyle F_r} & & \downarrow{\scriptstyle G_r} \\ Y_{(0)} & \xrightarrow[h]{} & X_{(0)} & \xrightarrow[c]{} & X \cup_h cY_{(0)} \end{array}$$

PROPOSITION 2.7. *In the decomposition* $H^i(Y)=\oplus V_k^i$ *of Definition* 2.6 *we assume that*

(1) $V_k^i=0$ *for* $k\geqq 2$, (2) V_0^i *is the image of* h^*,

then $X\underset{h}{\cup}cY$ *is a formal space.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\longleftarrow & H^i(X) & \xleftarrow{\;c^*\;} & H^i(X\underset{h}{\cup}cY) & \xleftarrow{\;\delta\;} & H^{i-1}(Y) & \xleftarrow{\;h^*\;} & \\
 & \downarrow F_r^* & & \downarrow G_r^* & & \downarrow P_r^* & & \\
\longleftarrow & H^i(X) & \xleftarrow{\;c^*\;} & H^i(X\underset{h}{\cup}cY) & \xleftarrow{\;\delta\;} & H^i(Y) & \xleftarrow{\;h^*\;} &
\end{array}$$

Then we have a decomposition $H^i(X\underset{h}{\cup}cY)=A^i\oplus\operatorname{Im}\delta$, where A^i is naturally isomorphic to $Hom_Q(\operatorname{Im}c_*, Q)$, and from the commutativity of the long homology exact sequence we have $G_r^*|A^i=r^iId$. On the other hand by the assumption $\operatorname{Im}\delta=\delta(V_1^i)$ we also have $G_r^*=r^iId$ on $\operatorname{Im}\delta$. 'Hence, by Theorem 2.2 the proof is completed. q. e. d.

EXAMPLE 2.8. Consider the cofibrations

(1) $BU(d-1)\xrightarrow{\;i\;}BU(d)\xrightarrow{\;c\;}MU(d)$

(2) $BSO(d-1)\xrightarrow{\;i\;}BSO(d)\xrightarrow{\;c\;}MSO(d)$

In (1), since $BU(*)$ is formal and i is a formal map and furthermroe $H^*(BU(d-1))$ is the image of i^*, assumptions of Proposition 2.7 are satisfied. Therefore $MU(d)$ is formal and c is a formal map. In (2) if d is even we see similarly that $MSO(d)$ is formal. If d is odd, then the euler class $e\in H^*(BSO(d-1))$ is not in the image of i^*. But, by considering the map $P_r: BSO(d-1)_{(0)}\to BSO(d-1)_{(0)}$ such that $P_r^*(\mathscr{P}_k)=r^{4k}\mathscr{P}_k$ $k=1, (d-3)/2$, $P_r^*(e)=r^de$, where $\mathscr{P}_k$ is the k-th Pontrjagin class of universal bundle. Then we can know that assumptions of Prop. 2.7 are satisfied. Therefore $MSO(d)$ is formal.

§ 3. Sphere bundles over a formal space

Let E be a $2n$ dimensional vector bundle over a formal space M, and

$$\pi: S(E)\longrightarrow M$$

be its associated sphere bundle. Then rational-homotopically $S(E)$ is principal; namely $S(E)$ is induced by a classifying map $l: M_{(0)}\to K(Q, 2n)$,

$$\begin{array}{ccc} S^{2n-1}_{(0)} & = & K(Q, 2n-1) \\ \downarrow & & \downarrow \\ S(E)_{(0)} & \longrightarrow & PK(Q, 2n) \\ \downarrow & & \downarrow \\ M_{(0)} & \xrightarrow[l]{} & K(Q, 2n) \end{array}$$

Therefore $S(E)$ is transgressive (for details see p. 10 and p. 59 of [2]). Thus we may apply Hirsh Lemma for $S(E)$ (see [1] p. 257), then we know that the minimal model of $S(E)$ is the minimal model of $m^*(M)\otimes_d \Lambda(\alpha)$ where $|\alpha|=2n-1$ and $d\alpha$ represents the euler class of E. Since M is formal, by Remark 2.3 we fix a decomposition $m^i(M)=\bigoplus_{k\geq 0} W_k^i$ and we may choose α so that $d\alpha\in W_0^{2n}$.

Then the correspondence defined by

$$P_r(x)=r^{i+k}x\,, \qquad x\in W_k^i\subset m^i(M)\,; \qquad P_r(\alpha)=r^{2n}\alpha$$

can be extended to a $D.G.A.$ automorphism of $m^*(M)\otimes_d \Lambda(\alpha)$ and P_r is lifted, uniquely up to homotopy, to a $D.G.A.$ automorphism P_r of the minimal model $m^*(S(E))$.

LEMMA 3.1. *There is a decomposition*

$$H^i(S(E))=V_0^i\oplus V_1^i\,, \qquad i=1, 2, \cdots.$$

such that $P_r^*(y)=r^iy$ *if* $y\in V_0^i$ $P_r^*(y)=r^{i+1}y$ *if* $y\in V_1^i$ *and* V_0^i *is the image of* π^*.

Proof. Since $\alpha^2=0$, any element of $m^*(M)\otimes_d \Lambda(\alpha)$ can be written as

$$z=\alpha x+y \qquad x,\ y\in m^*(M)$$

and

$$y=y_0+y_1 \qquad y_0\in W_0^*,\ y_1\in \bigoplus_{k\geq 1} W_k^*$$

Suppose z is closed, then

$$dz=d\alpha\cdot x-\alpha\cdot dx+dy_1=-\alpha\cdot dx+(d\alpha\cdot x+dy_1)=0$$

Therefore $dx=0$ and $d\alpha\cdot x+y_1=0$. If x is exact, we may assume $x=du$, $u\in \bigoplus_{k\geq 1} W_k^*$. Then

$$d\alpha\cdot x+dy_1=d(d\alpha\cdot u+y_1)=0$$

therefore $d\alpha\cdot u+y_1\in \bigoplus_{k\geq 1} W_k$ is closed, Since any closed element in $\bigoplus_{k\geq 1} W_k^*$ is exact, we see $d\alpha\cdot u+y_1$ is exact. Setting $d\alpha\cdot u+y_1=dv$, we have

$$z=\alpha\cdot x+y=d(\alpha\cdot u+v)+y_0$$

thus z is cohomologus to y_0. Therefore any closed element is represented by

$$z=\alpha x+y_1+y_0, \qquad x,\ y_0\in W_0^* \quad y_1\in \bigoplus_{k\geq 1} W_k^*$$

Since we know that $d\alpha$, $x\in W_0^*$, $dy_1=-d\alpha\cdot x\in W_0^*$ we can take y_1 from W_1. Then $[z]=[\alpha x+y_1]+[y_0]$, $[\alpha x+y_1]\in V_1^{|z|}$, $[y_0]\in V_0^{|z|}$. Thus the proof is completed.

By Lemma 3.1 the cofibration $D(E)\to D(E)/S(E)$ satisfies the conditions of Proposition 2.7. If E is odd dimensional, $S(E)$ is rationally homotopically trivial and a similar map exists. Thus we have

PROPOSITION 3.2. *Let E be a vector bundle over a formal space, then the Thom space $D(E)/S(E)$ is formal.*

Now consider a D. G. A. map f between formal minimal models. If f does not map any closed element to zero then f is formal. Hence the following is obtained from similar arguments as in [5].

PROPOSITION 3.3. *Let M^{2n-2d} be a closed submanifold of N^{2n} with a complex normal bundle. Suppose that N is simply connected formal space and the i-th Chern class of the normal bundle of M is trivial for $i\leqq n-2d$. Then the corresponding Thom map is formal. Especially if all Chern classes of the normal bundle are trivial, then the corresponding Thom map is null-homotopic.*

Remark. Let $CP^1\times CP^2\subset CP^5$ be the Segre imbedding and M be the intersection of its affine cone and S^{11} centered at the origin. Then the Thom map of M is not formal.

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