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ON THE ASYMPTOTIC BEHAVIOUR OF ALGEBROID FUNCTIONS OF EXTREMAL GROWTH

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1. Valiron [5] and Wahlund [7] established that an entire function $g(z)$ of order λ satisfies

$$\overline{\lim_{r\to\infty}} \frac{N(r, 0)}{\log M(r, g)} \geqq \frac{\sin \pi\lambda}{\pi\lambda} \quad (0 \leqq \lambda < 1)$$

This classical result was extended by Ozawa [3] to n-valued entire algebroid functions. In a recent paper of Williamson [8] it has been sharpened to the following:

Let g be an entire function of lower order $\mu<1$.

Then

$$\overline{\lim_{r\to\infty}} \frac{N(r, 0)}{\log M(r, g)} \geqq \frac{\sin \pi\mu}{\pi\mu} (0 \leqq \mu < 1). \tag{1.1}$$

And Williamson has also obtained a complete answer to the question, what can be said about the asymptotic behaviour of an entire function for which equality holds in (1.1).

The purpose of this paper is to extend Williamson's theorem to n-valued entire algebroid functions of lower order $\mu<1$. Let $f(z)$ be an n-valued entire algebroid function, $f_\nu(z)$ the ν-th determination of $f(z)$, $N(r; a, f)$ the counting-function of $f(z)$ and $M(r, f)$ the maximum modulus of f on $|z|=r$ such that

$$\log M(r, f) = \max_{|z|=r} \max_{1\leqq\nu\leqq n} \log|f_\nu(z)|.$$

We shall prove the following extension of Williamson's theorem:

THEOREM 1. *Let $f(z)$ be an n-valued transcendental entire algebroid function of lower order $\mu<1$. Then there is at least one a_ν among n different finite numbers a_j, $j=1, \cdots, n$, satisfying*

$$\overline{\lim_{r\to\infty}} \frac{nN(r; a_\nu, f)}{\log M(r, f)} \geqq \frac{\sin \pi\mu}{\pi\mu}. \tag{1.2}$$

Next we shall get the following theorem that gives a precise meaning to the asymptotic behaviour of an n-valued entire algebroid function for which equality holds in (1.2).

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THEOREM 2. *Let $f(z)$ be an n-valued transcendental entire algebroid function of lower order $\mu<1$ and assume that the equality*

$$\overline{\lim_{r\to\infty}} \frac{nN(r;a_\nu,f)}{\log M(r,f)}=\frac{\sin\pi\mu}{\pi\mu} \tag{1.3}$$

holds for all n different finite numbers a_ν, $\nu=1,\cdots,n$. Let $\{r_m\}$ be a sequence of Pólya peaks of order μ of $\log M(r,f)$. *Then there exist three positive sequences $\{R'_m\}$, $\{R''_m\}$, $\{\tilde{\varepsilon}_m\}$ such that*

$$\lim_{m\to\infty} R'_m=\lim_{m\to\infty}\frac{R''_m}{r_m}=\infty\,;\qquad \lim_{m\to\infty}\frac{R''_m}{r_m}=\lim_{m\to\infty}\tilde{\varepsilon}_m=0\,, \tag{1.4}$$

and such that the inequalities

$$R'_m\leqq t\leqq R''_m \qquad (m>m_0)\,,$$

imply the inequalities

$$\left(\frac{t}{r_m}\right)^\mu(1+\tilde{\varepsilon}_m)^{-1}\leqq\frac{\log M(t,f)}{\log M(r_m,f)}\leqq\left(\frac{t}{r_m}\right)^\mu(1+\tilde{\varepsilon}_m)\,, \tag{1.5}$$

$$\frac{\sin\pi\mu}{\pi\mu}-\tilde{\varepsilon}_m\leqq\frac{nN(t;a_\nu,f)}{\log M(t,f)}\leqq\frac{\sin\pi\mu}{\pi\mu}+\tilde{\varepsilon}_m \tag{1.6}$$

and

$$\frac{\sin\pi\mu}{\pi\mu}-\tilde{\varepsilon}_m\leqq\frac{nn(t;a_\nu,f)}{\log M(t,f)}\leqq\frac{\sin\pi\mu}{\pi\mu}+\tilde{\varepsilon}_m \tag{1.7}$$

for some a_ν, $\nu=1,\cdots,n$.

2. Preliminaries. Let $f(z)$ be an n-valued transcendental algebroid function defined by an irreducible equation

$$f^n+A_1(z)f^{n-1}+\cdots+A_{n-1}(z)f+A_n(z)=0\,,$$

where $A_1,\cdots,A_n$ are entire functions without common zeros. Let $f_\nu(z)$ be the ν-th determination of $f(z)$.

We put

$$A(z)=\max(1,\ |A_1|,\cdots,|A_n|)$$

$$g(z)=\max(1,\ |g_1|,\cdots,|g_n|)$$

$$g_\nu(z)=F(z,a_\nu)\,,\qquad \nu=1,\cdots,n$$

where $F(z,f)=0$ is the defining equation of f.

Then we have

$$\log M(r,f)=\max_{|z|=r}\max_{1\leqq\nu\leqq n}\log|f_\nu(z)|$$

$$\leqq\max_{|z|=r}\max_{1\leqq\nu\leqq n}\log^+|f_\nu(z)|$$

$$\leqq \max_{|z|=r} \sum_{1}^{n} \log^+ |f_\nu(z)|$$

and by Valiron's argument [6]

$$\sum_{1}^{n} \log^+ |f_\nu(z)| \leqq \log A(z) + O(1) \leqq \log g(z) + O(1) .$$

Further we have

$$\max_{|z|=r} \log g(z) = \log \max_{|z|=r} g(z) = \log \max_{1 \leqq \nu \leqq n} \max_{|z|=r} |g_\nu(z)|$$

$$= \max_{1 \leqq \nu \leqq n} \log M(r, g_\nu) .$$

Hence we get

$$\log M(r, f) \leqq \max_{1 \leqq \nu \leqq n} \log M(r, g_\nu) + O(1) . \tag{2.1}$$

And we remark

$$\log M(r, g_\nu) \leqq \log M(r, f) + O(1) .$$

3. Proof of Theorem 1. We shall give a proof of Theorem 1 according to Edrei's idea [1], using his well-known representation. Let g_ν be $F(z, a_\nu)$. Denote its zero by $\{b_k\}$. Then we can write

$$\log |g_\nu(re^{i\theta})| \leqq \sum_{0<|b_k|\leqq R} \log \left| 1 - \frac{re^{i\theta}}{b_k} \right| + A \frac{r}{R} \log M(2R, g_\nu) ,$$

provided that $|z| = r \leqq R/2$. Then we have

$$\log M(r, g_\nu) \leqq r \int_0^R N(t; 0, g_\nu) \frac{dt}{(t+r)^2} + A \frac{r}{R} \log M(2R, g_\nu) + O(1) .$$

Hence we obtain from (2.1) that

$$\log M(r, f) \leqq r \max_{1 \leqq \nu \leqq n} \int_0^R N(t; 0, g_\nu) \frac{dt}{(t+r)^2} + A \frac{r}{R} \log M(2R, f)$$

$$= r \max_{1 \leqq \nu \leqq n} n \int_0^R N(t; a_\nu, f) \frac{dt}{(t+r)^2} + A \frac{r}{R} \log M(2R, f) . \tag{3.1}$$

Assume that for all ν

$$\varlimsup_{r \to \infty} \frac{nN(r; a_\nu, f)}{\log M(r, f)} < \frac{\sin \pi\mu}{\pi\mu} .$$

Then

$$\frac{nN(r; a_\nu, f)}{\log M(r, f)} < \frac{\sin \pi\mu}{\pi\mu} - \varepsilon = U , \qquad \varepsilon > 0$$

for $r \geqq r_0$. Thus

$$\log M(r, f) < rU \int_{r_0}^R \log M(t, f) \frac{dt}{(t+r)^2} + A \frac{r}{R} \log M(2R, f) + O(1) . \tag{3.2}$$

Now we make use of the notion of Pólya peaks of order μ of $\log M(t, f)$. It is possible to find three positive sequences $\{r'_m\}$, $\{r''_m\}$, $\{\varepsilon_m\}$ such that

$$\lim_{m\to\infty} r'_m = \lim_{m\to\infty} \frac{r''_m}{r_m} = \infty, \qquad \lim_{m\to\infty} \frac{r'_m}{r_m} = \lim_{m\to\infty} \varepsilon_m = 0 \tag{3.3}$$

and such that the inequalities

$$r'_m \leqq t \leqq r''_m \qquad (m > m_0)$$

imply that

$$\log M(t, f) \leqq (1+\varepsilon_m)\left(\frac{t}{r_m}\right)^{\mu} \log M(r_m, f). \tag{3.4}$$

We deduce from (3.2) on setting

$$r = r_m, \qquad R = \frac{r''_m}{2}$$

that

$$\log M(r_m, f) < r_m U \int_{r'_m}^{r''_m} \log M(t, f) \frac{dt}{(t+r_m)^2}$$
$$+ r_m U \int_{r_0}^{r'_m} \log M(t, f) \frac{dt}{(t+r_m)^2} + A \frac{r_m}{r''_m} \log M(r''_m, f) + O(1).$$

By (3.3) the second integral and third term are dominated respectively by

$$U \log M(r'_m, f) \int_{r_0/r_m}^{r'_m/r_m} \frac{dx}{(x+1)^2} \leqq U \frac{r'_m}{r_m} \log M(r'_m, f) = o(\log M(r_m, f)) \quad (m\to\infty),$$

$$\frac{r_m}{r''_m} \log M(r''_m, f) \leqq (1+\varepsilon_m) \frac{r_m}{r''_m} \left(\frac{r''_m}{r_m}\right)^{\mu} \log M(r_m, f) = o(\log M(r_m, f)) \quad (m\to\infty)$$

since $\mu<1$. Thus in veiw of (3.4) we have

$$\log M(r_m, f) < r_m U (1+\varepsilon_m) \log M(r_m, f) \int_{r'_m}^{r''_m} \left(\frac{t}{r_m}\right)^{\mu} \frac{dt}{(t+r_m)^2}$$
$$+ o(\log M(r_m, f)). \tag{3.5}$$

Dividing this by $\log M(r_m, f)$, setting $x = t/r_m$ in the integral, and letting $m\to\infty$, we obtain

$$1 \leqq U \int_0^{\infty} x^{\mu} \frac{dx}{(x+1)^2} = U \frac{\pi\mu}{\sin \pi\mu}$$

and by definition of U we have

$$1 \leqq U \frac{\pi\mu}{\sin\pi\mu} = \left(\frac{\sin\pi\mu}{\pi\mu} - \varepsilon\right) \frac{\pi\mu}{\sin\pi\mu}$$
$$= 1 - \varepsilon \frac{\pi\mu}{\sin\pi\mu} < 1,$$

which is a contradiction. Hence Theorem 1 follows.

4. Example. Now we can consider equality parts in the inequality (2.1). Let $f(z;\lambda)$ be the Lindelöf function

$$f(z;\lambda)=\prod_{\nu=1}^{\infty}\left(1+\frac{z}{b_\nu}\right), \qquad b_\nu=\nu^{1/\lambda}, \quad \nu=1, 2, 3, \cdots.$$

Let $h_\alpha(z)=f(\alpha^{1/\lambda}(z+c);\mu)$. The asymptotic behaviour of $f(z;\mu)$ is completely known (cf. [4, p. 18]). In particular, it is easy to verify that $f(z;\mu)$ has order λ, lower order $\mu=\lambda$, and further that

$$\log h_\alpha(z)=\frac{\pi\alpha}{\sin\pi\mu}z^\mu(1+\varepsilon(z)),$$

where $\varepsilon(z)\to 0$, uniformly as $z\to\infty$ in the angle

$$|\arg z|\leqq\pi-\eta \qquad (0<\eta<\pi),$$

and hence

$$\log M(r, h_\alpha)\sim\frac{\pi\alpha}{\sin\pi\mu}r^\mu \qquad (r\to\infty). \tag{4.1}$$

Let $n(r, 0)$ denote the counting-function associated with $h_\alpha(z)$; clearly

$$n(r, 0)\sim\alpha r^\mu \qquad (r\to\infty),$$

and hence

$$N(r, 0)\sim\frac{\alpha r^\mu}{\mu} \qquad (r\to\infty).$$

Now we consider

$$F(z, f)\equiv f^n-h_\alpha(z)f^{n-1}+h_\alpha(z)-e^{i\theta}=0 \qquad (\theta \text{ is irrational.}) \tag{4.2}$$

which is irreducible.

The irreducibility of this equation (4.2) can be showed the following:

Suppose that

$$F(z, f)\equiv A(z, f)B(z, f),$$

then

$$F(z, a_\nu)=A(z, a_\nu)B(z, a_\nu)=\text{const.}$$

Hence we can have two cases.

Case 1) $A(z, a_\nu)$, $B(z, a_\nu)$ are constants together.

Or

Case 2) $A(z, a_\nu)=c_1e^H$, $B(z, a_\nu)=c_2e^{-H}$,

where H is an entire function. For if $A(z, a_\nu)=0$ has zeros, then $B(z, a_\nu)=0$ must have poles, which contradicts to that $B(z, a_\nu)$ is an entire algebroid function.

Suppose that $p\leqq q$, then $A(z, f)$, $B(z, f)$ have exceptional values with respective numbers $2p-1$, $2q-1$ with exceptional values of first kind and second kind combined.

Hence, assume that $2p-1<n-1$, then a contradiction follows, we have

$$2p-1\geqq n-1$$

thus

$$2p\geqq n\,.$$

On the other hand $p\leqq q$ and $p+q=n$. This leads us to the following fact:

$$p=q=\frac{n}{2}\,.$$

This implies

$$2p-1=n-1\,.$$

Thus exceptional values of first kind number $2p-1$, $A(z, f)=0$ has the number $p-1$ of exceptional values of first kind. Consequently f has a positive integral order in view of the estimation of $K(f)$ by Toda.

Similarly we can deduce that the order of f is a positive integer from $B(z, f)=0$.

By the positive integrity of order of f we have a contradiction. This contradiction gives that the equation (4.2) is irreducible.

Let $f_\alpha(z)$ be an entire algebroid function defined by (4.2). Then we have the lower order of $f_\alpha(z)$ is $\mu=\lambda$. Denoting by $f_\nu(z)$ the ν-th determination of $f_\alpha(z)$ and noting that $-h_\alpha(z)=\Sigma f_\nu(z)$, we have $|h_\alpha(z)|\leqq n \max_{1\leqq\nu\leqq n}|f_\nu(z)|$ and consequently

$$\log^+ M(r,\ h_\alpha)\leqq\log^+ M(r,\ f_\alpha)+\log n\,. \tag{4.3}$$

Now by choosing a_ν suitably, for example

$$a_\nu=\exp(2\pi\nu i/n+\theta\imath/n)\,,\qquad \nu=1,\cdots,n\,.$$

we can say that the defining equation $g_\nu\equiv F(z,\ a_\nu)$ gives

$$N(r\,;a_\nu,\ f_\alpha)=\frac{1}{n}N(r\,;0,\ g_\nu)=\frac{1}{n}N(r\,;0,\ h_\alpha)\sim\frac{\alpha r^\mu}{n\mu}\qquad (r\to\infty) \tag{4.4}$$

for all ν. Therefore it follows from (4.1), (4.3) and (4.4) that

$$\frac{\sin\pi\mu}{\pi\mu}=\varlimsup_{r\to\infty}\frac{N(r\,;0,\ h_\alpha)}{\log M(r,\ h_\alpha)}\geqq\varlimsup_{r\to\infty}\frac{nN(r\,;a_\nu,\ f_\alpha)}{\log M(r,\ f_\alpha)}\,. \tag{4.5}$$

On the other hand we can get

$$\varlimsup_{r\to\infty}\frac{nN(r\,;a_\nu,\ f_\alpha)}{\log M(r,\ f_\alpha)}\geqq\frac{\sin\pi\mu}{\pi\mu}$$

in view of (4.4) combining with Theorem 1. Thus from (4.5) we obtain

$$\varlimsup_{r\to\infty}\frac{nN(r\,;a_\nu,\ f_\alpha)}{\log M(r,\ f_\alpha)}=\frac{\sin\pi\mu}{\pi\mu}\,,\qquad \nu=1,\cdots,n\,.$$

Therefore there exist n different finite numbers a_ν, $\nu=1,\cdots,n$, which always satisfy the result that equality holds in (1.2) respectively.

5. The proof of Theorem 2. Edrei-Williamson's argument does work on our case. We shall sketch very briefly the steps of the proof. First, we show that if equality holds in (1.2) for all a_ν ($\nu=1, \cdots, n$), then, for every fixed $\sigma>0$,

$$\overline{\lim_{m\to\infty}} \frac{\log M(\sigma r_m, f)}{\log M(r_m, f)}=\sigma^{\mu}. \tag{5.1}$$

Note that (3.4) implies that

$$\overline{\lim_{m\to\infty}} \frac{\log M(\sigma r_m, f)}{\log M(r_m, f)}\leqq\sigma^{\mu}. \tag{5.2}$$

Thus, if (5.1) were false, there would exist some $\sigma>0$, some δ $(0<\delta<1)$, and some unbounded sequence Λ of positive integers such that

$$\frac{\log M(\sigma r_m, f)}{\log M(r_m, f)}<\sigma^{\mu}\delta^{2\mu} \qquad (m\in\Lambda). \tag{5.3}$$

Now by (3.1), in place of (3.2),

$$\log M(r, f)\leqq r(\chi+\xi_m)\int_0^R \log M(t, f)\frac{dt}{(t+r_m)^2}+A\frac{r}{R}\log M(2R, f)+O(1)$$

in view of

$$\chi=\frac{\sin \pi\mu}{\pi\mu}=\overline{\lim_{r\to\infty}}\frac{nN(r\,;a_\nu, f)}{\log M(r, f)} \qquad (\nu=1, 2, \cdots, n),$$

where $\{\xi_m\}$ is a suitable sequence tending to zero as $m\to\infty$.

Similar to the case of (3.5), we obtain

$$\log M(r_m, f)\leqq r_m(\chi+\xi_m)\int_{r'_m}^{r''_m} \log M(t, f)\frac{dt}{(t+r_m)^2}+o(\log M(r_m, f)) \quad (m\to\infty) \tag{5.4}$$

on using of the hypothesis of Theorem 2.

Let $I_1(r_m)$, $I_2(r_m)$, $I_3(r_m)$ respectively denote the portion of the integral in (5.4) over the intervals $[r'_m, \sigma\delta r_m]$, $[\sigma\delta r_m, \sigma r_m]$, $[\sigma r_m, r''_m]$. Then, by the same manner of the discussion [8],

$$I_1(r_m)+I_3(r_m)\leqq(1+\varepsilon_m)\log M(r_m, f)\left\{\int_0^{\sigma\delta}+\int_\sigma^\infty\right\}x^{\mu}\frac{dx}{(x+1)^2}.$$

And (5.3) implies that, for $m\in\Lambda$,

$$I_2(r_m)<\log M(r_m, f)\delta^{2\mu}\sigma^{\mu}\int_{\sigma\delta}^{\sigma}\frac{dx}{(x+1)^2}.$$

Thus, for $m\in\Lambda$, by combining the above two inequalities, (5.4) yields

$$1+o(1)\leqq(1+\varepsilon_m)(\chi+\xi_m)\left\{\frac{\pi\mu}{\sin \pi\mu}-K\right\},$$

where

$$K=\sigma^{\mu}\delta^{\mu}(1-\delta^{\mu})\int_{\sigma\delta}^{\sigma}\frac{dx}{(x+1)^2}>0.$$

Letting $m\to\infty$ $(m\in\Lambda)$ here, we find that

$$1\leqq\chi\left\{\frac{\pi\mu}{\sin\pi\mu}-K\right\},$$

which is a contradiction. Thus, assumption (5.3) is false and (5.1) true.

It now follows from (5.1) and Edrei's lemma [2] that there exist positive sequences $\{R'_m\}$, $\{R''_m\}$, $\{\tilde{\varepsilon}_m\}$ satisfying (1.4) such tnat for $R'_m\leqq t\leqq R''_m$ $(m>m_0)$, (1.5) is true.

Next, we show that

$$\lim_{m\to\infty}\frac{nN(r_m\,;a_\nu,f)}{\log M(r_m,f)}=\frac{\sin\pi\mu}{\pi\mu} \tag{5.5}$$

for some ν $(\nu=1, 2, \cdots, n)$. By the definition of χ,

$$\overline{\lim_{m\to\infty}}\frac{nN(r_m\,;a_\nu,f)}{\log M(r_m,f)}\leqq\chi.$$

Thus, if (5.5) is false there exists ε $(0<\varepsilon<\chi)$ and an unbounded sequence Λ of positive integers such that for all a_ν, $\nu=1, 2, \cdots, n$,

$$\frac{nN(r_m\,;a_\nu,f)}{\log M(r_m,f)}<\chi-\varepsilon \qquad (m\in\Lambda). \tag{5.6}$$

Hence applying the reasoning of [8, pp. 230~231] to (3.1) on setting $r=r_m$, $R=r''_m/2$, we can get

$$1\leqq1-\frac{\varepsilon}{3}\int_\omega^1 x^\mu\frac{dx}{(x+1)^2}, \qquad (m\to\infty\,;\ m\in\Lambda),$$

where $\omega=\{(\chi-\varepsilon)/(\chi-\varepsilon/2)\}^{1/\mu}$, which is contradiction. Thus, assertion (5.5) is true.

Now we make use of the same process as in [2]. Thus we have, by using (5.1) and (5.5),

$$\frac{nN(r_m\,;a_\nu,f)}{\log M(r_m,f)}(1+\tilde{\varepsilon}_m)^{-2}\leqq\frac{nN(t\,;a_\nu,f)}{\log M(t,f)}\leqq\frac{nN(r_m\,;a_\nu,f)}{\log M(r_m,f)}(1+\tilde{\varepsilon}_m)^2,$$

provided that t satisfies

$$R'_m\leqq t\leqq R''_m \qquad (m>m_0).$$

Consequently, (5.5) leads to (1.6).

The deduction of (1.7) from (1.6) is a straightforward Tauberian argument.

References

[1] Edrei, A., The deficiencies of meromorphic functions of finite lower order. Duke Math. J., **31** (1946), 1-22.

[2] Edrei, A., Locally tauberian theorems for meromorphic functions of lower order less than one. Trans. Amer. Math. Soc., **140** (1969), 309-332.

[3] Ozawa, M., On the growth of algebroid functions with several deficiencies, II. Kōdai Math. Sem. Rep. **22** (1970), 129-137.

[4] NEVANLINNA, R., Le théorème de Picard-Borel et la théorie des fonctions méromorphes. Paris (1929).

[5] VALIRON, G., Sur le minimum du module des fonctions entières d'order inférieur à un. Mathmatica, **11** (1935), 264-269.

[6] VALIRON, G., Sur la dérivée des fonctions algebroides. Bull. Soc. Math. **59** (1931), 17-39.

[7] WAHLUND, A., Uber einen Zusammenhang zwishen Maximalbetrage der ganzen Funktionen und seiner Grenze nach den Jansenschen Theoreme. Arkiv för Mat. Astr. Fys., 21 A, **23** (1929), 1-34.

[8] WILLIAMSON, J., The asymptotic behaviour of a class entire functions of entremal growth. J. London Math. Soc. (2), **7** (1973), 225-234.

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