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THE CURVATURE TENSORS OF $Sp(2)/SU(2)$ AND $SU(5)/Sp(2)\times S^1$ OF BERGER

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Introduction.

In Riemannian Geometry, one of the most interesting problems is to find all Einsteinian manifolds. But this problem is not so easy, because there exist Einsteinian manifolds of various sorts such as symmetric, homogeneous but not symmetric (cf. [8], [11], [12]) and non-homogeneous ones (cf. [2], [4], [13]). Then A. Besse has suggested the research for Einsteinian manifolds under the more restricted condition (see [3], p. 165),

$$(*)\qquad R_{ipqr}R_j{}^{pqr}=\text{constant}\ g_{ij}\,,$$

where $g=(g_{ij})$ is the Riemannian metric and $R=(R^i{}_{jkl})$ the curvature tensor. Its typical examples are an irreducible locally symmetric space and a harmonic Riemannian manifold, because the former satisfies $\nabla_k(R_{ipqr}R_j{}^{pqr})=0$ and the latter does

$$(**)\qquad R^p{}_{ijq}R^q{}_{klp}X^iX^jX^kX^l=\text{constant}^{1)}\,,$$

for any unit tangent vector $X=(X^i)$ (cf. [10]). Then we are interested in an Einsteinian manifold satisfying the condition (*), which is neither harmonic nor locally symmetric.

In this paper, we prove the following two theorems.

THEOREM 1. *$Sp(2)/SU(2)$ is an Einsteinian manifold which satisfies the condition (*), but does not the condition (**).*

THEOREM 2. *Let $M=G/H$ be a simply connected normal homogeneous Einsteinian manifold of strictly positive curvature, which satisfies the condition (**). Then M is homemorphic to a Riemannian symmetric space of rank one.*

Because M. Berger [1] has classified all simply connected normal homogeneous Riemannian manifolds of strictly positive curvature, and with the ex-

1) Prof. Berger and Prof. Vanhecke have informed me that P. Carpenter, A. Gray and T. J. Willmore have obtained some symmetric spaces satisfying (**).

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ceptions of two, viz, $Sp(2)/SU(2)$ and $SU(5)/Sp(2)\times S^1$, all are homeomorphic to a Riemannian symmetric space of rank one.

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1. A normal homogeneous Riemannian manifold.

G will always be a Lie group, H a closed subgroup, G/H the space of left cosets of H, $\pi: G\to G/H$ the natural projection. The Lie algebras of G and H will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. G/H is said to be a normal homogeneous Riemannian manifold if the metric on G/H is obtained as follows: Let there exists a positive definite inner product $\langle\,,\rangle$ on $\mathfrak{g}$ satisfying $\langle[X, Y], Z\rangle=\langle X, [Y, Z]\rangle$ for all $X, Y, Z\in\mathfrak{g}$, and let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$. Then the decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ is reductive, that is, $[\mathfrak{m}, \mathfrak{h}]\subset\mathfrak{m}$, and the restriction of the inner product to $\mathfrak{m}$, (which is identified with the tangent space at $\pi(H)$), induces a Riemannian metric on G/H by the action of G on G/H. The curvature tensors of normal homogeneous Riemannian manifolds are well known (see [9]); for $X, Y, Z\in\mathfrak{m}$,

$$(1.1)\qquad R(X, Y)Z=[[X, Y]_{\mathfrak{h}}, Z]+1/2[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} +1/4[[Y, Z]_{\mathfrak{m}}, X]_{\mathfrak{m}}+1/4[[Z, X]_{\mathfrak{m}}, Y]_{\mathfrak{m}},$$

where $[X, Y]_{\mathfrak{h}}$ (resp. $[X, Y]_{\mathfrak{m}}$) denotes the $\mathfrak{h}$ (resp. $\mathfrak{m}$)-components of $[X, Y]$.

Now choosing a basis $\{e_\alpha\}$ of $\mathfrak{m}$, we obtain, for $Y\in\mathfrak{m}$,

$$(1.2)\qquad R(e_\alpha, Y)Y=[[e_\alpha, Y]_{\mathfrak{h}}, Y]+1/4[[e_\alpha, Y]_{\mathfrak{m}}, Y]_{\mathfrak{m}},$$

from which

$$(1.3)\qquad \langle R(e_\alpha, Y)Y, e_\alpha\rangle=\langle[[e_\alpha, Y]_{\mathfrak{h}}, Y], e_\alpha\rangle+1/4\langle[e_\alpha, Y]_{\mathfrak{m}}, e_\alpha\rangle$$
$$=-\langle[e_\alpha, Y]_{\mathfrak{h}}, [e_\alpha, Y]_{\mathfrak{h}}\rangle-1/4\langle[e_\alpha, Y]_{\mathfrak{m}}, [e_\alpha, Y]_{\mathfrak{m}}\rangle,$$

since $\langle[X, Y], Z\rangle=\langle X, [Y, Z]\rangle$.

2. $Sp(2)/SU(2)$.

2.1. $Sp(2)$ is the sympletic 2-group and $SU(2)$ is the special unitary 2-group. Now an element of Lie algebra $sp(2)$ is skew-Hermitian of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ -\bar{a}_{12} & -a_{11} & \bar{a}_{14} & -\bar{a}_{13} \\ -\bar{a}_{13} & -a_{14} & a_{33} & a_{34} \\ -\bar{a}_{14} & a_{13} & -\bar{a}_{34} & -\bar{a}_{33} \end{pmatrix}$$

where a_{11}, a_{22} are pure imaginary, and the rest are arbitrary complex numbers. Let $P_1, P_2, P_3, Q_1, Q_2, Q_3, R_1, R_2, R_3, S_0$ be matrices in $sp(2)$ (cf. [5]) such that

$$P_1:\ a_{11}=-a_{33}=i, \qquad \text{otherwise } a_{ij}=0,$$
$$P_2:\ a_{13}=-a_{31}=1, \qquad \text{otherwise } a_{ij}=0,$$
$$P_3:\ a_{13}=a_{31}=i, \qquad \text{otherwise } a_{ij}=0,$$
$$Q_1:\ a_{22}=-a_{44}=i, \qquad \text{otherwise } a_{ij}=0,$$
$$Q_2:\ a_{24}=-a_{42}=1, \qquad \text{otherwise } a_{ij}=0,$$
$$Q_3:\ a_{24}=-a_{42}=i, \qquad \text{otherwise } a_{ij}=0,$$
$$R_1:\ a_{12}=a_{21}=-a_{34}=-a_{43}=i, \qquad \text{otherwise } a_{ij}=0,$$
$$R_2:\ a_{14}=a_{23}=-a_{32}=-a_{41}=1, \qquad \text{otherwise } a_{ij}=0,$$
$$R_3:\ a_{14}=a_{23}=a_{32}=a_{41}=i, \qquad \text{otherwise } a_{ij}=0,$$
$$S_0:\ a_{12}=-a_{21}=a_{34}=-a_{43}=1, \qquad \text{otherwise } a_{ij}=0,$$

For completeness, we list the Lie multiplication table: We denote a permutation of numbers 1, 2, 3 by β and its signum by $\mathrm{sgn}(\beta)$. Then we have

$$(2.1)\qquad [P_{\beta(1)}, P_{\beta(2)}]=2\,\mathrm{sgn}(\beta)P_{\beta(3)}, \quad [Q_{\beta(1)}, Q_{\beta(2)}]=2\,\mathrm{sgn}(\beta)Q_{\beta(3)},$$
$$[P_{\beta(1)}, R_{\beta(2)}]=\mathrm{sgn}(\beta)R_{\beta(3)}, \quad [Q_{\beta(1)}, R_{\beta(2)}]=\mathrm{sgn}(\beta)R_{\beta(3)},$$
$$[R_{\beta(1)}, R_{\beta(2)}]=2\,\mathrm{sgn}(\beta)(Q_{\beta(3)}+P_{\beta(3)}), \quad [P_j, Q_k]=0, \quad j, k=1, 2, 3,$$
$$[R_j, S_0]=2(Q_j-P_j), \quad [P_j, S_0]=R_j, \quad [Q_j, S_0]=-R_j,$$
$$[P_j, R_j]=-S_0, \quad [Q_j, R_j]=S_0.$$

Setting

$$H_1=3/2P_1+1/2Q_1, \quad H_2=Q_2+\sqrt{3}/2S_0, \quad H_3=Q_3+\sqrt{3}/2R_1,$$
$$E_1=1/2P_1-3/2Q_1, \quad E_2=\sqrt{3/2}\,Q_2-1/\sqrt{2}\,S_0, \quad E_3=\sqrt{3/2}\,Q_3-1/\sqrt{2}\,R_1,$$
$$E_4=\sqrt{5/2}\,P_2, \quad E_5=\sqrt{5/2}\,P_3, \quad E_6=\sqrt{5}/2R_2, \quad E_7=\sqrt{5}/2R_3,$$

we can get the Lie multiplication table:

$$(2.2)\qquad [H_1, H_2]=H_3, \quad [H_1, H_3]=-H_2, \quad [H_2, H_3]=H_1,$$
$$[H_1, E_1]=0, \quad [H_1, E_2]=E_3, \quad [H_1, E_3]=-E_2,$$
$$[H_1, E_4]=3E_5, \quad [H_1, E_5]=-E_4, \quad [H_1, E_6]=2E_7,$$
$$[H_1, E_7]=-2E_6, \quad [H_2, E_1]=\sqrt{6}E_3, \quad [H_1, E_2]=\sqrt{5/2}E_6,$$

$[H_2, E_3]=-\sqrt{6}E_1+\sqrt{5/2}E_7$, $[H_2, E_4]=-\sqrt{3/2}E_6$,

$[H_2, E_5]=-\sqrt{3/2}E_7$, $[H_2, H_6]=-\sqrt{5/2}E_2+\sqrt{3/2}E_4$,

$[H_2, E_7]=-\sqrt{5/2}E_3+\sqrt{3/2}E_5$, $[H_3, E_1]=-\sqrt{6}E_2$,

$[H_3, E_2]=\sqrt{6}E_1+\sqrt{5/2}E_7$, $[H_3, E_3]=-\sqrt{5/2}E_6$,

$[H_3, E_4]=\sqrt{3/2}E_7$, $[H_3, E_5]=-\sqrt{3/2}E_6$,

$[H_3, E_6]=\sqrt{5/2}E_3+\sqrt{3/2}E_5$, $[H_3, E_7]=-\sqrt{5/2}E_2-\sqrt{3/2}E_4$,

$[E_1, E_2]=-E_3-\sqrt{6}H_3$, $[E_1, E_3]=E_2+\sqrt{6}H_2$,

$[E_1, E_4]=E_5$, $[E_1, E_5]=-E_4$, $[E_1, E_6]=-E_7$,

$[E_1, E_7]=E_6$, $[E_2, E_3]=-E_1+H_1$, $[E_2, E_4]=E_6$,

$[E_2, E_5]=E_7$, $[E_2, E_6]=-E_4+\sqrt{5/2}H_2$

$[E_2, E_7]=-E_5+\sqrt{5/2}H_3$, $[E_3, E_4]=-E_7$,

$[E_3, E_5]=E_6$, $[E_3, E_6]=-E_5-\sqrt{5/2}H_3$,

$[E_3, E_7]=E_4+\sqrt{5/2}H_2$, $[E_4, E_5]=E_1+3H_1$,

$[E_4, E_6]=E_2-\sqrt{3/2}H_2$, $[E_4, E_7]=-E_3+\sqrt{3/2}H_3$,

$[E_5, E_6]=E_3-\sqrt{3/2}H_3$, $[E_5, E_7]=E_2-\sqrt{3/2}H_2$,

$[E_6, E_7]=-E_1+2H_1$.

Here we note: (i) $\{H_1, H_2, H_3, E_1, \cdots, E_7\}$ are linearly independent and therefore form a basis of $sp(2)$. (ii) Furthermore, if as an inner product on $sp(2)$ we take $\langle X, Y\rangle=-(1/5)$ trace (XY), then $\{H_1, H_2, H_3, E_1, \cdots, E_7\}$ is an orthonormal basis of $sp(2)$ and also (iii) the inner product is invariant under $Ad(Sp(2))$. (iv) Finally, one can show that $su(2)=$linear span$\{H_1, H_2, H_3\}$.

2.2. By (1.1) and (2.2), we calculate $R(E_\alpha, E_\beta)E_\gamma$:

(2.3) $R(E_1, E_2)E_1=25/4E_2$, $R(E_1, E_2)E_2=-25/4E_1-\sqrt{15}E_7$,

$R(E_1, E_2)E_3=\sqrt{15}E_6$, $R(E_1, E_2)E_4=-2E_7$,

$R(E_1, E_2)E_5=2E_6$, $R(E_1, E_2)E_6=-\sqrt{15}E_3-2E_5$,

$R(E_1, E_2)E_7=\sqrt{15}E_2+2E_4$.

(2.4) $R(E_1, E_3)E_1=25/4E_3$, $R(E_1, E_3)E_2=\sqrt{15}E_6$,

$R(E_1, E_3)E_3=-25/4E_1+\sqrt{15}E_7$, $R(E_1, E_3)E_4=-2E_6$,

$R(E_1, E_3)E_5=-2E_7$, $R(E_1, E_3)E_6=-\sqrt{15}E_2+2E_4$,

$R(E_1, E_3)E_7=-\sqrt{15}E_3-2E_5$.

(2.5) $R_1(E_1, E_4)E_1=1/4E_4$, $R(E_1, E_4)E_2=-E_7$,

$R(E_1, E_4)E_3=-E_6$, $R(E_1, E_4)E_4=-1/4E_1$,

$R(E_1, E_4)E_5=0$, $R(E_1, E_4)E_6=E_3$,

$R(E_1, E_4)E_7=E_2$.

(2.6) $R(E_1, E_5)E_1=1/4E_5$, $R(E_1, E_5)E_2=E_6$,

$R(E_1, E_5)E_3=-E_7$, $R(E_1, E_5)E_4=0$,

$R(E_1, E_5)E_5=-1/4E_1$, $R(E_1, E_5)E_6=-E_2$,

$R(E_1, E_5)E_7=E_3$.

(2.7) $R(E_1, E_6)E_1=1/4E_6$, $R(E_1, E_6)E_2=-E_5$,

$R(E_1, E_6)E_3=E_4$, $R(E_1, E_6)E_4=-E_3$,

$R(E_1, E_6)E_5=E_2$, $R(E_1, E_6)E_6=-1/4E_1$,

$R(E_1, E_6)E_7=0$.

(2.8) $R(E_1, E_7)E_1=1/4E_7$, $R(E_1, E_7)E_2=E_4$,

$R(E_1, E_7)E_3=E_5$, $R(E_1, E_7)E_4=-E_2$,

$R(E_1, E_7)E_5=-E_3$, $R(E_1, E_7)E_6=0$,

$R(E_1, E_7)E_7=-1/4E_1$.

(2.9) $R(E_2, E_3)E_1=0$, $R(E_2, E_3)E_2=5/4E_3$,

$R(E_2, E_3)E_3=-5/4E_2$, $R(E_2, E_3)E_4=2E_5$,

$R(E_2, E_3)E_5=-2E_4$, $R(E_2, E_3)E_6=3E_7$,

$R(E_2, E_3)E_7=-3E_6$.

(2.10) $R(E_2, E_4)E_1=E_7$, $R(E_2, E_4)E_2=1/4E_4$,

$R(E_2, E_4)E_3=E_5$, $R(E_2, E_4)E_4=-1/4E_2$,

$R(E_2, E_4)E_5=-E_3$, $R(E_2, E_4)E_6=0$,

$R(E_2, E_4)E_7=-E_1$.

(2.11) $R(E_2, E_5)E_1=-E_6$, $R(E_2, E_5)E_2=1/4E_5$,

$R(E_2, E_5)E_3=-E_4$, $R(E_2, E_5)E_4=E_3$,

$R(E_2, E_5)E_5=-1/4E_2$ $\quad R(E_2, E_5)E_6=E_1,$

$R(E_2, E_5)E_7=0.$

(2.12) $R(E_2, E_6)E_1=\sqrt{15}E_3+E_5,$ $\quad R(E_2, E_6)E_2=11/4E_6,$

$R(E_2, E_6)E_3=-\sqrt{15}E_1+3/2E_7,$ $\quad R(E_2, E_6)E_4=-\sqrt{15}/2E_6,$

$R(E_2, E_6)E_5=-\sqrt{15}/2E_7-E_1,$ $\quad R(E_2, E_6)E_6=-11/4E_2+\sqrt{15}/2E_4,$

$R(E_2, E_6)E_7=-3/2E_3+\sqrt{15}/2E_5.$

(2.13) $R(E_2, E_7)E_1=-\sqrt{15}E_2-E_4,$ $\quad R(E_2, E_7)E_2=\sqrt{15}E_1+11/4E_7,$

$R(E_2, E_7)E_3=-3/2E_6,$ $\quad R(E_2, E_7)E_4=E_1+\sqrt{15}/2E_7,$

$R(E_2, E_7)E_5=-\sqrt{15}/2E_6,$ $\quad R(E_2, E_7)E_6=3/2E_3+\sqrt{15}/2E_5,$

$R(E_2, E_7)E_7=-11/4E_2-\sqrt{15}/2E_4.$

(2.14) $R(E_3, E_4)E_1=E_6,$ $\quad R(E_3, E_4)E_2=-E_5,$

$R(E_3, E_4)E_3=1/4E_4,$ $\quad R(E_3, E_4)E_4=-1/4E_3,$

$R(E_3, E_4)E_5=E_2,$ $\quad R(E_3, E_4)E_5=-E_1,$

$R(E_3, E_4)E_7=0.$

(2.15) $R(E_3, E_5)E_1=E_7,$ $\quad R(E_3, E_5)E_2=E_4,$

$R(E_3, E_5)E_3=1/4E_5,$ $\quad R(E_3, E_5)E_4=-E_2,$

$R(E_3, E_5)E_5=-1/4E_3,$ $\quad R(E_3, E_5)E_6=0,$

$R(E_3, E_5)E_7=-E_1.$

(2.16) $R(E_3, E_6)E_1=\sqrt{15}E_2-E_4,$ $\quad R(E_3, E_6)E_2=-\sqrt{15}E_1-3/2E_7,$

$R(E_3, E_6)E_3=11/4E_6,$ $\quad R(E_3, E_6)E_4=-\sqrt{15}E_7+E_1,$

$R(E_3, E_6)E_5=\sqrt{15}/2E_6,$ $\quad R(E_2, E_6)E_6=-11/4E_3-15/2E_5,$

$R(E_3, E_6)E_7=3/2E_2+\sqrt{15}/2E_4.$

(2.17) $R(E_3, E_7)E_1=\sqrt{15}E_3+E_5,$ $\quad R(E_3, E_7)E_2=3/2E_6,$

$R(E_3, E_7)E_3=-\sqrt{15}E_1+11/4E_7,$ $\quad R(E_3, E_7)E_4=-\sqrt{15}/2E_6,$

$R(E_3, E_7)E_5=E_1-\sqrt{15}E_7,$ $\quad R(E_3, E_7)E_6=-3/2E_2+\sqrt{15}/2E_4,$

$R(E_3, E_7)E_7=-11/4E_3+\sqrt{15}/2E_5.$

(2.18) $R(E_4, E_5)E_1=0,$ $\quad R(E_4, E_5)E_2=2E_3,$

$R(E_4, E_5)E_3=-2E_2$, $R(E_4, E_5)E_4=37/4E_5$,

$R(E_4, E_5)E_5=-37/4E_4$, $R(E_4, E_5)E_6=5E_7$,

$R(E_4, E_5)E_7=-5E_6$.

(2.19) $R(E_4, E_6)E_1=-2E_3$, $R(E_4, E_6)E_2=-\sqrt{15}/2E_6$,

$R(E_4, E_6)E_3=2E_1-\sqrt{15}/2E_7$, $R(E_4, E_6)E_4=7/4E_6$,

$R(E_4, E_6)E_5=5/2E_7$, $R(E_4, E_6)E_6=\sqrt{15}/2E_2-7/4E_4$,

$R(E_4, E_6)E_7=\sqrt{15}/2E_2-5/2E_5$.

(2.20) $R(E_4, E_7)E_1=-2E_2$, $R(E_4, E_7)E_2=2E_1+\sqrt{15}/2E_7$,

$R(E_4, E_7)E_3=-\sqrt{15}/2E_6$, $R(E_4, E_7)E_4=7/4E_7$,

$R(E_4, E_7)E_5=-5/2E_6$, $R(E_4, E_7)E_6=\sqrt{15}/2E_3+5/2E_5$,

$R(E_4, E_7)E_7=-\sqrt{15}/2E_2-7/4E_4$.

(2.21) $R(E_5, E_6)E_1=2E_2$, $R(E_5, E_6)E_2=-2E_1-\sqrt{15}/2E_7$,

$R(E_5, E_6)E_3=\sqrt{15}/2E_6$, $R(E_5, E_6)E_5=-5/2E_7$,

$R(E_5, E_6)E_5=7/4E_6$, $R(E_5, E_6)E_6=-\sqrt{15}/2E_3-7/4E_5$,

$R(E_5, E_6)E_7=\sqrt{15}/2E_2+5/2E_4$.

(2.22) $R(E_5, E_7)E_1=-2E_3$, $R(E_5, E_7)E_2=-\sqrt{15}/2E_6$,

$R(E_5, E_7)E_3=2E_1-\sqrt{15}/2E_7$, $R(E_5, E_7)E_4=5/2E_6$,

$R(E_5, E_7)E_5=7/4E_7$, $R(E_5, E_7)E_6=\sqrt{15}/2E_2-5/2E_4$,

$R(E_5, E_7)E_7=\sqrt{15}/2E_3-7/4E_5$.

(2.23) $R(E_6, E_7)E_1=0$, $R(E_6, E_7)E_2=3E_3$,

$R(E_6, E_7)E_3=-3E_2$, $R(E_6, E_7)E_4=5E_5$,

$R(E_6, E_7)E_5=-5E_4$, $R(E_6, E_7)E_6=17/4E_7$,

$R(E_6, E_7)E_7=-17/4E_6$.

2.3. For the computation of Ricci curvature, we first give $[E_\alpha, Y]$ ($\alpha=1, \cdots, 7$), $Y=\sum a_\beta E_\beta \in (su(2))^\perp$: By (2.2), we have

$$[E_1, Y]=\sqrt{6}a_3H_2-\sqrt{6}a_2H_3$$

$$+a_3E_2-a_2E_3-a_5E_4+a_4E_5+a_7E_6-a_6E_7,$$

$$[E_2, Y]=a_3H_1+\sqrt{5/2}a_6H_2+(\sqrt{6}a_1+\sqrt{5/2}a_7)H_3$$
$$-a_3E_1+a_1E_3-a_6E_4-a_7E_5+a_4E_6+a_5E_7\,,$$
$$[E_3, Y]=-a_2H_1+(-\sqrt{6}a_1+\sqrt{5/2}a_7)H_2-\sqrt{5/2}a_6H_3$$
$$+a_2E_1-a_1E_2+a_7E_4-a_6E_5+a_5E_6-a_4E_7\,,$$
$$[E_4, Y]=3a_5H_1-\sqrt{3/2}a_6H_2+\sqrt{3/2}a_7H_3$$
$$+a_5E_1-a_6E_2-a_7E_3-a_1E_5-a_2E_6+a_3E_7\,,$$
$$[E_5, Y]=-3a_1H_1-\sqrt{3/2}a_7H_2-\sqrt{3/2}a_6H_3$$
$$-a_4E_1+a_7E_2-a_6E_3+a_1E_4-a_3E_6-a_2E_7\,,$$
$$[E_6, Y]=2a_7H_1+(-\sqrt{5/2}a_2+\sqrt{3/2}a_4)H_2+(\sqrt{5/2}a_3+\sqrt{3/2}a_5)H_3$$
$$-a_7E_1-a_4E_2-a_5E_3+a_2E_4+a_3E_5+a_1E_7\,,$$
$$[E_7, Y]=-2a_6H_1+(\sqrt{3/2}a_5-\sqrt{5/2}a_3)H_2+(-\sqrt{5/2}a_2-\sqrt{3/2}a_4)H_3$$
$$+a_6E_1-a_5E_2+a_4E_3-a_3E_4+a_2E_5-a_1E_3\,.$$

Hence by using (1.3), we have

$$\begin{aligned}\mathrm{Ric}\,(Y, Y)&=-\sum\langle R(E_\alpha, Y)Y, E_\alpha\rangle\\&=\sum\{\langle E_\alpha, Y]_{\mathfrak{h}}, [E_\alpha, Y]_{\mathfrak{h}}\rangle+1/4\langle[E_\alpha, Y]_{\mathfrak{m}}, [E_\alpha, Y]_{\mathfrak{m}}\rangle\}\\&=12\sum a_\alpha^2+(27/2)\sum a_\alpha^2=(51/2)\langle Y, Y\rangle\,.\end{aligned}$$

This shows that $Sp(2)/SU(2)$ is Einsteinian.[2)]

Next, after a long calculation using all equations of subsection 2.2, we obtain

$$\sum\langle R(E_\alpha, E_\gamma)E_\delta, R(E_\beta, E_\gamma)E_\delta\rangle=(1299/4)\delta_{\alpha\beta}$$

This concludes that $Sp(2)/SU(2)$ satisfies the condition (*).

Finally, we note that $Sp(2)/SU(2)$ does not satisfy the condition (**), because for example, we have

$$\sum\langle R(E_\alpha, E_1)E_1, R(E_\alpha, E_1)E_1\rangle=627/8$$

and

$$\sum\langle R(E_\alpha, E_7)E_7, R(E_\alpha, E_7)E_7\rangle=435/8\,,$$

taking account of a part of the equations of subsection 2.2. Thus, summing up these facts, we have

2) Prof. Berger has pointed out that this fact was shown by D'Atri and Ziller ([6]). I wish to express my thanks to him.

THEOREM 1. *$Sp(2)/SU(2)$ is an Einsteinian manifold which satisfies the condition (*), but does not the condition (**).*

Remark. The pinching (1/37) of $Sp(2)/SU(2)$ has been calculated by H. I. Eliasson [7].

3. $SU(5)/Sp(2)\times S^1$.

We show that $SU(5)/Sp(2)\times S^1$ is not Einsteinian. $SU(5)$ is the special unitery 5-group and $Sp(2)$ is the sympletic 2-group. Then we denote the Lie algebra of $SU(5)$ by $su(5)$ and the Lie algebra of $Sp(2)\times S^1$ by $sp(2)+\boldsymbol{R}$.

Let ε_{jk} be the matrix with a 1 in the jth row and kth column and 0 elsewhere: $\varepsilon_{jk}=(\delta_{jr}\delta_{ks})$. Then setting

$$i^2=-1,\quad A_{jk}=i(\varepsilon_{jj}-\varepsilon_{kk}),\quad B_{jk}=\varepsilon_{jk}-\varepsilon_{kj},\quad C_{jk}=i(\varepsilon_{jk}+\varepsilon_{kj}),$$

we have

$$[A_{rj}, A_{kl}]=-\delta_{rk}B_{rk}+\delta_{rl}B_{rl}+\delta_{jk}B_{jk}-\delta_{jl}B_{jl},$$

$$[A_{rj}, B_{kl}]=\delta_{rk}C_{rl}-\delta_{rl}C_{rk}-\delta_{jk}C_{jl}+\delta_{jl}C_{jk},$$

$$[A_{rj}, C_{kl}]=-\delta_{rk}B_{rl}-\delta_{rl}B_{rk}+\delta_{jk}B_{jl}+\delta_{jl}B_{jk},$$

$$[B_{rj}, B_{rl}]=\delta_{jk}B_{rl}-\delta_{jl}B_{rk}-\delta_{rk}B_{jl}+\delta_{rl}B_{jk},$$

$$[B_{rj}, C_{kl}]=\delta_{jl}C_{rk}+\delta_{jk}C_{rl}-\delta_{rl}C_{jk}-\delta_{rk}C_{jl},$$

$$[C_{rj}, C_{kl}]=-\delta_{jk}B_{rl}-\delta_{jl}B_{rk}-\delta_{rk}B_{jl}-\delta_{rl}B_{jk}.$$

Furthermore, setting $\langle X, Y\rangle=-(1/2)\operatorname{trace}(XY)$, $X, Y\in su(5)$, we obtain an orthogonal decomposition (cf. [5]):

$$S_1=A_{12},\quad S_2=B_{12},\quad S_3=C_{12},\quad S_4=1/\sqrt{2}(B_{13}-B_{24})$$

$$S_5=1/\sqrt{2}(C_{13}+C_{24}),\quad S_6=1/\sqrt{2}(B_{14}+B_{23}),\quad S_7=1/\sqrt{2}(C_{14}-C_{23}),$$

$$S_8=A_{34},\quad S_9=B_{34},\quad S_{10}=C_{34},\quad S_{11}=1/\sqrt{3}(2A_{15}-A_{12}),$$

$$Q_1=\sqrt{3/2}\,A_{13}-1/\sqrt{6}\,A_{12}-1/\sqrt{6}\,A_{15},\quad Q_2=1/\sqrt{2}(B_{13}+B_{24}),$$

$$Q_3=1/\sqrt{2}(C_{13}-C_{24}),\quad Q_4=1/\sqrt{2}(B_{14}-B_{23}),\quad Q_5=1/\sqrt{2}(C_{14}+C_{23}),$$

$$Q_6=B_{15},\quad Q_7=C_{15},\quad Q_8=B_{25},\quad Q_9=C_{25},\quad Q_{10}=B_{35},$$

$$Q_{11}=C_{35},\quad Q_{12}=B_{45},\quad Q_{13}=C_{45},$$

where $S_1, \cdots, S_{11}$ span $sp(2)+\boldsymbol{R}$ and $Q_1, \cdots, Q_{13}$ span $(sp(2)\oplus\boldsymbol{R})^\perp$.

Now for the purpose of calculations of Ricci curvature $\operatorname{Ric}(Q_2, Q_2)$ and $\operatorname{Ric}(Q_{13}, Q_{13})$, we need the following identities:

(3.1) $[Q_1, Q_2]=5/2\sqrt{6}S_5+\sqrt{6}/4Q_3$, $[Q_2, Q_3]=S_1-S_8$.

$[Q_2, Q_4]=S_2-S_9$, $[Q_2, Q_5]=S_3-S_{10}$,

$[Q_2, Q_6]=-1\sqrt{2}Q_{10}$, $[Q_2, Q_7]=-1/\sqrt{2}Q_{11}$,

$[Q_2, Q_8]=-1\sqrt{2}Q_{12}$, $[Q_2, Q_9]=-1/\sqrt{2}Q_{13}$,

$[Q_2, Q_{10}]=1/\sqrt{2}Q_6$, $[Q_2, Q_{11}]=1/\sqrt{2}Q_7$,

$[Q_2, Q_{12}]=1/\sqrt{2}Q_8$, $[Q_2, Q_{13}]=1/\sqrt{2}Q_9$.

(3.2) $[Q_1, Q_{13}]=1/\sqrt{6}Q_{12}$, $[Q_3, Q_{13}]=1/\sqrt{2}Q_8$,

$[Q_4, Q_{13}]=1/\sqrt{2}Q_7$, $[Q_5, Q_{13}]=-1/\sqrt{2}Q_6$,

$[Q_6, Q_{13}]=1/\sqrt{2}(S_7+Q_5)$, $[Q_7, Q_{13}]=-1/\sqrt{2}(S_6+Q_4)$,

$[Q_8, Q_{13}]=1/\sqrt{2}(S_5-Q_3)$, $[Q_9, Q_{13}]=1/\sqrt{2}(S_4-Q_2)$,

$[Q_{10}, Q_{13}]=S_{10}$, $[Q_{11}, Q_{13}]=-S_9$,

$[Q_{12}, Q_{13}]=-S_8+1/\sqrt{3}S_{11}-\sqrt{2/3}Q_1$.

(1.3), (3.1) and (3.2) imply that

$$\mathrm{Ric}(Q_2, Q_2)\neq\mathrm{Ric}(Q_{13}, Q_{13}).$$

This shows that $SU(5)/Sp(2)\times S^1$ is not Einsteinian.

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