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A DUALITY RELATION FOR HARMONIC DIMENSIONS AND ITS APPLICATIONS

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Consider an *end* Ω in the sense of Heins [4]. Denote by $\mathcal{P}(\Omega)$ the class of nonnegative harmonic functions on Ω with vanishing boundary values on $\partial\Omega$. A nonzero function $h\in\mathcal{P}(\Omega)$ is said to be *minimal* if any $g\in\mathcal{P}(\Omega)$ dominated by h is a constant multiple of h. The cardinal number of normalized minimal functions is referred to as the *harmonic dimension* of Ω ([4]) which will be denoted by $\dim\mathcal{P}(\Omega)$.

In [4], Heins showed that there exists an end with any given integral harmonic dimension and asked whether there exist ends with infinite harmonic dimensions. Subsequently, the existence of ends Ω with $\dim\mathcal{P}(\Omega)=\mathcal{A}$ (the countablly infinite cardinal number) and with $\dim\mathcal{P}(\Omega)=\mathcal{C}$ (the cardinal number of continuum) were shown by Kuramochi [6] and Constantinescu-Cornea [1], respectively.

We are particularly interested in the following criterion of Heins [4]: *The harmonic dimension of Ω is one if and only if every bounded harmonic function on $\bar{\Omega}$ has a limit at the ideal boundary.* Motivated by this criterion we consider the quotient space $\mathcal{B}(\Omega)=HB(\bar{\Omega})/HB_0(\bar{\Omega})$ where $HB(\bar{\Omega})$ is the linear space of bounded harmonic functions on $\bar{\Omega}$ and $HB_0(\bar{\Omega})$ the subspace of $HB(\bar{\Omega})$ consisting of u such that u has the limit 0 at the ideal boundary. In terms of the dimension of the linear space $\mathcal{B}(\Omega)$, $\dim\mathcal{B}(\Omega)$ in notation, the above criterion may be restated as follows: $\dim\mathcal{P}(\Omega)=1$ *if and only if* $\dim\mathcal{B}(\Omega)=1$. The Heins criterion in this formulation can be generalized as follows which is the main achievement of the present paper:

THEOREM 1. *If either* $\dim\mathcal{P}(\Omega)$ *or* $\dim\mathcal{B}(\Omega)$ *is finite, then* $\dim\mathcal{P}(\Omega)=\dim\mathcal{B}(\Omega)$.

The proof will be given in no. 1.3 in a more general setting. Two applications, which may have their own interests, of Theorem 1 will be discussed in the rest. The first is concerned with a relation between harmonic dimensions and moduli conditions which, in a sense, generalizes a result in [4] p. 215. As the second application, an example of ends Ω with $\dim\mathcal{P}(\Omega)=\mathcal{A}$ will be given.

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1.1. A relatively noncompact subregion Ω of an open Riemann surface is referred to as an *end* ([4]) if Ω satisfies the following conditions: (i) the relative boundary $\partial\Omega$ consists of a finite number of analytic Jordan curves, (ii) there exist no nonconstant bounded harmonic functions on Ω with vanishing boundary values on $\partial\Omega$, (iii) Ω has a single ideal boundary component. In this section, let Ω be a relatively noncompact subregion satisfying the condition (i). Denote by β the ideal boundary of Ω. Without loss of generality, we may assume that there exists an open Riemann surface R with the exhaustion $\{R_n\}_{n=0}^{\infty}$ such that $\Omega=R-\bar{R}_0$. For $u\in HB(\bar{\Omega})$, let u_n be the harmonic function on $\Omega\cap R_n$ with boundary values $u|_{\partial\Omega}$ on $\partial\Omega$ and 0 on ∂R_n. Observe that $\lim_{n\to\infty}u_n$ exists and belongs to $HB(\bar{\Omega})$. Set $HB(\bar{\Omega}\,;\beta)=\{u\in HB(\bar{\Omega})\,;\, u=\lim_{n\to\infty}u_n\}$. For $v\equiv 1\in HB(\bar{\Omega})$, set $e=\lim_{n\to\infty}v_n$. Consider the linear space $B(\Omega\,;\beta)=\{u/e\,;\, u\in HB(\bar{\Omega}\,;\beta)\}$, its subspace $B_0(\Omega\,;\beta)=\{w\in B(\Omega\,;\beta)\,;\,\lim_{p\to\beta}w(p)=0\}$, and the quotient space $\mathscr{B}(\Omega\,;\beta)=B(\Omega\,;\beta)/B_0(\Omega\,;\beta)$. Denote by dim $\mathscr{B}(\Omega\,;\beta)$ the dimension of the linear space $\mathscr{B}(\Omega\,;\beta)$. Then we have the following duality relation (cf. [4], Hayashi [3], and Nakai [7]):

THEOREM 2. *If either* $\dim\mathscr{P}(\Omega)$ *or* $\dim\mathscr{B}(\Omega\,;\beta)$ *is finite, then* $\dim\mathscr{P}(\Omega)=\dim\mathscr{B}(\Omega\,;\beta)$.

The above theorem implies Theorem 1. In fact, if Ω satisfies the condition (ii) then $HB(\bar{\Omega}\,;\beta)=HB(\bar{\Omega})$ and $e\equiv 1$. Hence $B(\Omega\,;\beta)=HB(\bar{\Omega})$ and $B_0(\Omega\,;\beta)=HB_0(\bar{\Omega})$, and a fortiori $\mathscr{B}(\Omega\,;\beta)=\mathscr{B}(\Omega)$.

1.2. Consider the linear space $\mathscr{E}(\Omega)=\{h_1-h_2: h_1,\, h_2\in\mathscr{P}(\Omega)\}$ and the bilinear functional $(w, h)\mapsto\langle w, h\rangle=-\int_{\partial\Omega}w^*dh=\int_{\partial\Omega}w(\partial h/\partial n)ds$ on $B(\Omega\,;\beta)\times\mathscr{E}(\Omega)$ where $\partial/\partial n$ is the inner normal derivative. Set $Q_\Omega=\left\{\{h\in\mathscr{P}(\Omega)\,;\, -\int_{\partial\Omega}{}^*dh=1\right\}$.

LEMMA 1. *Every* $w\in B(\Omega\,;\beta)$ *satisfies the following equalities*:

(1) $$\limsup_{p\to\beta} w(p)=\sup\langle w, Q_\Omega\rangle\,,\qquad \liminf_{p\to\beta} w(p)=\inf\langle w, Q_\Omega\rangle\,.$$

Although the essence of the proof of this lemma is found in [3] and [4], we give the proof for the sake of completeness.

Given an arbitrary cluster value α of w at β and a sequence $\{p_n\}$ in Ω such that $\lim_{n\to\infty}p_n=\beta$ and $\lim_{n\to\infty}w(p_n)=\alpha$. Observe that

(2) $$u(p_n)=-\frac{1}{2\pi}\int_{\partial\Omega}u^*dg(\cdot,\, p_n)$$

for $u\in HB(\bar{\Omega}\,;\beta)$, where $g(\cdot,\, p_n)$ is the Green's function on Ω with pole p_n.

Applying (2) to ew and e, we see that $w(p_n)=-\int_{\partial\Omega} w^*d(g(\cdot, p_n)/2\pi e(p_n))$ and $1=-\int_{\partial\Omega} {}^*d(g(\cdot, p_n)/2\pi e(p_n))$. Therefore a subsequence of $\{g(\cdot, p_n)/2\pi e(p_n)\}$ has a limiting function g, which belongs to Q_Ω, and $\alpha=-\int_{\partial\Omega} w^*dg=\langle w, g\rangle$. Thus

$$\inf \langle w, Q_\Omega\rangle \leqq \liminf_{p\to\beta} w(p), \qquad \limsup_{p\to\beta} w(p) \leqq \sup \langle w, Q_\Omega\rangle.$$

Next, given an arbitrary $h\in Q_\Omega$ and let h_{mn} be the harmonic function on $R_m-\bar{R}_n$ $(m>n)$ with boundary values $h|_{\partial R_n}$ on R_n and 0 on ∂R_m. Set $h_n=\lim_{m\to\infty} h_{mn}$. Observe that

$$\left\langle \frac{u}{e}, h\right\rangle=-\int_{\partial\Omega} u^*dh=\int_{\partial R_n} u^*d(h-h_n) \tag{3}$$

for $u\in HB(\bar{\Omega}\,;\beta)$. Applying (3) to ew and e, we see that

$$\langle w, h\rangle=\int_{\partial R_n} ew^*d(h-h_n) \quad\text{and}\quad 1=\int_{\partial R_n} e^*d(h-h_n).$$

Since $h-h_n\geqq 0$ on $R-R_n$, this implies that $\inf_{\partial R_n} w \leqq \langle w, h\rangle \leqq \sup_{\partial R_n} w$ and a fortiori

$$\liminf_{p\to\beta} w(p) \leqq \inf \langle w, Q_\Omega\rangle, \qquad \sup \langle w, Q_\Omega\rangle \leqq \limsup_{p\to\beta} w(p).$$

This completes the proof.

1.3. Proof of Theorem 2. We first remark that the dimension of the linear space $\mathcal{E}(\Omega)$, $\dim \mathcal{E}(\Omega)$ in notation, coincides with $\dim \mathcal{P}(\Omega)$ if either $\dim \mathcal{E}(\Omega)$ or $\dim \mathcal{P}(\Omega)$ is finite (cf. e.g. [4]).

Consider the $\mathcal{E}(\Omega)$-kernel ($B(\Omega\,;\beta)$-kernel resp.)

$$K_1=\bigcap_{h\in\mathcal{E}(\Omega)} \{w\in B(\Omega\,;\beta)\,;\ \langle w, h\rangle=0\},$$

$$(K_2=\bigcap_{w\in B(\Omega;\beta)} \{h\in\mathcal{E}(\Omega)\,;\ \langle w, h\rangle=0\} \text{ resp.})$$

of the bilinear functional $(w, h)\mapsto\langle w, h\rangle$. By means of (1), we have that $K_1=B_0(\Omega\,;\beta)$. Let h be in K_2. Then $\langle w, h\rangle=\int_{\partial\Omega} w(\partial h/\partial n)ds=0$ for any $w\in B(\Omega\,;\beta)$ and hence $\partial h/\partial n\equiv 0$ on $\partial\Omega$. By the fact that $h\equiv 0$ on $\partial\Omega$, we conclude that $h\equiv 0$ on Ω and especially $K_2=\{0\}$. Therefore $\mathcal{B}(\Omega\,;\beta)=B(\Omega\,;\beta)/K_1$ and $\mathcal{E}(\Omega)=\mathcal{E}(\Omega)/K_2$ can be considered to be subspaces of $\mathcal{E}(\Omega)^*$ and $\mathcal{B}(\Omega\,;\beta)^*$ (conjugate spaces of $\mathcal{E}(\Omega)$ and $\mathcal{B}(\Omega\,;\beta)$) respectively and in particular

$$\dim \mathcal{B}(\Omega\,;\beta) \leqq \dim \mathcal{E}(\Omega)^*, \qquad \dim \mathcal{E}(\Omega) \leqq \dim \mathcal{B}(\Omega\,;\beta)^*.$$

Since finite dimensional linear spaces are isomorphic to their conjugate spaces, it follows from the above inequalities that $\dim \mathcal{B}(\Omega\,;\beta)=\dim \mathcal{E}(\Omega)=\dim \mathcal{P}(\Omega)$.

2.1. Let Ω be an end. As in no. 1.1, Ω can be considered to be a subregion of a null boundary Riemann surface R with a *normal* exhaustion $\{R_n\}_{n=0}^{\infty}$ (i. e. $R-\bar{R}_n$ has no relatively compact components) such that $\Omega=R-\bar{R}_0$. Denote by ω_n the harmonic measure of ∂R_{2n} with respect to $A_n=R_{2n}-\bar{R}_{2n-1}$. The modulus of A_n, mod A_n in notation, is the quantity $2\pi/\left(\int_{\partial\Omega_{2n}} {}^*d\omega_n\right)$. Consider the following conditions:

(A.1) *For every $n\in\boldsymbol{N}$, there exists a unique $N\in\boldsymbol{N}$ such that A_n consists of N disjoint annuli $A_{n1}, A_{n2}, \cdots, A_{nN}$;*

(A.2) $\sum_{n=1}^{\infty} \operatorname{mod} A_n=+\infty$.

Then we prove (cf. [4] and Kawamura [5])

THEOREM 3. *If $\{A_n\}$ satisfies* (A.1) *and* (A.2), *the harmonic dimension of Ω is at most N.*

2.2. Set $\mu_n=\operatorname{mod} A_n$. The function $z_n=x_n+iy_n=\mu_n(\omega_n+i\omega_n^*)$ (ω_n^* is the conjugate harmonic function of ω_n) maps $\bar{A}_n$, less suitable slits on which ω_n^* is constant, conformally into the horizontally sliced rectangle $\{x_n+iy_n\,;\ 0\leqq x_n\leqq\mu_n,\ 0\leqq y_n\leqq 2\pi\}$. Consider closed curves $l_{ni}(x_n)=\{p\in A_{ni}\,;\ \operatorname{Re} z_n(p)=x_n\}$ ($i=1, \cdots, N$; $0\leqq x_n\leqq\mu_n$) and set $l_n(x_n)=\bigcup_{i=1}^{N} l_{ni}(x_n)$. Given arbitrary $N+1$ functions $u_1, \cdots, u_{N+1}$ in $HB(\bar{\Omega})$. Denote by $\delta_{nij}(x_n)$ the oscillation of u_j on $l_{ni}(x_n)$ and set

$$\delta_n(x_n)=\sum_{i=1}^{N}\sum_{j=1}^{N+1}\delta_{nij}(x_n)\,.$$

We assume that $\delta_n(x_n)$ attains its minimum when $x_n=t_n$. Then we have

$$\delta_n=\delta_n(t_n)\leqq\sum_{j=1}^{N+1}\int_0^{2\pi}\left|\frac{\partial u_j}{\partial y_n}\right|dy_n \qquad (0\leqq x_n\leqq\mu_n)\,.$$

The Schwarz inequality yields

$$\delta_n^2\leqq 2\pi(N+1)\sum_{j=1}^{N+1}\int_0^{2\pi}\left|\frac{\partial u_j}{\partial y_n}\right|^2 dy_n\,.$$

Integrating both sides of the above from 0 to μ_n with respect to dx_n, we obtain

$$\delta_n^2\mu_n\leqq 2\pi(N+1)\sum_{j=1}^{N+1}\int_0^{\mu_n}\int_0^{2\pi}\left|\frac{\partial u_j}{\partial y_n}\right|^2 dx_n dy_n\leqq 2\pi(N+1)\sum_{j=1}^{N+1} D_{A_n}(u_j)\,,$$

where $D_{A_n}(u_j)$ denotes the Dirichlet integral of u_j on A_n. Since each u_j has the finite Dirichlet integral on Ω, we see that $\sum_{n=1}^{\infty}\delta_n^2\mu_n$ converges. By means of (A.2), this yields

$$\liminf_{n\to\infty}\delta_n=0\,. \tag{4}$$

2.3. By virtue of Theorem 1, we have only to show that there exists a nontrivial linear combination $\sum_{j=1}^{N+1} c_j u_j$ ($c_j\in\boldsymbol{R}$) belonging to $HB_0(\bar{\Omega})$, i.e.

$\dim \mathscr{B}(\Omega)\leqq N$.

By (4), we can find a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ and $c_{ij}\in \boldsymbol{R}$ $(i=1, \cdots, N;$ $j=1, \cdots, N+1)$ such that

(5) $$\lim_{k\to\infty}\Big(\max_{p\in l_{n_k}{}^{i}(t_{n_k})}|u_j(p)-c_{ij}|\Big)=0\,.$$

For $\boldsymbol{u}_j=(c_{1j}, \cdots, c_{Nj})\in \boldsymbol{R}^N$ $(j=1, \cdots, N+1)$, choose $(\alpha_1, \cdots, \alpha_{N+1})$ $(\neq(0, \cdots, 0))\in \boldsymbol{R}^{N+1}$ such that $\sum_{j=1}^{N+1}\alpha_j \boldsymbol{u}_j=(0, \cdots, 0)$. Then (5) yields

$$\lim_{k\to\infty}\Big(\max_{p\in l_{n_k}(t_{n_k})}\Big|\sum_{j=1}^{N+1}\alpha_j u_j(p)\Big|\Big)=0\,.$$

Since $l_n(t_n)$ separates $l_m(t_m)$ $(m=1, \cdots, n-1)$ from the ideal boundary β, this implies that $\sum_{j=1}^{N+1}\alpha_j u_j\in HB_0(\bar{\Omega})$.

3.1. Consider the mapping $(m, n)\mapsto\mu=\mu(m, n)=2^{m-1}(2n-1)$ of $\boldsymbol{N}^2$ to $\boldsymbol{N}$. It is clear that $(m, n)\mapsto\mu(m, n)$ is bijective, $\mu(m, n)\leqq\mu(m', n')$ if $m\leqq m'$ and $n\leqq n'$, and that $\mu(m, n)\to\infty$ if $m\to\infty$ or $n\to\infty$.

Let D_μ $(\mu=\mu(m, n)\in \boldsymbol{N})$ be the disk $\{|z-3\cdot 2^{2\mu-2}|<2^{2\mu-2}\}$ and S_μ a slit in D_μ. Set

$$R_0=\{1<|z|<\infty\}-\bigcup_{\mu=1}^{\infty}S_\mu\,,\qquad F_0=R_0-\bigcup_{\mu=1}^{\infty}D_\mu$$

and

$$R_n=\{|z|<\infty\}-\bigcup_{m=1}^{\infty}S_{\mu(m,n)}\,,\qquad F_n=R_n-\bigcup_{m=1}^{\infty}D_{\mu(m,n)}\qquad (n\in \boldsymbol{N})\,.$$

Denote by g_0 the Green's function on R_0 and by ω the bounded harmonic function on R_0 with boundary values 1 on $|z|=1$ and -1 on $\bigcup_{\mu=1}^{\infty}S_\mu$. By choosing S_μ sufficiently small we may assume that

(S.1) $$\limsup_{z\to\infty}\omega(z)>0$$

and

(S.2) $$\liminf_{z\to\infty,\, z\in F_0}g_0(\cdot, z)>0\,.$$

Join R_0 and R_n crosswise along $S_{\mu(m,n)}$ for every $(m, n)\in \boldsymbol{N}^2$. The resulting surface Ω is a covering surface of $\{|z|<\infty\}$ with the relative boundary $\partial\Omega=\{z\in R_0;\ |z|=1\}$. It is easily checked that Ω is an end. We will prove that *the harmonic dimension of Ω is $\aleph$.*

3.2. Let π be the projection of Ω. For an arbitrarily given $N\in \boldsymbol{N}$, set $\Omega_N=\Omega-\bigcup_{n=1}^{N}(R_n\cap\pi^{-1}(\{|z|\leqq 1\}))$ and $C_n=R_n\cap\pi^{-1}(\{|z|=1\})$ $(n=1, \cdots, N)$. Then Ω_N is a subend of Ω with the relative boundary $\partial\Omega_N=\partial\Omega\cup(\bigcup_{n=1}^{N}C_n)$. Consider harmonic measures w_n $(n=1, \cdots, N)$ of C_n with respect to Ω_N and an arbitrary nontrivial linear combination $w=\sum_{n=1}^{N}a_n w_n$ of $\{w_1, \cdots, w_N\}$. Choose

a_ι such that $|a_\iota|=\max\{|a_1|, \cdots, |a_N|\}(\neq 0)$. Observe that $w/a_\iota=1$ on C_ι and $w/a_\iota \geqq -1$ on $R_i\cap\pi^{-1}(\{|z|<1\})$. By means of (S.1) this implies that $\limsup_{p\to\beta} w(p)/a_\iota>0$, i. e. $w\notin HB_0(\bar{\Omega}_N)$. Hence, from Theorem 1, it follows that $\dim \mathcal{P}(\Omega_N)\geqq N$. Since $\dim \mathcal{P}(\Omega)=\dim \mathcal{P}(\Omega_N)$ (cf. [4]) and N is arbitrary, we conclude that $\dim \mathcal{P}(\Omega)\geqq \mathcal{A}$.

3.3. Consider the Martin compactification $\Omega^*=\Gamma\cup\bar{\Omega}$ of $\bar{\Omega}$ where Γ is the *Martin ideal boundary* of $\bar{\Omega}$. Denote by Δ the set of minimal points in Γ. In the theory of Martin compactification, it is well-known that $\dim \mathcal{P}(\Omega)$ coincides with $\sharp\Delta$ (the cardinal number of Δ). Let $\{\zeta_\iota\}$ be a sequence in Ω such that $\{\zeta_i\}$ converges to $q\in\Delta$. Then $k_q=\lim_{\iota\to\infty} g(\cdot, \zeta_\iota)$ is in $\mathcal{P}(\Omega)$ and minimal, where g is the Green's function on Ω. For a closed set F in Ω, let

$$(k_q)_F(\zeta)=\inf_{v\in\Phi(k_q, F)} v(\zeta)\,,$$

where $\Phi(k_q, F)$ is the class of nonnegative superharmonic functions v on Ω such that $v\geqq k_q$ on F except for a polar set.

LEMMA 2. *If U is a neighborhood of q, then $(k_q)_{\Omega-U}$ is a potential and moreover there exists a unique relatively noncompact component G of $U\cap\Omega$ such that $(k_q)_{\Omega-U}<k_q$ on G.*

For the proof we refer to e. g. Constantinescu-Cornea [2].

3.4. We are in the stage to show that $\dim \mathcal{P}(\Omega)$ does not exceed $\mathcal{A}$.

Consider two sequences $\{\zeta_\iota^{(j)}\}$ $(j=1, 2)$ in F_0 $(\subset\Omega)$ such that $\lim_{\iota\to\infty}\zeta_\iota^{(j)}=\beta$ (i. e. $\lim_{\iota\to\infty}\zeta_i^{(j)}=\infty$ in $\{|z|<\infty\}$). Choosing subsequences, if necessary, we may assume that there exist limiting functions $k_j=\lim_{\iota\to\infty} g(\cdot, \zeta_i^{(j)})$ and $h_j=\lim_{\iota\to\infty} g_0(\cdot, \zeta_i^{(j)})$ $(j=1, 2)$. From (S.2), it follows that h_1/h_2 is constant (cf. e. g. [2]). Setting $h_j\equiv 0$ on $\Omega-R_0$, h_j are subharmonic and $0\leqq h_j\leqq k_j$ on Ω. Hence there exist least harmonic majorants $\hat{h}_j$ of h_j. If k_j are minimal, then k_1/k_2 is constant since $0\leqq\hat{h}_j\leqq k_j$ and $\hat{h}_1/\hat{h}_2$ is constant. This implies that $Cl(F_0)\cap\Delta$ consists of at most a single point, where Cl denotes the closure in Ω^*. The similar argument yields that $\sharp(Cl(F_n)\cap\Delta)\leqq 1$ for every $n\in \boldsymbol{N}$. Consequently we see that

$$\sharp\left(\bigcup_{n=0}^{\infty}(Cl(F_n)\cap\Delta)\right)\leqq \mathcal{A}. \tag{6}$$

Next, suppose that there exists a $q\in\Delta-\Delta_1$ where $\Delta_1=\bigcup_{n=0}^{\infty}(Cl(F_n)\cap\Delta)$. Let $\{\zeta_i\}$ $(\subset\Omega)$ be a sequence converging to q and $k_q=\lim_{\iota\to\infty} g(\cdot, \zeta_\iota)$. Since Ω^*-F_0 is a neighborhood of q, by Lemma 2, there exists a unique component G of $\Omega-F_0$ such that $(k_q)_{F_0}<k_q$ on G. Set

$$G_n=\bar{R}_n\cup(R_0\cap\pi^{-1}(\bigcup_{m=1}^{\infty} D_{\mu(m,n)})) \qquad (n\in \boldsymbol{N})\,.$$

Observe that each G_n is a subregion of $\Omega-F_0$, $G_n \cap G_{n'}=\phi$ if $n \neq n'$, and that $\Omega-F_0=\bigcup_{n=1}^{\infty} G_n$. Hence $G=G_n$ for an $n \in \boldsymbol{N}$. Since $\Omega^*-(F_0 \cup F_n)$ is also a neighborhood of q, by Lemma 2, there exists a unique component G' of $\Omega-(F_0 \cup F_n)$ such that $(k_q)_{F_0 \cup F_n} < k_q$ on G'. From the fact that $(k_q)_{F_0} \leqq (k_q)_{F_0 \cup F_n}$, it follows that G' is a component of $G-F_n=G_n-F_n$. Observe that $G_n-F_n=(\bar{R}_n \cup R_0) \cap \pi^{-1}(\bigcup_{m=1}^{\infty} D_{\mu(m,n)})$ is a union of mutually disjoint relatively compact subregions. This contradicts the relative noncompactness of G'. Thus $\Delta=\Delta_1$ and therefore, by virtue of (6), we conclude that $\dim \mathscr{P}(\Omega)=\#\Delta=\#\Delta_1 \leqq \mathcal{A}$.

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