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KODAI MATH. J.
4 (1981), 457—479

AN EXTREMAL PROBLEM FOR SUBHARMONIC FUNCTIONS OF $\mu_* < 1/2$

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0. Introduction. Let f be an entire function. We denote the order and the lower order of f by λ and μ, respectively. And we set

$$m^*(r, f) = \min_{|z|=r} |f(z)|, \qquad M(r, f) = \max_{|z|=r} |f(z)|.$$

Then the classical $\cos \pi\lambda$ theorem of Valiron and Wiman asserts

$$\varlimsup_{r\to\infty} \frac{\log m^*(r, f)}{\log M(r, f)} \geqq \cos \pi\lambda, \tag{1}$$

provided that $0 \leqq \lambda < 1$. In 1960 Kjellberg [15] showed that if $0 \leqq \mu < 1$, then the above assertion (1) is valid with λ replaced by μ.

In [5], [6] Drasin and Shea considered those functions for which (1) is the best, and discussed the "global" asymptotic behavior of such functions. Their argument involves solving a convolution inequality. Baernstein [1] made use of their study on the convolution inequality to prove two theorems complementing the spread relation.

On the other hand, Edrei [9] also considered the extremal functions of the $\cos \pi\mu$ theorem, and discussed the "local" asymptotic behavior of such functions. His idea in [9] lies in adapting the work of Cartwright [2] on a sinusoidal indicator to his local one introduced in [8]. Further he showed in [9] that his method is applicable to the following extremal problem:

For meromorphic functions, assume $0 \leqq \mu < 1/2$, $k = \delta(\infty, f) - 1 + \cos \pi\mu > 0$. *Then*

$$\varlimsup_{r\to\infty} \frac{\log m^*(r, f)}{T(r, f)} \geqq k \cdot \frac{\pi\mu}{\sin \pi\mu}. \tag{2}$$

This inequality is best possible. The problem is to characterize those functions for which (2) is the best (See [9, Theorem 1].).

In connection with (2), it is natural to consider the quantity:

$$\varlimsup_{r\to\infty} \frac{\log m^*(r, f)}{m_2(r, f)}$$

Received May 6, 1980

for an entire or a meromorphic function f, where

$$m_2(r, f)=\left(\frac{1}{2\pi}\int_{-\pi}^{+\pi}\{\log|f(re^{i\theta})|\}^2 d\theta\right)^{1/2}.$$

In [19] we defined the local indicator for a sequence of subharmonic functions, and considered the above problem more generally for δ-subharmonic functions, that is, those functions $v(z)$ which can be represented as

$$v(z)=u^{(1)}(z)-u^{(2)}(z)\,, \tag{3}$$

where $u^{(1)}(z)$, $u^{(2)}(z)$ are subharmonic functions in $\boldsymbol{C}$. For a δ-subharmonic function (3), we put

$$m^*(r, v)=\inf_{|z|=r} v(z), \quad M(r, v)=\sup_{|z|=r} v(z), \quad N(r, v)=\frac{1}{2\pi}\int_{-\pi}^{+\pi} V(re^{i\theta})d\theta\,.$$

Then the characteristic function of v is defined by

$$T(r, v)=N(r, v^+)+N(r, u^{(2)})=N(r, \max(u^{(1)}, u^{(2)}))\,.$$

With the above $T(r, v)$, we consider the following four quantities:

$$\lambda=\varlimsup_{r\to\infty}\frac{\log T(r, v)}{\log r} \qquad \text{(the order of } v)\,,$$

$$\mu=\varliminf_{r\to\infty}\frac{\log T(r, v)}{\log r} \qquad \text{(the lower order of } v)\,,$$

$$\lambda_*=\sup\left\{\rho:\ \varlimsup_{A, r\to\infty}\frac{T(Ar, v)}{A^\rho T(r, v)}=\infty\right\} \quad \text{(the upper index of Pólya peaks for } v)\,.$$

$$\mu_*=\inf\left\{\rho:\ \varliminf_{A, r\to\infty}\frac{T(Ar, v)}{A^\rho T(r, v)}=0\right\} \quad \text{(the lower index of Pólya peaks for } v)\,.$$

It is easy to see that $\mu_*\leqq\mu\leqq\lambda\leqq\lambda_*$. Drasin and Shea [7] proved that the Pólya peaks of order ρ for v exist iff $\rho\in[\mu_*, \lambda_*]$, $\rho<\infty$. We remark that there exists a subharmonic function satisfying $\mu_*<\mu$ or $\lambda<\lambda_*$. Further we define $\delta(\infty, v)$ and $m_2(r, v)$ as follows:

$$\delta(\infty, v)=1-\varlimsup_{r\to\infty}\frac{N(r, u^{(2)})}{T(r, v)}, \qquad m_2(r, v)=\{N(r, v^2)\}^{1/2}\,.$$

One of our results in [19] can now be described.

THEOREM A. *Let v be δ-subharmonic defined by* (3). *Assume that* $\mu_*<1/2$ *and* $N(r, u^{(1)})\sim T(r, v)$ $(r\to\infty)$. *Further let ρ satisfy the following three conditions:*

(i) $\mu_*\leqq\rho\leqq\lambda_*$, (ii) $0\leqq\rho<1/2$,

(iii) $k_2(\rho)=\cos\pi\rho-1+\delta(\infty, v)/(2-\delta(\infty, v))>0$.

Then

$$\overline{\lim_{r\to\infty}}\frac{m^*(r, v)}{m_2(r, v)} \geqq \frac{k_2(\rho)}{\sqrt{1/2+\sin 2\pi\rho/4\pi\rho}} \equiv C_2(\rho). \tag{4}$$

In particular, if v is subharmonic, then the assumption: $N(r, u^{(1)}) \sim T(r, v)$ can be dropped.

If $\delta(\infty, v)=1$, the estimate (4) is best possible. And an elementary but somewhat lengthy computation shows that

$$\sup_{\substack{\mu_*\leqq\rho\leqq\lambda_* \\ \rho<1/2}} C_2(\rho)=C_2(\mu_*),$$

provided that $\delta(\infty, v)=1$.

In this paper we shall make use of Edrei's idea stated above to obtain the following result.

THEOREM. *Let $u(z)$ be a subharmonic function in $\boldsymbol{C}$ and have $\mu_*<1/2$. Assume that $u(z)$ satisfies*

$$\overline{\lim_{r\to\infty}}\frac{m^*(r, u)}{m_2(r, u)}=C_2(\mu_*). \tag{5}$$

Then there exists a positive, increasing, unbounded sequence $\{y_k\}_1^\infty$ having all the following properties:

I.

$$\lim_{k\to\infty}\frac{m^*(y_k, u)}{m_2(y_k, u)}=C_2(\mu_*), \tag{6}$$

$$\lim_{k\to\infty}\frac{N(y_k, u)}{m_2(y_k, u)}=\frac{\sin\pi\mu_*}{\pi\mu_*}\frac{1}{\sqrt{1/2+\sin 2\pi\mu_*/4\pi\mu_*}}\equiv C_1(\mu_*), \tag{7}$$

$$\lim_{k\to\infty}\frac{N(y_k, u)}{T(y_k, u)}=1. \tag{8}$$

II. *There also exist three positive sequences $\{y_k'\}_1^\infty$, $\{y_k'''\}_1^\infty$, $\{\varepsilon_k\}_1^\infty$ such that, as $k\to\infty$,*

$$y_k/y_k' \longrightarrow \infty, \qquad y_k'''/y_k \longrightarrow \infty, \qquad \varepsilon_k \longrightarrow 0, \tag{9}$$

and such that

$$y_k'\leqq r\leqq y_k''', \qquad k\geqq k_0 \tag{10}$$

implies

$$(1-\varepsilon_k)\left(\frac{r}{y_k}\right)^{\mu_*}\leqq\frac{N(r, u)}{N(y_k, u)}\leqq(1+\varepsilon_k)\left(\frac{r}{y_k}\right)^{\mu_*}, \tag{11}$$

$$(1-\varepsilon_k)\left(\frac{r}{y_k}\right)^{\mu_*} \leqq \frac{T(r, u)}{T(y_k, u)} \leqq (1+\varepsilon_k)\left(\frac{r}{y_k}\right)^{\mu_*}, \tag{12}$$

$$\frac{m_2(r, u)}{m_2(y_k, u)} \leqq (1+\varepsilon_k)\left(\frac{r}{y_k}\right)^{\mu_*}. \tag{13}$$

III. *Further if* $\mu_*>0$, *then there exist three sequences* $\{y_k''\}_1^\infty$, $\{\delta_k\}_1^\infty$, $\{\theta_k\}_1^\infty$ *such that as* $k\to\infty$,

$$y_k''/y_k \longrightarrow \infty, \qquad y_k'''/y_k'' \longrightarrow \infty, \qquad \delta_k \longrightarrow 0, \tag{14}$$

and such that

$$y_k'' \leqq r \leqq y_k''', \qquad k \geqq k_1 \tag{15}$$

implies

$$N(r, u; \mathcal{S}_k) < \varepsilon_k N(r, u), \tag{16}$$

where

$$\mathcal{S}_k = \{z : \delta_k \leqq \arg z - \theta_k \leqq 2\pi - \delta_k\}. \tag{17}$$

From the statement of I and II in Theorem, we can easily derive the following facts:

If $u(z)$ has $\mu_*<1/2$ and satisfies (5), then

$$\lim_{\substack{r\to\infty\\ r\in G}} \frac{N(Kr, u)}{N(r, u)} = K^{\mu_*}, \qquad \lim_{\substack{r\to\infty\\ r\in G}} \frac{T(Kr, u)}{T(r, u)} = K^{\mu_*}, \tag{18}$$

uniformly for K in any interval $A^{-1} \leqq K \leqq A$ $(A>1)$, with

$$G = \bigcup_{n=1}^{\infty} [a_n, b_n] \qquad (a_n\to\infty,\ b_n/a_n\to\infty).$$

Further,

$$\lim_{\substack{r\to\infty\\ r\in G}} \frac{N(r, u)}{T(r, u)} = 1, \tag{19}$$

and

$$\varliminf_{\substack{r\to\infty\\ r\in G}} \frac{N(r, u)}{m_2(r, u)} \geqq C_1(\mu_*) \tag{20}$$

hold.

The above estimates (18)—(20) are no longer true if we omit the restriction $r\in G$. To see this, we make use of the concept of a flexible proximate order which was introduced by Drasin [4].

Let $\lambda(r)$ $(r>0)$ be a continuous, nonnegative function which is continuously differentiable off a discrete set D, such that

$$r\lambda'(r) \longrightarrow 0 \qquad (r\to\infty,\ r\notin D)\,. \tag{21}$$

Let E and E_1 be sets of the form

$$E=\bigcup_{n=1}^{\infty}[a_n,\ b_n]\,, \qquad E_1=\bigcup_{n=1}^{\infty}[k_n^{-1}a_n,\ k_nb_n]\,, \tag{22}$$

where

$$(1<)k_n\uparrow\infty \quad (n\to\infty)\,, \qquad [k_n^{-1}a_n,\ k_nb_n]\cap[k_m^{-1}a_m,\ k_mb_m]=\emptyset \quad (m\neq n)\,,$$

$$\int_{E_1\cap[1,r]} t^{-1}dt=o(\log r) \qquad (r\to\infty)\,. \tag{23}$$

Now, suppose that $\lambda(r)$ satisfies

$$0<\rho_1\leqq\lambda(r)\leqq\rho_2<1\,, \tag{24}$$

$$\lambda(r)=\begin{cases}\rho_1 \quad (<1/2) & (r\in\bigcup_{n=1}^{\infty}[a_{2n},\ b_{2n}])\,,\\ \rho_2 & (r\in\bigcup_{n=1}^{\infty}[a_{2n-1},\ b_{2n-1}])\,,\\ \rho \quad (\rho_1<\rho<\rho_2) & (r\notin E_1)\,,\end{cases} \tag{25}$$

and let $\lambda(r)$ be extended to E_1-E so that it is continuous and

$$t\lambda'(t)=\begin{cases}-(\rho-\rho_1)/\log k_{2n} & t\in(k_{2n}^{-1}a_{2n},\ a_{2n})\,,\\ (\rho_2-\rho)/\log k_{2n-1} & t\in(k_{2n-1}^{-1}a_{2n-1},\ a_{2n-1})\,,\\ (\rho-\rho_1)/\log k_{2n} & t\in(b_{2n},\ k_{2n}b_{2n})\,,\\ -(\rho_2-\rho)/\log k_{2n-1} & t\in(b_{2n-1},\ k_{2n-1}b_{2n-1})\,.\end{cases}$$

Then (21) holds, and by (23), (25) it is clear that

$$(\log r)^{-1}\int_1^r\lambda(t)t^{-1}dt \longrightarrow \rho \qquad (r\to\infty)\,. \tag{26}$$

Let $f(z)$ be a canonical product with negative zeros with counting function

$$n(r)=\left[\exp\left(\int_1^r\lambda(t)t^{-1}dt\right)\right]. \tag{27}$$

Then (26) implies that f is of order ρ (<1) (cf. [2, Theorem 1.11.]) and so for a suitable branch of $\log f(z)$

$$\log f(z)=z\int_0^{\infty}\frac{n(t)}{t(t+z)}dt \qquad (|\arg z|<\pi)\,. \tag{28}$$

Using the reasoning of the proof of Proposition in [4, p. 133], we have from (21), (24), (25), (26) and (28)

$$\log f(z)=\left\{\frac{\pi}{\sin \pi\lambda(r)} e^{i\lambda(r)\theta}+o(1)\right\}n(r), \tag{29}$$

where the $o(1)$ in (29) tends to zero uniformly as $z\to\infty$ in any sector: $|\theta|\leqq\pi-\eta$. From (29) we easily obtain for $u(z)=\log|f(z)|$,

$$N(r, u)=\frac{n(r)}{\lambda(r)}(1+o(1)), \tag{30}$$

$$T(r, u)=\frac{n(r)}{\lambda(r)}(1+o(1))\,(\lambda(r)\leqq 1/2), \quad =\frac{n(r)}{\lambda(r)\sin \pi\lambda(r)}(1+o(1))\ \ (\lambda(r)>1/2), \tag{31}$$

$$m_2(r, u)=\frac{\pi n(r)}{\sin \pi\lambda(r)}\sqrt{1/2+\sin 2\pi\lambda(r)/4\pi\lambda(r)}\cdot(1+o(1)), \tag{32}$$

$$m^*(r, u)\leqq\frac{\pi n(r)}{\sin \pi\lambda(r)}\cos \pi\lambda(r)\cdot(1+o(1)). \tag{33}$$

If $K\ (>0)$ is fixed, (21) implies that $\lambda(Kr)=\lambda(r)+o(1)$ $(r\to\infty)$, and so by (27) $n(Kr)\sim K^{\lambda(r)}n(r)$ $(r\to\infty)$. Hence by (31)

$$\frac{T(Kr, u)}{K^{\lambda(r)}T(r, u)}=1+o(1) \qquad (r\to\infty).$$

Thus (25) and the definition of μ_* (λ_*) imply

$$\mu_*=\rho_1 \qquad (\lambda_*=\rho_2). \tag{34}$$

On the other hand, we have by (32), (33)

$$\frac{m^*(r, u)}{m_2(r, u)}\leqq C_2(\lambda(r))(1+o(1))\leqq C_2(\rho_1)(1+o(1)). \tag{35}$$

It follows from (34) and (35) that $u(z)$ satisfies (5). However, by (30)—(33) we have

$$\frac{N(Kr, u)}{N(r, u)}=K^{\lambda(r)}(1+o(1)), \tag{36}$$

$$\frac{T(Kr, u)}{T(r, u)}=K^{\lambda(r)}(1+o(1)), \tag{37}$$

$$\frac{N(r, u)}{m_2(r, u)}=C_1(\lambda(r))(1+o(1)), \qquad \frac{N(r, u)}{T(r, u)}=\begin{cases}1+o(1) & (\lambda(r)<1/2)\\ \sin \pi\lambda(r)\cdot(1+o(1)) & (\lambda(r)\geqq 1/2).\end{cases} \tag{38}$$

(36)—(38) illustrate our assertion which we have stated above in relation to (18)—(20). And from (22), (23) we have log dens $G=0$ in this case. This fact is worth while to be compared with the result of Drasin and Shea in [6, pp.

281-283]. Further we note from (26), (27), (31) and (34) that $\rho_1=\mu_*<\mu=\rho=\lambda<\lambda_*=\rho_2$.

Our theorem is unsatisfactory in one respect, that is, we cannot answer whether in addition to (13) an estimate from below such as

$$(1-\varepsilon_k)\left(\frac{r}{y_k}\right)^{\mu_*}\leqq\frac{m_2(r,\ u)}{m_2(y_k,\ u)}\qquad(y_k'\leqq r\leqq y_k''')$$

holds or not.

Now, we conclude § 0 by describing our plan for the proof of our theorem. First, in § 1, we shall state the definition and the elementary properties of the local indicator for a sequence $\{B_m(z)\}_1^\infty$ of subharmonic functions such that $B_m(z)$ is subharmonic in the annulus: $r_m'\leqq|z|\leqq r_m''$ $(m=1, 2, \cdots)$ (cf. [19]). Next, we remark that, roughly speaking, Edrei's idea in [9] is supported by two facts—the one is the Boutroux—Cartan Lemma and the other is a lemma due to Edrei and Fuchs [10, p. 322]. So in § 2, we shall extend these two lemmas for subharmonic functions. In § 3, in relation to an extremal function $u(z)$ satisfying (5), we define a sequence $\{B_m(z)\}_1^\infty$ of subharmonic functions and show that the local indicator of $\{B_m(z)\}_1^\infty$ is sinusoidal. This fact implies that Edrei's idea is applicable to our problem. In § 4, we shall prove a lemma which is essential to the proof of I and II of our theorem. To do this, we need an estimate of Miles and Shea [16] and an estimate due to Gol'dberg [11]. In § 5, we follow Edrei's procedure in [9] to obtain I and II of our theorem. In § 6, combining some estimates obtained in § 4 and § 5 and the reasoning of Miles and Shea in [17], III of our theorem will be proved.

For background material on subharmonic functions, see [13] or [18].

1. Definition of the local indicator of order ρ of a sequence $\{B_m(z)\}_1^\infty$ of subharmonic functions.

We now prepare several notations and their properties in order to define the local indicator of order ρ of a sequence $\{B_m(z)\}_1^\infty$ of the given subharmonic functions.

(i) three infinite sequences of positive numbers $\{r_m'\}_1^\infty$, $\{r_m\}_1^\infty$, $\{r_m''\}_1^\infty$ such that $r_m'<r_m<r_m''<r_{m+1}'$ $(m=1, 2, \cdots)$, and such that, as $m\to\infty$

$$r_m/r_m'\longrightarrow\infty\,,\qquad r_m''/r_m\longrightarrow\infty\,.$$

(ii) a sequence $\{B_m(z)\}_1^\infty$ such that $B_m(z)$ is subharmonic in the annulus:

$$r_m'<|z|<r_m''\,.$$

(iii) a strictly positive sequence $\{V(r_m)\}_1^\infty$ and a quantity ρ $(0<\rho<\infty)$. We then define a sequence $\{V_m(z)\}_1^\infty$ of analytic comparison functions:

$$V_m(z)=V_m(r)e^{i\rho\theta}=V(r_m)\left(\frac{r}{r_m}\right)^\rho e^{i\rho\theta}\qquad(z=re^{i\theta})\,.$$

The symbol $V_m(r)$ always refers to the choice of $\theta=0$.

(iv) Consider the intervals $I_m=[r'_m, r''_m]$ $(m=1, 2, \cdots)$ as well as the intervals $I_m(s)=[r_m e^{-s}, r_m e^{s}]$ $(m=1, 2, \cdots; s=1, 2, \cdots)$, and let

$$\Lambda=\bigcup_{m=1}^{\infty} I_m\,, \qquad \Lambda(s)=\bigcup_{m=1}^{\infty} I_m(s) \qquad (s=1, 2, \cdots)\,.$$

(v) Let the sequence $\{B_m(z)\}_1^\infty$ be chosen so that

$$\overline{\lim_{\substack{r\to\infty\\ r\in\Lambda}}}\frac{M(r, B)}{V(r)}<\infty\,,$$

where $B(z)$ stands for $B_m(z)$ in the annulus: $r'_m<|z|<r''_m$ $(m=1, 2, \cdots)$. With these preparations we now define the local indicator. Firstly we set for every real value of θ,

$$h_s(\theta)=\overline{\lim_{\substack{r\to\infty\\ r\in\Lambda(s)}}}\frac{B(re^{i\theta})}{V(r)} \qquad (s=1, 2, \cdots)\,,$$

and consider

$$h(\theta)=\lim_{s\to\infty} h_s(\theta)\,.$$

The real function $h(\theta)$ is called the local indicator of order ρ of $\{B_m(z)\}_1^\infty$ at the peaks $\{r_m\}_1^\infty$. With this definition, Edrei's Fundamental Lemma can be extended straightforwardly for the sequence $\{B_m(z)\}_1^\infty$ of subharmonic functions.

FUNDAMENTAL LEMMA. *Let $h(\theta)$ be the local indicator of order ρ $(0<\rho<\infty)$ of $\{B_m(z)\}_1^\infty$ at the peaks $\{r_m\}_1^\infty$. Let θ_1, θ_2 be given such that $0<\theta_2-\theta_1<\pi/\rho$, and let the constants a, b be such that the sinusoid $H(\theta)=a\cdot\cos\rho\theta+b\cdot\sin\rho\theta$ satisfies the conditions: $h(\theta_j)\leqq H(\theta_j)$ $(j=1, 2)$. Then given $\varepsilon>0$ and any integer $s>0$, there exists a bound $r_0=r_0(\varepsilon, s, a, b, \theta_1, \theta_2)$, independent of θ, such that*

$$B(re^{i\theta})\leqq(H(\theta)+\varepsilon)\cdot V(r)\,,$$

for $r\in\Lambda(s)$, $\theta_1\leqq\theta\leqq\theta_2$, $r\geqq r_0$.

From Fundamental Lemma, we immediately have $h(\theta)\leqq H(\theta)$ $(\theta_1\leqq\theta\leqq\theta_2)$, that is, the subtrigonometric character of $h(\theta)$. It is known that many important properties of an indicator depend only on its subtrigonometric character (cf. [3]). For example, we have the following three facts:

1. The subtrigonometric inequality (cf. [3, p. 44]). If $h(\theta)$ is of order ρ and if $0<\theta_2-\theta_1<\pi/\rho$, $0<\theta_3-\theta_2<\pi/\rho$, then

$$(1.1) \qquad \begin{vmatrix} h(\theta_1) & \cos\rho\theta_1 & \sin\rho\theta_1 \\ h(\theta_2) & \cos\rho\theta_2 & \sin\rho\theta_2 \\ h(\theta_3) & \cos\rho\theta_3 & \sin\rho\theta_3 \end{vmatrix} \geqq 0\,.$$

In particular, if $0 \leqq \theta < \pi/\rho$, then

$$\frac{h(-\theta)+h(\theta)}{2} \geqq h(0)\cos\rho\theta\,. \tag{1.2}$$

2. Continuity (cf. [3, p. 37]). If $h(\theta_0) \neq -\infty$ for some θ_0, then $h(\theta)$ is uniformly bounded and continuous in $[-\pi, \pi]$.

3. Uniformity (cf. [3, p. 46]). If $h(\theta_0) \neq -\infty$ for some θ_0, then it is possible, given $\varepsilon > 0$ and $s > 0$, to find $r_0 = r_0(\varepsilon, s)$ such that $r > r_0$, $r \in \Lambda(s)$ imply

$$B(re^{i\theta}) \leqq (h(\theta)+\varepsilon)\cdot V(r) \qquad (-\pi \leqq \theta \leqq \pi)\,.$$

2. Some lemmas on subharmonic or δ-subharmonic functions.

First, we shall extend the Boutroux-Cartan Lemma for positive Borel measures. For the original form of this lemma, cf. [9, p. 39].

LEMMA 1. *Suppose that μ is a positive Borel measure defined in the disk: $|w| \leqq \sigma < 1$ such that $\mu\,(|w| \leqq \sigma) < \infty$. Then given ξ $(0 < \xi \leqq 1)$, there exist a finite or countable set of disks, say $\{\Gamma_k\}$, whose radii $\{\rho_k\}$ satisfy*

$$\sum_k \rho_k \leqq 2\xi\sigma\,,$$

such that

$$w \notin \bigcup_k \Gamma_k\,, \qquad |w| \leqq \sigma$$

imply

$$\int_{|\zeta|<\sigma} \log|w-\zeta|\, d\mu(\zeta) \geqq \mu(|w| \leqq \sigma)(1+2e)\log\left(\frac{\xi\sigma}{2e}\right).$$

Proof. For each fixed positive integer ν, we construct a maximal number of mutually disjoint closed disks $\Gamma_k^{(\nu)} = \Gamma(x_k^{(\nu)}, r_\nu/2)$, $k=1, \cdots, k_\nu$ such that $r_\nu = 2\xi\sigma 2^{-\nu}e^{-\nu}$ and $\mu(\Gamma_k^{(\nu)}) \geqq \mu(|z| \leqq \sigma)e^{-\nu}$, where r is the radius and x is the center of $\Gamma = \Gamma(x, r)$. Clearly $k_\nu \leqq [e^\nu]$. Hence

$$\sum_{\nu=1}^{\infty} \sum_{k=1}^{k_\nu} r_\nu \leqq \sum_{\nu=1}^{\infty} e^\nu \cdot 2\xi\sigma \cdot 2^{-\nu}e^{-\nu} = 2\xi\sigma\,.$$

Now, suppose that w $(|w| \leqq \sigma)$ is a point outside all the disks $\Gamma(x_k^{(\nu)}, r_\nu)$ $(\nu = 1, 2, \cdots;\ k=1, \cdots, k_\nu)$. Then it is easy to see that

$$\mu(\Gamma(w, r_\nu/2)) < \mu(|\zeta| \leqq \sigma)\cdot e^{-\nu} \qquad (\nu=1, 2, \cdots)\,.$$

Thus

$$\int_{|\zeta|<\sigma} \log|w-\zeta|\, d\mu(\zeta) = \int_{\{|\zeta-w|\geqq r_1/2\}\cap\{|\zeta|<\sigma\}} \log|\zeta-w|\, d\mu(\zeta)$$

$$+\sum_{\nu=1}^{\infty}\int_{\{r_{\nu+1}/2\leqq|\zeta-w|<r_\nu/2\}\cap\{|\zeta|<\sigma\}} \log|\zeta-w|\, d\mu(\zeta)$$

$$\geqq \log\left(\frac{r_1}{2}\right)\cdot\mu(|\zeta|\leqq\sigma)+\sum_{\nu=1}^{\infty}\log\left(\frac{r_{\nu+1}}{2}\right)\mu(|\zeta|\leqq\sigma)\cdot e^{-\nu}$$

$$=\mu(|\zeta|\leqq\sigma)e\sum_{\nu=1}^{\infty}\log\left(\frac{r_\nu}{2}\right)\cdot e^{-\nu}$$

$$=\mu(|\zeta|\leqq\sigma)e\left\{\log\xi\sigma\sum_{\nu=1}^{\infty}e^{-\nu}-\log 2e\cdot\sum_{\nu=1}^{\infty}\nu\cdot e^{-\nu}\right\}$$

$$\geqq\mu(|\zeta|\leqq\sigma)\left\{\frac{e}{e-1}\log\xi\sigma-(1+2e)\log 2e\right\}$$

$$\geqq\mu(|\zeta|\leqq\sigma)(1+2e)\log\left(\frac{\xi\sigma}{2e}\right).$$

This completes the proof.

Once the Boutroux-Cartan Lemma is established for positive Borel measures, it is not difficult to prove the following fact which is an extension of Lemma 1 in [9, pp. 35-42] for subharmonic functions.

LEMMA 2. *Let $u(\zeta)$ ($\zeta=te^{i\omega}$) be subharmonic in the sector*:

$$\Sigma=\left\{\zeta:\ e^{-s}<t<e^{s},\ |\omega|<\frac{\pi}{\gamma}\right\}\qquad (s>0,\ \gamma\geqq 1),$$

and let $u(\zeta)\leqq 0$ ($\zeta\in\Sigma$), $u(1)\neq-\infty$. Consider the sector

$$\Sigma'=\left\{\zeta:\ e^{-s'}<t<e^{s'},\ |\omega|<\frac{\pi}{\gamma'}\right\}\qquad (0<s'<s,\ \gamma'>\gamma).$$

Then there exist two positive constants H_j ($j=1, 2$) depending only on s, s', γ, γ', and having the following properties:

Given ξ ($0<\xi\leqq 1$), it is possible to associate a set $\Omega(\xi)$ such that means $\Omega(\xi)<\pi\xi$, and such that the conditions $\omega\notin\Omega(\xi)$, $te^{i\omega}\in\Sigma'$ imply

$$u(te^{i\omega})\geqq\left(H_2+H_1\log\frac{1}{\xi}\right)u(1).$$

Next, we shall extend a lemma of Edrei and Fuchs [10, p. 322] to δ-subharmonic functions in $\boldsymbol{C}$.

LEMMA 3. *Let $v=u^{(1)}-u^{(2)}$ be a δ-subharmonic function in $\boldsymbol{C}$. Then if $I(r)$*

is any measurable subset of $[-\pi, \pi]$, *it is possible to find absolute constants* K_j $(j=1, 2)$ *such that* $0<K_1<K_2$ *and*

$$\int_{I(r)} |v(re^{i\theta})|\, d\theta \leqq K_1 m(I(r))\left\{1+\log^+ \frac{1}{m(I(r))}\right\} T(4r, v),$$

$$\int_{I(r)} |v(re^{i\theta})|^2 d\theta \leqq K_2 m(I(r))\left\{1+\left(\log \frac{1}{m(I(r))}\right)^2\right\}(T(4r, v))^2.$$

Proof. Let μ_j $(j=1, 2)$ be the Riesz mass associated with $u^{(j)}(z)$. For $|z|=r<t<\infty$, we have

$$u^{(j)}(z)=h^{(j)}(z)+\int_{|\zeta|<t} \log \frac{|z-\zeta|}{2t}\, d\mu_j(\zeta)+n^{(j)}(t)\log 2t,$$

where $u^{(j)}(t)=\mu_j(|\zeta|<t)$ and $h^{(j)}(z)$ is harmonic in $|z|<t$. Here we put

$$v^{(1)}(z)=v(z)+\int_{|\zeta|<t} \log \frac{|z-\zeta|}{2t}\, d\mu_2(\zeta)=u^{(1)}(z)-h^{(2)}(z)-n^{(2)}(t)\log 2t. \tag{2.1}$$

Clearly $v^{(1)}(z)$ is subharmonic and satisfies $v^{(1)}(z)\leqq v(z)$ in $|z|<t$. Hence the Poisson-Jensen formula for subharmonic functions (See [13, Theorem 3.14]) gives

$$v^{(1)}(re^{i\theta})=\frac{1}{2\pi}\int_{-\pi}^{+\pi} v^{(1)}(te^{i\phi})\, \frac{t^2-r^2}{t^2-2tr\cos(\phi-\theta)+r^2}\, d\phi+\int_{|\zeta|<t} \log \left|\frac{t(z-\zeta)}{t^2-\bar{z}\zeta}\right| d\mu_1(\zeta)$$

$$\leqq \frac{t+r}{t-r} N(t, v^+),$$

so that

$$(v^{(1)}(z))^+ \leqq \frac{t+r}{t-r} N(t, v^+). \tag{2.2}$$

We deduce from (2.1) and (2.2) that

$$(v(z))^+ \leqq \frac{t+r}{t-r} N(t, v^+)+\int_{|\zeta|<t} \log \frac{2t}{|z-\zeta|}\, d\mu_2(\zeta). \tag{2.3}$$

If we write $z=re^{i\theta}$ and $\zeta=se^{i\beta}$, we easily have

$$|z-\zeta| \geqq \begin{cases} r|\sin(\theta-\beta)| & (|\beta-\theta|\leqq\pi/2), \\ r & (\pi/2<|\beta-\theta|\leqq\pi). \end{cases}$$

It follows from this and (2.3) that

$$(v(z))^+ \leqq \frac{t+r}{t-r} N(t, v^+)+n^{(2)}(t)\log \frac{2t}{r} + \int_{\{\zeta, s<t, |\beta-\theta|\leqq\pi/2\}} \log \frac{1}{|\sin(\theta-\beta)|}\, d\mu_2(\zeta). \tag{2.4}$$

Now, we prove the second inequality of our lemma. The proof of the first one is contained in the proof of the second one. From (2.4) we have

$$
\begin{aligned}
\int_{I(r)}\{v^+(re^{i\theta})\}^2 d\theta \leqq m(I(r))&\left\{\frac{t+r}{t-r}N(t,\, v^+)+n^{(2)}(t)\log\frac{2t}{r}\right\}^2 \\
&+2\left\{\frac{t+r}{t-r}N(t,\, v^+)+n^{(2)}(t)\log\frac{2t}{r}\right\} \\
&\cdot\int_{I(r)}d\theta\int_{\{\zeta.s<t,\,|\beta-\theta|\leqq\pi/2\}}\log\frac{1}{|\sin(\theta-\beta)|}d\mu_2(\zeta) \\
&+\int_{I(r)}d\theta\left\{\int_{\{\zeta:s<t,\,|\beta-\theta|\leqq\pi/2\}}\log\frac{1}{|\sin(\theta-\beta)|}d\mu_2(\zeta)\right\}^2 .
\end{aligned}
\tag{2.5}
$$

In order to estimate the integrals in the right hand side of (2.5), we put

$$H=\inf\left\{\frac{\pi}{2},\ \frac{m(I(r))}{2}\right\}.$$

Then

$$
\begin{aligned}
\int_{I(r)\cap\{\theta:|\theta-\beta|\leqq\pi/2\}}\log\frac{1}{|\sin(\theta-\beta)|}d\theta &\leqq 2\int_0^H\log\frac{1}{\sin\theta}d\theta \\
\leqq 2\int_0^H\left(\log\frac{1}{\theta}+\log\frac{\pi}{2}\right)d\theta& \\
=2H\left(1+\log\frac{\pi}{2}+\log\frac{1}{H}\right)\leqq m(I(r))\left(1+\log^+\frac{\pi}{m(I(r))}\right).&
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{I(r)}d\theta\int_{\{\zeta:s<t,\,|\beta-\theta|\leqq\pi/2\}}\log\frac{1}{|\sin(\theta-\beta)|}d\mu_2(\zeta) \\
&=\int_{|\zeta|<t}d\mu_2(\zeta)\int_{I(r)\cap\{\theta.|\theta-\beta|\leqq\pi/2\}}\log\frac{1}{|\sin(\theta-\beta)|}d\theta \\
&\leqq n^{(2)}(t)m(I(r))\cdot\left\{1+\log^+\frac{\pi}{m(I(r))}\right\}.
\end{aligned}
\tag{2.6}
$$

In the same way we have

$$
\begin{aligned}
&\int_{I(r)\cap\{\theta:|\theta-\beta|\leqq\pi/2\}}\left\{\log\frac{1}{|\sin(\theta-\beta)|}\right\}^2 d\theta \\
&\qquad\leqq m(I(r))\left\{(\log H)^2-2\log H+2-\log\frac{\pi}{2}\log H+\left(\log\frac{\pi}{2}\right)^2\right\},
\end{aligned}
$$

so that by Schwarz's inequality in continuous form

$$\int_{I(r)} d\theta \left\{ \int_{\{\zeta, s<t, |\beta-\theta|\leq\pi/2\}} \log \frac{1}{|\sin(\theta-\beta)|} d\mu_2(\zeta) \right\}^2$$

$$\leqq n^{(2)}(t) \int_{I(r)} d\theta \int_{\{\zeta, s<t, |\beta-\theta|\leq\pi/2\}} \left\{ \log \frac{1}{|\sin(\theta-\beta)|} \right\}^2 d\mu_2(\zeta)$$

$$(2.7) \qquad = n^{(2)}(t) \int_{|\zeta|<t} d\mu_2(\zeta) \int_{I(r)\cap\{\theta, |\theta-\beta|\leq\pi/2\}} \left\{ \log \frac{1}{|\sin(\theta-\beta)|} \right\}^2 d\theta$$

$$\leqq (n^{(2)}(t))^2 m(I(r)) \cdot \left\{ (\log H)^2 - 2\log H + 2 - \log\frac{\pi}{2} \cdot \log H + \left(\log\frac{\pi}{2}\right)^2 \right\}.$$

Substituting (2.6) and (2.7) into (2.5), we obtain

$$\int_{I(r)} \{v^+(re^{i\theta})\}^2 d\theta \leqq m(I(r)) \left\{ \left(\frac{t+r}{t-r} N(t, v^+) + n^{(2)}(t) \log\frac{2t}{r} \right)^2 \right.$$

$$(2.8) \qquad + 2\left(\frac{t+r}{t-r} N(t, v^+) + n^{(2)}(t) \log\frac{2t}{r} \right) n^{(2)}(t) \left(1 + \log^+ \frac{\pi}{m(I(r))} \right)$$

$$\left. + (n^{(2)}(t))^2 \left((\log H)^2 - 2\log H + 2 + \log\frac{\pi}{2} \log H + \left(\log\frac{\pi}{2}\right)^2 \right) \right\}.$$

We remark that (2.8) also holds if we replace v^+ and $n^{(2)}$ by v^- and $n^{(1)}$, respectively. And it is clear that

$$N(t, v^+) + N(t, v^-) \leqq 2T(t, v),$$

$$n^{(1)}(t) + n^{(2)}(t) \leqq \frac{N(2t, u^{(1)}) + N(2t, u^{(2)})}{\log 2} \leqq \frac{2}{\log 2} T(2t, v).$$

All the above estimates combine to show the desired inequality (with $t=2r$).

3. The indicator associated with Theorem.

Let $u(z)$ be an extremal subharmonic function satisfying (5). Take a sequence $\{r_m\}_1^\infty$ of Pólya peaks of order μ_* for $u(z)$, and let $\{r'_m\}_1^\infty$, $\{r''_m\}_1^\infty$, $\{\varepsilon_m\}_1^\infty$ be the associated sequences. We define comparison functions:

$$V_m(z) = \left(\frac{z}{r_m}\right)^{\mu_*} (1+\varepsilon_m) T(r_m, u) = \left(\frac{r}{r_m}\right)^{\mu_*} e^{i\mu_*\theta} (1+\varepsilon_m) T(r_m, u)$$

$$= V(r_m) \left(\frac{r}{r_m}\right)^{\mu_*} e^{i\mu_*\theta} \qquad (m=1, 2, \cdots).$$

And let $\nu(\zeta)$ be the positive Borel measure associated with $u(z)$, and put $n(t) = \nu(|\zeta|<t)$. With these notations, we define

$$B_m(z) = \int_0^{r''_m/4} \log\left|1 + \frac{z}{t}\right| dn(t) \qquad (m=1, 2, \cdots).$$

Let $h(\theta)$ be the local indicator for the sequence $\{B_m(z)\}_1^\infty$ at the peaks $\{r_m\}_1^\infty$, with comparison functions $\{V_m(z)\}_1^\infty$. In [19] we have shown the existence of $h(\theta)$ and $h(0)\geqq 1$. In this section, we shall prove the following lemma.

LEMMA 4. $$h(\theta)=h(0)\cos \mu_*\theta \qquad (|\theta|\leqq\pi).$$

Proof. We define two sequences $\{u_{1,m}(z)\}_1^\infty$ and $\{u_{3,m}(z)\}_1^\infty$ of subharmonic functions as follows:

$$u_{1,m}(z)=\int_{|\zeta|\leqq r''_m/4}\log\left|1-\frac{z}{\zeta}\right|d\nu(\zeta), \qquad u_{3,m}(z)=u(z)-u_{1,m}(z).$$

As Kjellberg showed in [15, p. 192], we have

$$|u_{3,m}(z)|<16\frac{M(r''_m/2,\ u)}{r''_m}\cdot r \qquad (r<r''_m/8). \tag{3.1}$$

And the Poisson-Jensen formula for subharmonic functions gives

$$M(r''_m/2,\ u)\leqq 3T(r''_m,\ u). \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$|u_{3,m}(z)|<48T(r''_m,\ u)\frac{r}{r''_m}\leqq 48V(r)\left(\frac{r}{r''_m}\right)^{1-\mu_*} \qquad (r<r''_m/8). \tag{3.3}$$

Let $\eta>0$ (small enough) be given, and determine s (>0) so that $h_s(\pi)>h(\pi)-\eta/6$. By the definition of $h_s(\pi)$, there exists a sequence $\{\chi_n\}\subset\Lambda(s)$, tending to ∞, such that

$$B(-\chi_n)>(h_s(\pi)-\eta/3)V(\chi_n)>(h(0)\cos\pi\mu_*-\eta/2)\cdot V(\chi_n)>0. \tag{3.4}$$

Using (3.3), we have for $n>n_0(\eta,\ s)$

$$\begin{aligned} m^*(\chi_n,\ u)&\geqq m^*(\chi_n,\ u_1)+m^*(\chi_n,\ u_3)\\ &\geqq m^*(\chi_n,\ B)+m^*(\chi_n,\ u_3)>(h(\pi)-\eta)\cdot V(\chi_n). \end{aligned} \tag{3.5}$$

On the other hand, by Schwarz's inequality

$$m_2(\chi_n,\ u)=m_2(\chi_n,\ u_1+u_3)\leqq m_2(\chi_n,\ u_1)+\frac{\eta}{2}V(\chi_n) \qquad (n>n_0). \tag{3.6}$$

We now use an estimate due to Miles and Shea [16, p. 378], that is,

$$m_2(\chi_n,\ u_1)\leqq m_2(\chi_n,\ B). \tag{3.7}$$

In order to estimate $m_2(\chi_n,\ B)$ from above, we may note (3.4) and appeal to the Fundamental Lemma, so that

$$0<B(\chi_n e^{i\theta})<(H(\theta)+\varepsilon)V(\chi_n) \qquad (n>n_1(\varepsilon,\ s)), \tag{3.8}$$

where

$$H(\theta)=\frac{h(0)\sin(\pi-\theta)\mu_*+h(\pi)\sin\theta\mu_*}{\sin\pi\mu_*}. \tag{3.9}$$

Combining (3.5)—(3.9) we have

$$\frac{m^*(\chi_n, u)}{m_2(\chi_n, u)}>\frac{(1-E(\varepsilon, \eta))\cdot t}{\left\{(1+t^2)\left(\frac{1}{2}-\frac{\sin 2\pi\mu_*}{4\pi\mu_*}\right)+t\left(\frac{\sin\pi\mu_*}{\pi\mu_*}-\cos\pi\mu_*\right)\right\}^{1/2}}, \tag{3.10}$$

where $t=h(\pi)/h(0)\geqq\cos\pi\mu_*$, $E(\varepsilon, \eta)\to 0$ as $\varepsilon\to 0$ and $\eta\to 0$. The right hand side increases at t increases, and so, it is not smaller than $C_2(\mu_*)(1-E(\varepsilon, \eta))$. It follows from this and our assumption (5) that $t=\cos\pi\mu_*$. Hence $h(\pi)=h(-\pi)=h(0)\cdot\cos\pi\mu_*$. Substituting these into the subtrigonometric inequality (1.1), we have $h(\theta)\leqq h(0)\cdot\cos\theta\mu_*$. On the other hand, it is clear from (1.2) that $h(\theta)\geqq h(0)\cdot\cos\theta\mu_*$. Thus $h(\theta)=h(0)\cdot\cos\theta\mu_*$ $(|\theta|\leqq\pi)$.

4. A preliminary lemma for the proof of assertions I and II of Theorem.

LEMMA 5. *Let $u(z)$ be a subharmonic function satisfying* (5), *and let* $\{r_m\}$, $\{r'_m\}$, $\{r''_m\}$, $\{\varepsilon_m\}$, $\{V_m(z)\}$, $\{B_m(z)\}$, $h(\theta)$ *be defined as in* § 3. *Then given* ε $(0<\varepsilon<h(0)\cos\pi\mu_*)$, δ $(0<\delta<\delta_0$, *where* δ_0 *is a fixed positive number satisfying* $4^{2\mu_*}\cdot 2\delta_0 K_2(1+(\log 2\delta_0)^2)<\varepsilon/6)$ *and* L (>0), *it is possible to determine* $q=q(\varepsilon, \delta, L)$ (*a positive integer*), $\{l_m\}_1^\infty$ (*a sequence of unbounded, increasing integers*) *and* $\{R_{l_m}\}_1^\infty$ *such that*

$$e^{-q}r_{l_m}\leqq R_{l_m}\leqq e^q r_{l_m} \qquad (m=1, 2, \cdots),$$

$$B_{l_m}(R_{l_m})>(h(0)-\varepsilon)V_{l_m}(R_{l_m}) \qquad (m=1, 2, \cdots),$$

and such that for $e^{-L}R_{l_m}\leqq r\leqq e^L R_{l_m}$

$$(1-2\varepsilon)h(0)\frac{\sin\pi\mu_*}{\pi\mu_*}V_{l_m}(r)\leqq N(r, u)\leqq T(r, u)\leqq(1+\varepsilon)h(0)\frac{\sin\pi\mu_*}{\pi\mu_*}V_{l_m}(r),$$

$$m_2(r, u)\leqq(1+2\varepsilon)\sqrt{\frac{1}{2}+\frac{\sin 2\pi\mu_*}{4\pi\mu_*}}\,h(0)V_{l_m}(r).$$

Proof. We define s, s', γ, γ' as follows:

$$s=2L, \qquad s'=L, \qquad \gamma=1, \qquad \gamma'=\pi/(\pi-\delta).$$

And let η be a number satisfying the following inequalities.

$$\begin{cases} \eta^2+2\eta\leqq\varepsilon/2\,, \quad e^{L\mu_*}\Big(H_2+H_1\log\dfrac{1}{\eta}\Big)\leqq\varepsilon\,, \\ \dfrac{2\delta+\pi\eta}{2\pi}+K_1(\pi\eta+2\delta)\Big\{1+\log\dfrac{1}{\pi\eta+2\delta}\Big\}4^{\mu_*}\leqq\varepsilon\,, \\ 4^{2\mu_*}\cdot K_2(\pi\eta+2\delta)\Big\{1+\Big(\log\dfrac{1}{\pi\eta+2\delta}\Big)^2\Big\}<(\varepsilon/2)\Big(\dfrac{1}{2}+\dfrac{\sin 2\pi\mu_*}{4\pi\mu_*}\Big)h(0)\,. \end{cases}$$

Next, we determine $q=q(\eta)=q(\varepsilon,\,\delta,\,L)$ so that $h_q(0)>h(0)-\eta/2$. By the definition of $h_q(0)$, it is possible to find a sequence $\{R_{l_m}\}_1^\infty\subset A(q)$ tending to ∞ such that

$$B_{l_m}(R_{l_m})>(h_q(0)-\eta/2)V_{l_m}(R_{l_m}) \qquad (4.1)$$
$$>(h(0)-\eta)V_{l_m}(R_{l_m})>(h(0)-\varepsilon)V_{l_m}(R_{l_m})\,.$$

By the uniformity property of the local indicator (cf. § 1), we can determine m_0 so that the conditions $m>m_0$, $r\in A(2L+q)$ imply

$$B_{l_m}(re^{i\theta})\leqq(h(\theta)+\eta e^{-2L\mu_*})V_{l_m}(r) \qquad (|\theta|\leqq\pi)\,. \qquad (4.2)$$

Now, we introduce a sequence $\{u_{l_m}(\zeta)\}_1^\infty$ of subharmonic functions:

$$u_{l_m}(\zeta)=B_{l_m}(z)-h(\theta)V_{l_m}(r)-V_{l_m}(R_{l_m}) \qquad (\zeta=z/R_{l_m}=te^{i\omega})\,. \qquad (4.3)$$

Further we put

$$\Sigma=\{\zeta: e^{-2L}\leqq t\leqq e^{2L},\ |\omega|\leqq\pi\}, \qquad \Sigma'=\{\zeta: e^{-L}\leqq t\leqq e^{L},\ |\omega|\leqq\pi-\delta\}\,.$$

From (4.2) and (4.3) we deduce $u_{l_m}(\zeta)\leqq 0$ $(\zeta\in\Sigma)$. And it follows from (4.1) and (4.3) that $u_{l_m}(1)\neq-\infty$. Hence we can apply Lemma 2 to $u=u_{l_m}$. That is, it is possible to find positive two constants H_1, H_2, depending only on L, δ, and having the following properties.

Given ξ $(0<\xi\leqq1)$, there exists a set $\Omega(\xi)$ such that means $\Omega(\xi)<\pi\xi$ and such that the conditions $\omega\notin\Omega(\xi)$, $te^{i\omega}\in\Sigma'$ imply $u_{l_m}(te^{i\omega})\geqq(H_2+H_1\log(1/\xi))u_{l_m}(1)$. Returning to the variable z, we have from (4.1), (4.3) and the choice of η,

$$B_{l_m}(re^{i\theta})\geqq(h(\theta)-\varepsilon)V_{l_m}(r)>0 \qquad (4.4)$$
$$(e^{-L}R_{l_m}\leqq r\leqq e^{L}R_{l_m},\ \theta\notin\Omega(\eta),\ |\theta|\leqq\pi-\delta)\,.$$

Here we put $\Omega'(\eta)=[-\pi,\,-\pi+\delta]\cup[\pi-\delta,\,\pi]\cup\Omega(\eta)$. Then by Lemma 3,

$$\frac{1}{2\pi}\int_{\Omega'(\eta)}|B_{l_m}(re^{i\theta})|\,d\theta\leqq K_1(2\delta+\pi\eta)\Big(1+\log\frac{1}{2\delta+\pi\eta}\Big)T(4r,\,B_{l_m})$$
$$\leqq(2\delta+\pi\eta)4^{\mu_*}K_1\Big(1+\log\frac{1}{2\delta+\pi\eta}\Big)V_{l_m}(r) \qquad (m>m_1)\,,$$

so that from (4.4) and the choice of η we have

$$N(r, B_{l_m}) \geqq \frac{1}{2\pi}\int_{[-\pi,+\pi]\setminus\Omega'(\eta)} B_{l_m}(re^{i\theta})d\theta - \frac{1}{2\pi}\int_{\Omega'(\eta)} |B_{l_m}(re^{i\theta})|d\theta$$

$$(4.5) \qquad \geqq \left\{\frac{\sin\pi\mu_*}{\pi\mu_*}h(0) - h(0)\frac{2\delta+\pi\eta}{2\pi} - \varepsilon - 4^{\mu_*}K_1(2\delta+\pi\eta)\left(1+\log\frac{1}{2\delta+\pi\eta}\right)\right\}V_{l_m}(r)$$

$$\geqq \frac{\sin\pi\mu_*}{\pi\mu_*}h(0)(1-2\varepsilon)V_{l_m}(r) \qquad (e^{-L}R_{l_m} \leqq r \leqq e^L R_{l_m},\ m>m_1).$$

However, since Jensen's formula gives $N(r, u) = N(r, B_{l_m})$ for $r \leqq r''_{l_m}/4$, we have from (4.5)

$$(4.6) \qquad N(r, u) > (1-2\varepsilon)h(0)\frac{\sin\pi\mu_*}{\pi\mu_*}V_{l_m}(r) \qquad (e^{-L}R_{l_m} \leqq r \leqq e^L R_{l_m},\ m>m_1).$$

Now, we estimate $T(r, u)$ from above. For $r \leqq r''_{l_m}/8$, we have

$$(4.7) \qquad T(r, u) = T(r, u_{1,l_m} + u_{3,l_m}) \leqq T(r, u_{1,l_m}) + T(r, u_{3,l_m}).$$

Using an estimate due to Gol'dberg [11] and (4.2), we obtain for $e^{-L}R_{l_m} \leqq r \leqq e^L R_{l_m}$

$$(4.8) \qquad T(r, u_{1,l_m}) \leqq T(r, B_{l_m}) \leqq \frac{\sin\pi\mu_*}{\pi\mu_*}h(0)(1+\varepsilon/2)V_{l_m}(r).$$

And from (3.3) we have for $e^{-L}R_{l_m} \leqq r \leqq e^L R_{l_m}$

$$(4.9) \qquad T(r, u_{3,l_m}) < \frac{\sin\pi\mu_*}{\pi\mu_*}\cdot(\varepsilon/2)\cdot V_{l_m}(r) \qquad (m \geqq m_2(\varepsilon, L)).$$

Substituting (4.8) and (4.9) into (4.7) we have

$$(4.10) \qquad T(r, u) < \frac{\sin\pi\mu_*}{\pi\mu_*}(1+\varepsilon)h(0)V_{l_m}(r) \qquad (e^{-L}R_{l_m} \leqq r \leqq e^L R_{l_m},\ m \geqq m_2).$$

Finally we estimate $m_2(r, u)$ from above. First we estimate $m_2(r, B_{l_m})$. It follows from (4.2) and (4.4) that for $e^{-L}R_{l_m} \leqq r \leqq e^L R_{l_m}$

$$(4.11) \qquad \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus\Omega'(\eta)} \{B_{l_m}(re^{i\theta})\}^2 d\theta \leqq \frac{1}{2\pi}\int_{-\pi}^{+\pi}(h(\theta)+\eta)^2 d\theta\cdot(V_{l_m}(r))^2$$

$$\leqq (1+\varepsilon/2)(h(0))^2\left(\frac{1}{2}+\frac{\sin 2\pi\mu_*}{4\pi\mu_*}\right)(V_{l_m}(r))^2.$$

By Lemma 3 and the choice of η, we have

$$\frac{1}{2\pi}\int_{\Omega'(\eta)}\{B_{l_m}(re^{i\theta})\}^2 d\theta \leqq K_2(2\delta+\pi\eta)\left\{1+\left(\log\frac{1}{\pi\eta+2\delta}\right)^2\right\}(T(4r, B_{l_m}))^2$$

$$(4.12) \qquad \leqq 4^{2\mu_*}K_2(2\delta+\pi\eta)\left\{1+\left(\log\frac{1}{\pi\eta+2\delta}\right)^2\right\}(V_{l_m}(r))^2$$

$$\leqq (\varepsilon/2)\left(\frac{1}{2}+\frac{\sin 2\pi\mu_*}{4\pi\mu_*}\right)h(0)(V_{l_m}(r))^2 \quad (e^{-L}R_{l_m} \leqq r \leqq e^L R_{l_m}).$$

Combining (4.11) and (4.12), we obtain

$$m_2(r, B_{l_m}) \leqq (1+\varepsilon)h(0)\sqrt{\frac{1}{2}+\frac{\sin 2\pi\mu_*}{4\pi\mu_*}}\, V_{l_m}(r). \tag{4.13}$$

Remembering (3.6) and (3.7), we have

$$m_2(r, u) \leqq (1+2\varepsilon)h(0)\sqrt{\frac{1}{2}+\frac{\sin 2\pi\mu_*}{4\pi\mu_*}}\, V_{l_m}(r) \qquad (m \geqq m_0). \tag{4.14}$$

The restrictions: $m \geqq m_0$ in (4.14), $m > m_1$ in (4.6), $m \geqq m_2$ in (4.10) are not essential. Thus we have the desired results.

5. Diagonalization—Proof of assertions I and II of Theorem.

In this section we follow Edrei's procedure in [9, pp. 54-59]. We set $\varepsilon = \varepsilon_n \downarrow 0$ $(n \to \infty)$, $\delta = \delta_n \downarrow 0$ $(n \to \infty)$, $L = n$ in our previous Lemma 5. Then using the method of diagonalization, we easily obtain the following fact:

Let $u(z)$ be a subharmonic function satisfying (5). Then it is possible to find a sequence $\{t_n\}$ of Pólya peaks of order μ_* for u (As usual, we denote the associated sequences by $\{t'_n\}$, $\{t''_n\}$.), a positive unbounded sequence $\{x_n\}$, $\{\tilde{B}_n\} \subset \{B_{l_m}\}$ and $\{\tilde{V}_n\} \subset \{V_{l_m}\}$ such that

$$e^n t'_n < e^{-n} x_n, \qquad e^n x_n < e^{-n} t''_n, \qquad \tilde{B}_n(x_n) \geqq (h(0)-\varepsilon_n)\tilde{V}_n(x_n), \tag{5.1}$$

and such that for $e^{-n}x_n \leqq r \leqq e^n x_n$

$$\tilde{B}_n(re^{i\theta}) \leqq (h(\theta)+\varepsilon_n)\tilde{V}_n(r) \qquad (|\theta| \leqq \pi), \tag{5.2}$$

$$(1-2\varepsilon_n)h(0)\frac{\sin \pi\mu_*}{\pi\mu_*}\tilde{V}_n(r) \leqq N(r, u) \leqq T(r, u) \leqq (1+\varepsilon_n)h(0)\frac{\sin \pi\mu_*}{\pi\mu_*}\tilde{V}_n(r), \tag{5.3}$$

$$m_2(r, u) \leqq (1+2\varepsilon_n)\cdot\sqrt{\frac{1}{2}+\frac{\sin 2\pi\mu_*}{4\pi\mu_*}}\, h(0)\tilde{V}_n(r). \tag{5.4}$$

Now, consider the sequence $\{\tilde{B}_n\}$ of subharmonic functions in $\bigcup_{n=1}^{\infty}[e^{-n}x_n, e^n x_n] = \tilde{\Lambda}$, and set $\bigcup_{n=1}^{\infty}[e^{-s}x_n, e^s x_n] = \tilde{\Lambda}(s)$ $(s=1, 2, \cdots)$. We introduce the indicator $\tilde{h}(\theta)$ of order μ_* of $\{\tilde{B}_n\}$ with peaks $\{x_n\}$ and comparison functions $\{\tilde{V}_n\}$. By (5.1), (5.2) and (1.2) we have $\tilde{h}(\theta) = h(\theta) = h(0)\cos\mu_*\theta$ $(|\theta| \leqq \pi)$. Further we put

$$\tilde{u}_{1,n}(z) = \int_{|\zeta| \leqq t''_n/4} \log\left|1-\frac{z}{\zeta}\right| d\nu(\zeta), \qquad \tilde{u}_{3,n}(z) = u(z) - \tilde{u}_{1,n}(z).$$

Then as we have shown in (3.3),

$$|\tilde{u}_{3,n}(z)| < 48\tilde{V}_n(r)\left(\frac{r}{t''_n}\right)^{1-\mu_*}.$$

Hence if $r\in[e^{-n}x_n, e^n x_n]$ we have from (5.1)

$$|\tilde{u}_{3,n}(z)|<48e^{-n(1-\mu_*)}\tilde{V}_n(r).$$

Without loss of generality we may assume $48\,e^{-n(1-\mu_*)}\leqq\varepsilon_n$, so that

$$|\tilde{u}_{3,n}(z)|<\varepsilon_n\cdot\tilde{V}_n(r)\qquad(r\in[e^{-n}x_n, e^n x_n]). \tag{5.5}$$

Let $\eta>0$ be given. Determine $s=s(\eta)$ so that $\tilde{h}_s(-\pi)>\tilde{h}(-\pi)-\eta/2$. By the definition of $\tilde{h}_s(-\pi)$, there exists a positive sequence $\{\tau_n\}\subset\tilde{\Lambda}(s)$ tending to ∞ such that

$$\tilde{B}(-\tau_n)>(\tilde{h}_s(-\pi)-\eta/2)\tilde{V}(\tau_n)>(\tilde{h}(-\pi)-\eta)\tilde{V}(\tau_n),\qquad n\in N,$$

where N is a sequence of unbounded increasing integers. Therefore it follows from this and (5.5) that

$$m^*(\tau_n, u)\geqq(h(0)\cos\pi\mu_*-\eta-\varepsilon_n)\tilde{V}(\tau_n). \tag{5.6}$$

Take $\eta=\eta_n=n^{-1}$ in (5.6). Using the method of diagonalization, we give to k consecutive values, and at each stage select $n=n_k$ such that

$$n_k\in N(k),\ n_{k+1}>n_k,\qquad n_k>k+s\Big(\frac{1}{k}\Big),\qquad 2\varepsilon_{n_k}<1/k.$$

If we put

$$y_k=\tau_{n_k},\quad y_k'=e^{-k}y_k,\quad y_k'''=e^k y_k,\quad \tilde{\tilde{B}}_k=\tilde{B}_{n_k},\quad \tilde{\tilde{V}}_k=\tilde{V}_{n_k},$$

then we have by (5.1)—(5.4), and (5.6)

$$m^*(y_k, u)>h(0)\Big(\cos\pi\mu_*-\frac{2}{k}\Big)\tilde{\tilde{V}}_k(y_k), \tag{5.7}$$

$$\tilde{\tilde{B}}_k(re^{i\theta})\leqq h(0)\Big(\cos\theta\mu_*+\frac{1}{k}\Big)\tilde{\tilde{V}}_k(r)\qquad(|\theta|\leqq\pi,\ y_k'\leqq r\leqq y_k'''), \tag{5.8}$$

$$\Big(1-\frac{1}{k}\Big)h(0)\frac{\sin\pi\mu_*}{\pi\mu_*}\tilde{\tilde{V}}_k(r)\leqq N(r, u)\leqq T(r, u) \tag{5.9}$$

$$\leqq\Big(1+\frac{1}{k}\Big)h(0)\frac{\sin\pi\mu_*}{\pi\mu_*}\tilde{\tilde{V}}_k(r),$$

$$m_2(r, u)\leqq\Big(1+\frac{1}{k}\Big)\sqrt{\frac{1}{2}+\frac{\sin 2\pi\mu_*}{4\pi\mu_*}}\,h(0)\tilde{\tilde{V}}_k(r)\qquad(y_k'\leqq r\leqq y_k'''). \tag{5.10}$$

Now, we prove the assertions I and II of our theorem. First by assumption (5), we easily see that given $\varepsilon>0$ there exists a bound k_0 such that $k\geqq k_0(\varepsilon)$ implies

$$m_2(y_k, u)>(1-\varepsilon)\sqrt{\frac{1}{2}+\frac{\sin 2\pi\mu_*}{4\pi\mu_*}}\,h(0)\tilde{\tilde{V}}_k(y_k). \tag{5.11}$$

Hence, by (5.10), (5.11) we have (6) and (13). And from (5.10), (5.11) and (5.9), (7) follows. (8), (11) and (12) are immediate consequences of (5.9).

6. Proof of assertion III of Theorem.

Choose $a=a(\mu_*)\in(0, 1/2)$ such that

$$a^{\mu_*}<\mu_*\log 2/2^{\mu_*}, \qquad a^{-\mu_*}-1+(2^{\mu_*}/\log 2)\log a>0. \tag{6.1}$$

Let

$$u_{1,k}(z)=\int_{y_k'<|\zeta|<a y_k'''}\log\left|1-\frac{z}{\zeta}\right|d\nu(\zeta), \qquad u_{3,k}(z)=u(z)-u_{1,k}(z).$$

Further put

$$u_{2,k}(z)=\int_{y_k'}^{a y_k'''}\log\left|1+\frac{z}{t}\right|dn(t).$$

Similar computations as in [19] yield

$$|u_{3,k}(z)|\leqq K_3\left\{T(2y_k', u)\log\left(\frac{r}{y_k'}\right)+T(2a y_k''', u)\left(\frac{r}{y_k'''}\right)\right\} \quad \left(2y_k'\leqq r\leqq\frac{a y_k'''}{2}\right). \tag{6.2}$$

From (11) we have

$$N(a y_k''', u)\leqq(1+\varepsilon_k)\left(\frac{y_k'''}{y_k}\right)^{\mu_*}a^{\mu_*}N(y_k, u), \tag{6.3}$$

$$n(a y_k''')\leqq(1+\varepsilon_k)(2a)^{\mu_*}\frac{N(y_k, u)}{\log 2}\left(\frac{y_k'''}{y_k}\right)^{\mu_*}.$$

On the other hand, by the choice of a of (6.1), we have

$$\left(\frac{y_k'''}{y_k}\right)^{\mu_*}a^{\mu_*}+\frac{(2a)^{\mu_*}}{\log 2}\left(\frac{y_k'''}{y_k}\right)^{\mu_*}\log\left(\frac{r}{a y_k'''}\right)\leqq\left(\frac{r}{y_k}\right)^{\mu_*}(r>y_k'''). \tag{6.4}$$

Combining (6.3) and (6.4), we obtain

$$N(a y_k''', u)+n(a y_k''')\log\left(\frac{r}{a y_k'''}\right)\leqq(1+\varepsilon_k)N(y_k, u)\left(\frac{r}{y_k}\right)^{\mu_*} \quad (r>y_k''').$$

It follows from this and (11) that

$$N_k(r)\equiv N(r, u_{1,k})=N(r, u_{2,k})\leqq(1+\varepsilon_k)N(y_k, u)\left(\frac{r}{y_k}\right)^{\mu_*} \quad (0<r<\infty). \tag{6.5}$$

Next, we introduce auxiliary functions:

$$B_k(z)=\int_{|\zeta|<\infty}\log\left|1+\frac{z}{t}\right|dn_k(t) \qquad (k=1, 2, \cdots), \tag{6.6}$$

where

$$(6.7)\qquad n_k(t)=\mu_*\left(\frac{t}{y_k}\right)^{\mu_*}N(y_k,\ u)\qquad (0<t<\infty).$$

The convergence of the right hand side of (6.6) is due to Heins [14]. If we put for a subharmonic function $u(z)$

$$c_m(r,\ u)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}u(re^{i\theta})e^{-im\theta}d\theta\qquad (m=0,\ \pm1,\ \cdots),$$

we easily have from (6.5)—(6.7)

$$(6.8)\qquad |c_m(y_k,\ B_k)|=N(y_k,\ u)\frac{\mu_*^2}{|m^2-\mu_2^*|},$$

$$(6.9)\qquad \frac{N(y_k,\ u)}{m_2(y_k,\ B_k)}=C_1(\mu_*)\qquad (k=1,\ 2,\ \cdots),$$

$$(6.10)\qquad |c_m(y_k,\ u_{2,k})|\leqq(1+2\varepsilon_k)\cdot|c_m(y_k,\ B_k)|.$$

By (6.2), (11), (8) and (7) we have for $|z|=y_k$

$$|u_{3,k}(z)|\leqq K_3T(y_k,\ u)(1+2\varepsilon_k)\left\{\left(\frac{y_k'}{y_k}\right)^{\mu_*}\log\left(\frac{y_k}{y_k'''}\right)+\left(\frac{y_k}{y_k'''}\right)^{1-\mu_*}(2a)^{\mu_*}\right\}$$

$$\leqq K_3C_1(\mu_*)(1+2\varepsilon_k)\varepsilon_k m_2(y_k,\ u)\qquad (k\geqq k_0),$$

so that with a suitable $\{\delta_k\}\downarrow 0$,

$$(6.11)\qquad 1-\delta_k<\frac{m_2(y_k,\ u_{1,k})}{m_2(y_k,\ u)}<1+\delta_k.$$

Further an estimate due to Miles and Shea gives

$$(6.12)\qquad |c_m(r,\ u_{1,k})|\leqq|c_m(r,\ u_{2,k})|\qquad (k=1,\ 2,\ \cdots;\ m=0,\ \pm1,\ \cdots).$$

Hence by (7), (6.11), (6.12) and (6.10), we have

$$C_1(\mu_*)=\lim_{k\to\infty}\frac{N(y_k,\ u)}{m_2(y_k,\ u)}=\lim_{k\to\infty}\frac{N(y_k,\ u)}{m_2(y_k,\ u_{1,k})}$$

$$\geqq\varlimsup_{k\to\infty}\frac{N(y_k,\ u)}{m_2(y_k,\ u_{2,k})}\geqq\varlimsup_{k\to\infty}\frac{N(y_k,\ u)}{m_2(y_k,\ B_k)}=C_1(\mu_*).$$

This implies, in particular, that for $m=1,\ 2$

$$(6.13)\qquad |c_m(y_k,\ u_{1,k})|>(1-\eta_k)|c_m(y_k,\ u_{2,k})|\qquad (\eta_k\downarrow 0,\ k=1,\ 2,\ \cdots).$$

Now we appeal to the reasoning of Miles and Shea in [17, pp. 182-183]. In fact, their reasoning in it is applicable since (6.8), (6.10) and (6.13) hold. Hence it is possible to find a positive, increasing, unbounded sequence $\{M_k\}$ $(M_k>1)$ such that

$$M_k y_k \leqq y_k''' \qquad (k=1, 2, \cdots),$$

and such that $M_k^{3/4} y_k \leqq r \leqq M_k y_k$ implies

$$N(r, u; S_k) < \frac{1}{M_k^{\mu_*/8}} N(r, u)$$

for a suitable S_k. Therefore, if we put

$$x_k''' = M_k y_k, \quad x_k'' = M_k^{3/4} y_k, \quad x_k = y_k, \quad x_k' = y_k', \quad \varepsilon_k' = \max\left(\varepsilon_k, \frac{1}{M_k^{\mu_*/8}}\right),$$

then all the assertions I, II and III of our theorem are valid for $\{y_k\}$, $\{y_k'\}$, $\{y_k''\}$, $\{y_k'''\}$, $\{\varepsilon_k\}$ replaced by $\{x_k\}$, $\{x_k'\}$, $\{x_k''\}$, $\{x_k'''\}$, $\{\varepsilon_k'\}$, respectively. This completes the proof of our theorem.

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