T. KATO
KODAI MATH. J.
4 (1981), 443—456

ON RIEMANN SURFACES OF GENUS FOUR WITH NON-TRIVIAL AUTOMORPHISMS

BY TAKAO KATO

§ 1. Introduction.

Let S be a compact Riemann surface of genus greater than two. Generically S does not admit a non-trivial conformal automorphism (Baily [1]). (Henceforth, we shall use the term automorphism instead of conformal automorphism.) Hence, compact Riemann surfaces which admit non-trivial automorphisms have some restricted properties. These appear in the vanishing properties of the theta functions, in Weierstrass points, in defining equations of these surfaces and etc. Recently, Kuribayashi and Komiya [6] determined all defining equations of Riemann surfaces of genus three which admit non-trivial automorphisms.

In this paper we shall consider Riemann surfaces of genus four and shall determined all defining equations of surfaces whose automorphisms groups are of order three. First, we shall give defining equations of surfaces having automorphisms of prime orders. The admissible prime orders are 2, 3 and 5. To study the cases of order 3 and 5 carefully we obtain the main result.

§ 2. Statement of results.

Let S be a compact Riemann surface of genus 4. Suppose that ϕ is an automorphism of S of prime order N. Let t be the number of the fixed points of ϕ and let $\tilde{g}$ be the genus of $S/\langle\phi\rangle$, where $\langle\phi\rangle$ is the group generated by ϕ. Since N is prime, using the Riemann-Hurwitz relation we have

$$6=(2\tilde{g}-2)N+(N-1)t .$$

Thus there are the following 7 cases: (It is known that case $N=7$, $\tilde{g}=1$, $t=1$ does not occur.)

(I) $N=2$, $\tilde{g}=0$, $t=10$,
(II) $N=2$, $\tilde{g}=1$, $t=6$,
(III) $N=2$, $\tilde{g}=2$, $t=2$,
(IV) $N=3$, $\tilde{g}=0$, $t=6$,

Received April 18, 1980

(V) $N=3,\ \tilde{g}=1,\ t=3,$
(VI) $N=3,\ \tilde{g}=2,\ t=0,$
(VII) $N=5,\ \tilde{g}=0,\ t=4.$

Then we have a defining equation of S as follows:

THEOREM 1. Suppose that S is a Riemann surface of genus four which admits a non-trivial automorphism φ of prime order. Then, S *is defined by* one of (1)-(17). *Here,* $\alpha, \beta, \cdots$ *are complex numbers and* $A(x), B(x), \cdots$ *are polynomials in* x. *Although* $A(x), B(x), \cdots$ *and* $\alpha, \beta, \cdots$ *must satisfy so that the genus of* S *is four, the genera of the surfaces defined by* (1)-(17) *are generically four.*

Case (I).

(1) $$y^2=x(x-1)(x-\alpha_1)\cdots(x-\alpha_7).$$

Case (II).

(2) $$y^4-2A(x)y^2+A(x)^2-B(x)^2x(x-1)(x-\alpha)=0.$$

Here, if there is a fixed point P *of* ϕ *so that* $N(P)=\{4, 5, 6, 8\}$, *then* $\deg A\leqq 2$ *and* $\deg B=1$ *and if there is no such a point, then* $\deg A\leqq 3$, $\deg B=2$ *and* $A(x)^2-B(x)^2x(x-1)(x-\alpha)$ *has at least one double zero. The definition of* $N(P)$ *is given in the next section.*

Case (III). *If there is a fixed point* P *of* ϕ *so that* $N(P)=\{3, 5, 6, 8\}$, *then the equation is*

(3) $$y^3+(\alpha x^2-\beta)y+x(x^2-1)(x^2-\gamma)=0.$$

If there is a fixed point P *of* ϕ *so that* $N(P)=\{3, 6, 7, 8\}$, *then the equation is*

(4) $$y^3+(\alpha x^4+\beta x^2+\gamma)y+x(x^2-1)(x^2-\delta)(x^2-\varepsilon)=0.$$

If there is a fixed point P *of* ϕ *so that* $N(P)=\{4, 5, 7, 8\}$, *then the equation is*

(5) $$y^4-2A(x)y^2+A(x)^2-x(x-1)(x-\alpha)(x-\beta)(x-\gamma)=0.$$

Here, $\deg A\leqq 2$ *and* $A(x)^2-x(x-1)(x-\alpha)(x-\beta)(x-\gamma)$ *has at least one double zero.*

If both of the fixed points P, Q *of* ϕ *satisfy that* $N(P)=N(Q)=\{5, 6, 7, 8\}$ *and if* $3P+3Q$ *is a canonical divisor, then the equation is*

(6) $$y^2=(x^2-1)(x^2-\alpha_1)\cdots(x^2-\alpha_4)$$

or

(7) $$y^4+(x^3+\alpha x^2+\beta x)y^2+\gamma x^2(x-\delta)^2=0.$$

If $3P+3Q$ *is not a canonical divisor, then the equation is*

$$y^4+(x^3+\alpha x^2+\beta x+\gamma)y^2+\delta x(x-1)^2(x-\varepsilon)^2=0. \tag{8}$$

Case (IV). *If there is a fixed point* P *of* ϕ *so that* $N(P)=\{3, 5, 6, 8\}$, *then the equation is*

$$y^3=x(x-1)(x-\alpha)(x-\beta)(x-\gamma). \tag{9}$$

If there is no such a point, then the equation is

$$y^3=x^2(x-1)^2(x-\alpha)(x-\beta)(x-\gamma). \tag{10}$$

Case (V). *If* S *is hyperelliptic, then the equation is*

$$y^2=(x^3-1)(x^3-\alpha)(x^3-\beta). \tag{11}$$

If S *is non-hyperelliptic, then the equation is*

$$y^6-2(\beta x+\gamma)y^3+(\beta x+\gamma)^2-x(x-1)(x-\alpha)=0. \tag{12}$$

Case (VI).

$$y^6+(\alpha x^3+\beta x^2+\gamma x+\delta)y^3+1=0 \tag{13}$$

or

$$y^6+(\alpha x^3+\beta x^2+\gamma x+\delta)y^3+x^3=0. \tag{14}$$

Case (VII).

$$y^5=x(x-1)(x-\alpha), \tag{15}$$

$$y^5=x^3(x-1)^2(x-\alpha) \tag{16}$$

or

$$y^5=x^4(x-1)(x-\alpha). \tag{17}$$

Studying Cases (IV)-(VII) carefully, we have

THEOREM 2. *Let* S *be a compact Riemann surface of genus* 4. *Then the order of the automorphisms group of* S *is three if and only if* S *is defined by* (9), (10) *or* (12), *where* α, β *and* γ *are chosen generically.*

§ 3. Preparations.

In this section we shall state some known results which are used in the proof of our theorems. Let S be a compact Riemann surface of genus g ($\geqq 2$) and let P be a point on S. Then there are g orders n_i, $1=n_1<n_2<\cdots<n_g<2g$

such that there is no meromorphic function on S which has a pole of order n_i at P and is holomorphic elsewhere. The sequence $G(P)=\{n_1, \cdots, n_g\}$ is called the gap sequence at P. The sequence $N(P)=\{1, \cdots, 2g\}-G(P)$ is often called the Weierstrass sequence. If a positive integer m is not a member of $G(P)$, then m is called a non-gap value and there is a non-constant meromorphic function on S which has a pole of order m at P and is holomorphic elsewhere. If $G(P)$ is known, then we have a defining equation of S as follows:

LEMMA 1. [2, 5, 9] *Let m be the first non-gap at P and let n be the least non-gap which is prime to m. If x and y are meromorphic functions on S with poles of order m and n at P, respectively, and being holomorphic elsewhere, then S is defined by*

$$y^m+A_1(x)y^{m-1}+\cdots+A_{m-1}(x)y+A_m(x)=0,$$

where $A_i(x)$ $(i=1, \cdots, m)$ are polynomials in x and $\deg A_i \leqq \frac{ni}{m}$ $(i\neq m)$, $\deg A_m=n$.

Let ϕ be an automorphism of S of prime order N. Put

$$H_j=\{\theta \mid \theta \text{ is a holomorphic differential on } S \text{ satisfying } \theta\circ\phi=\mu^j\theta\},$$

$(j=0, \cdots, N-1)$, where $\mu=\exp(2\pi i/N)$. Let n_j be the dimension of H_j. Then Lewittes [7] proved

LEMMA 2. *Suppose that ϕ has t fixed points and that the genus of $S/\langle\phi\rangle$ is $\tilde{g}$. Then*

i) $n_0=\tilde{g}$.

ii) *If $t=0$, then $n_j=\tilde{g}-1$ for $1\leqq j\leqq N-1$.*

iii) *If $t>1$, then there is at least one index k, $1\leqq k\leqq N-1$, such that $n_k\neq 0$ and for any such index*

$$\tilde{g}-1+\frac{N-1}{N}t\geqq n_k\geqq \tilde{g}-1+\frac{t}{N}.$$

To examine the order of the automorphisms group of surfaces the following lemma due to Igusa is convenient.

LEMMA 3. [4] *Let S and S' be compact Riemann surfaces of positive genera. Let $P(x, y)=0$ and $P'(x, y)=0$ be defining equations of S and S', respectively. Suppose that P' is a specialization of P. Then the automorphisms group of S is isomorphic to a subgroup of the automorphisms group of S'.*

For a non-hyperelliptic surface of genus 4, we have

LEMMA 4. [3, 5, 8] *Let S be a non-hyperelliptic Riemann surface of genus 4. If S admits a half-canonical divisor of dimension 2, then, up to a linear transformation, S can be expressible as a 3-sheeted covering of $\boldsymbol{P}^1$ only in one way.*

If S admits no half-canonical divisor of dimension 2, then, up to a linear transformation, S can be expressible as a 3-sheeted covering of $\boldsymbol{P}^1$ in two ways. Furthermore, if D_1 and D_2 are divisors of degree 3 and of dimension 2 and if these are not linearly equivalent, then D_1+D_2 is a canonical divisor.

§4. Proof of Theorem 1.

Case (I). In this case, S is hyperelliptic. Then we easily have an equation

$$y^2=x(x-1)(x-\alpha_1)\cdots(x-\alpha_7), \tag{1}$$

where $\alpha_1, \cdots, \alpha_7$ are mutually distinct complex numbers.

Case (II). Let π be the natural projection of S onto $S/\langle\phi\rangle$. Let $P_1, \cdots, P_6$ be the fixed points of ϕ. Let $Y^2=x(x-1)(x-\alpha)$ be a defining equation of the torus $S/\langle\phi\rangle$ where $(x, Y)=(\infty, \infty)$ corresponds to $\pi(P_1)$. Since $x\circ\pi$ and $Y\circ\pi$ are meromorphic functions on S, $N(P_1)$ is $\{4, 5, 6, 8\}$ or $\{4, 6, 7, 8\}$. Let f be a meromorphic function on S whose polar divisor is $5P_1$ or $7P_1$. Put $y=f-f\circ\phi$. Then $y\circ\phi=-y$ and $y(P_j)=0$ $(j=2, \cdots, 6)$. If $(f)_\infty=5P_1$, then $(y)=P_2+\cdots+P_6-5P_1$ and if $(f)_\infty=7P_1$, then $(y)=P_2+\cdots+P_6+Q+\phi(Q)-7P_1$, where Q is a point on S. Since y^2 can be viewed as a meromorphic function on $S/\langle\phi\rangle$ whose polar divisor is $5\pi(P_1)$ or $7\pi(P_1)$, we have

$$y^2=A(x)+B(x)Y. \tag{18}$$

Here $A(x)$ and $B(x)$ are polynomials in x such that $\deg A\leqq 2$, $\deg B=1$ if $(f)_\infty=5P_1$ and that $\deg A\leqq 3$, $\deg B=2$ if $(f)_\infty=7P_1$. From (18) we have

$$y^4-2A(x)y^2+A(x)^2-B(x)^2x(x-1)(x-\alpha)=0. \tag{2}$$

In case of $(f)_\infty=7P_1$, since $\pi(Q)=\pi\circ\phi(Q)$,

$$A(x)^2-B(x)^2x(x-1)(x-\alpha)$$

has at least one double zero.

Case (III). Let P_1 and P_2 be the fixed points of ϕ. Since $S/\langle\phi\rangle$ is of genus 2, $N(P_1)$ is $\{3, 5, 6, 8\}$, $\{3, 6, 7, 8\}$, $\{4, 5, 7, 8\}$ or $\{5, 6, 7, 8\}$.

Suppose that $N(P_1)$ is $\{3, 5, 6, 8\}$. There are meromorphic functions x, y such that $x\circ\phi=-x$, $y\circ\phi=-y$ and that the polar divisors of x and y are $3P_1$ and $5P_1$, respectively. Then we have

$$y^3+A(x)y^2+B(x)y+C(x)=0, \tag{19}$$

where $A(x)$, $B(x)$ and $C(x)$ are polynomials in x such that $\deg A\leqq 1$, $\deg B\leqq 3$ and $\deg C=5$. Since $x\circ\phi=-x$ and $y\circ\phi=-y$,

$$-y^3+A(-x)y^2-B(-x)y+C(-x)=0. \tag{20}$$

Adding (19) and (20), we have

$$(A(x)+A(-x))y^2+(B(x)-B(-x))y+C(x)+C(-x)=0.$$

Since S is not hyperelliptic, $A(x)+A(-x)=0$, $B(x)-B(-x)=0$ and $C(x)+C(-x)=0$. Thus $A(x)=\alpha_1 x$, $B(x)=\beta_1 x^2+\beta_2$ and $C(x)=\gamma_1 x^5+\gamma_2 x^3+\gamma_3 x$, where $\gamma_1\neq 0$. In (19), replacing y to $y+\alpha_1 x/3$ and applying a suitable linear transformation in x, we have

$$y^3+(\alpha x^2-\beta)y+x(x^2-1)(x^2-\gamma)=0. \tag{3}$$

Suppose that $N(P_1)$ is $\{3, 6, 7, 8\}$. There are meromorphic functions x, y such that $x\circ\phi=-x$, $y\circ\phi=-y$ and that the polar divisors of x and y are $3P_1$ and $7P_1$, respectively. Then we have

$$y^3+A(x)y^2+B(x)y+C(x)=0,$$

where $A(x)$, $B(x)$ and $C(x)$ are polynomials in x such that $\deg A\leqq 2$, $\deg B\leqq 4$ and $\deg C=7$. As in the preceding paragraph we have $A(x)=\alpha_1 x$, $B(x)=\beta_1 x^4+\beta_2 x^2+\beta_3$ and $C(x)=\gamma_1 x^7+\gamma_2 x^5+\gamma_3 x^3+\gamma_4 x$, where $\gamma_1\neq 0$. Hence, applying suitable transformations we have

$$y^3+(\alpha x^4+\beta x^2+\gamma)y+x(x^2-1)(x^2-\delta)(x^2-\varepsilon)=0. \tag{4}$$

Suppose that $N(P_1)$ is $\{4, 5, 7, 8\}$. Let π be the natural projection of S onto $S/\langle\phi\rangle$. There are meromorphic functions x, y on S such that $x\circ\phi=x$, $y\circ\phi=-y$ and that the polar divisors of x and y are $4P_1$ and $5P_1$, respectively. Since $x\circ\phi=x$ and $y\circ\phi=-y$, x and y^2 can be viewed as meromorphic functions on $S/\langle\phi\rangle$ whose polar divisors are $2\pi(P_1)$ and $5\pi(P_1)$, respectively. Since $\pi(P_1)$ is a Weierstrass point of $S/\langle\phi\rangle$, we may assume that $S/\langle\phi\rangle$ is defined by

$$Y^2=x(x-1)(x-\alpha)(x-\beta)(x-\gamma).$$

Since the polar divisor of Y is $5\pi(P_1)$, multiplying a suitable constant we have

$$y^2=A(x)+Y,$$

where $A(x)$ is a polynomial in x such that $\deg A\leqq 2$. Thus we have

$$y^4-2A(x)y^2+A(x)^2-x(x-1)(x-\alpha)(x-\beta)(x-\gamma)=0. \tag{5}$$

Since $y(P_2)=0$ and x has the multiplicity at least 2 at $x(P_2)$,

$$A(x)^2-x(x-1)(x-\alpha)(x-\beta)(x-\gamma)$$

has at least one double zero.

Suppose that both $N(P_1)$ and $N(P_2)$ are $\{5, 6, 7, 8\}$. Let θ be a holomorphic differential on S such that $(\theta)\geqq 3P_1$. Considering $\theta+\theta\circ\phi$ we may assume that $\theta=\theta\circ\phi$. Then $(\theta)=3P_1+P_2+Q+\phi(Q)$, where $P_1\neq Q$.

First, assume that $P_2 \neq Q$. Since $l(3P_1+Q+\phi(Q))=3$ and $l(3P_1)=1$, we have $l(3P_1+Q)=2$, where $l(D)$ is the dimension of the space of meromorphic functions whose divisors are multiples of D. If there is a meromorphic function f on S whose polar divisor is $2P_1+Q$ or P_1+Q, then the polar divisor of $f-f\circ\phi$ is $P_1+Q+\phi(Q)$. If $(f)_\infty=2P_1+Q$, then using Lemma 4 we have that $(2P_1+Q)+(P_1+Q+\phi(Q))$ is a canonical divisor which contradicts the fact that $3P_1+P_2+Q+\phi(Q)$ is also a canonical divisor. If $(f)_\infty=P_1+Q$, then S is hyperelliptic and there is no meromorphic function whose polar divisor is of degree 3. This is a contradiction. Therefore, there is a meromorphic function f on S whose polar divisor is $3P_1+Q+\phi(Q)$. Put $x=f+f\circ\phi$ and $y=f-f\circ\phi$. Then $(x)_\infty=2P_1+Q+\phi(Q)$ and $(y)=P_2+D+\phi(D)-(3P_1+Q+\phi(Q))$, where D is a positive divisor of degree 2. Viewing x and y^2 as functions on $S/\langle\phi\rangle$, we have $(x)_\infty=\pi(P_1)+\pi(Q)$ and $(y^2)=\pi(P_2)+2\pi(D)-(3\pi(P_1)+2\pi(Q))$. Since $\deg(x)_\infty=2$, $S/\langle\phi\rangle$ is defined by $Y^2=A(x)$ where $A(x)$ is a polynomial in x whose degree is 6. Since y^2 is a meromorphic function on $S/\langle\phi\rangle$, there are rational functions $B(x)$ and $C(x)$ such that

$$y^2=B(x)+C(x)Y\,.$$

Let σ be the hyperelliptic involution of $S/\langle\phi\rangle$. Then $Y=-Y\circ\sigma$ and $x=x\circ\sigma$ and $\pi(Q)=\pi\circ\sigma(P_1)$. Therefore,

$$y^2\circ\sigma=B(x)-C(x)Y\,.$$

Hence,

$$(B(x))_\infty=(y^2+y^2\circ\sigma)_\infty=3\pi(P_1)+3\pi(Q)$$

and

$$(C(x)Y)_\infty=(y^2-y^2\circ\sigma)_\infty=3\pi(P_1)+3\pi(Q)\,.$$

Since $(Y)_\infty=3\pi(P_1)+3\pi(Q)$, B is a polynomial of degree 3 and C is a non-zero constant. Since

$$\begin{aligned}(B(x)^2-(CY)^2)&=(y^2\cdot y^2\circ\sigma)\\&=\pi(P_2)+\sigma\circ\pi(P_2)+2\pi(D)+2\sigma\circ\pi(D)-5\pi(P_1)-5\pi(Q)\,,\end{aligned}$$

$B(x)^2-(CY)^2$ is of degree 5 and at least 2 double zeros. Thus, noting that $D\neq 2P_2$ and applying suitable transformations we have

$$y^4+(x^3+\alpha x^2+\beta x+\gamma)y^2+\delta x(x-1)^2(x-\varepsilon)^2=0\,. \tag{8}$$

Next, assume that $P_2=Q$, i.e. $(\theta)=3P_1+3P_2$. Using the Riemann-Roch theorem we have $l(3P_1+P_2)\geqq 2$ and $l(2P_1+2P_2)\geqq 2$. If there is a meromorphic function x on S whose polar divisor is P_1+P_2, then S is hyperelliptic and defined by

$$y^2=(x^2-1)(x^2-\alpha)(x^2-\beta)(x^2-\gamma)(x^2-\delta)\,. \tag{6}$$

If there is a meromorphic function f on S whose polar divisor is $2P_1+P_2$, then the polar divisor of $x=f-f\circ\phi$ is P_1+P_2. This is absurd.

Hence, if there is no meromorphic function on S whose polar divisor is P_1+P_2, then there are meromorphic functions x and y whose polar divisors are $2P_1+2P_2$ and $3P_1+P_2$, respectively, and $x\circ\phi=x$, $y\circ\phi=-y$. Viewing x and y^2 as functions on $S/\langle\phi\rangle$, we have $(x)_\infty=\pi(P_1)+\pi(P_2)$ and $(y^2)_\infty=3\pi(P_1)+\pi(P_2)$. Let $S/\langle\phi\rangle$ be defined by $Y^2=A(x)$, where $A(x)$ is a polynomial in x of degree 6 and let $y^2=B(x)+C(x)Y$. As above, since $y^2\circ\sigma=B(x)-C(x)Y$, B is a polynomial in x of degree 3 and C is a non-zero constant. Putting $(y)=D+\phi(D)-(3P_1+P_2)$, where D is a positive divisor of degree 2, we have $(y^2)=2\pi(D)-(3\pi(P_1)+\pi(P_2))$ on $S/\langle\phi\rangle$ and $(y^2\cdot y^2\circ\sigma)=2\pi(D)+2\sigma\circ\pi(D)-4(\pi(P_1)+\pi(P_2))$. Hence,

$$B(x)^2-(CY)^2=a(x-b)^2(x-c)^2\,,$$

where a, b and c are constants. On the other hand, the divisor of $x-b$ is $D'+\phi(D')-2P_1-2P_2$ for a positive divisor D' of degree 2. Therefore, it is viewed as $2\pi(D')-\pi(P_1)-\pi(P_2)$ on $S/\langle\phi\rangle$. Thus $y^4-2A(b)y^2$ has two double zeros or one fourth order zero. Hence, $A(b)=0$. Applying suitable transformations we have

$$y^4+(x^3+\alpha x^2+\beta x)y^2+\gamma x^2(x-\delta)^2=0\,. \tag{7}$$

Case (IV). Let $P_1, \cdots, P_6$ be the fixed points of ϕ. Since $S/\langle\phi\rangle$ is of genus zero, $N(P_1)$ is $\{3, 5, 6, 8\}$ or $\{3, 6, 7, 8\}$. Then there are meromorphic functions x, y such that $x\circ\phi=x$, $y\circ\phi=\omega y$ or $\omega^2 y$ and that the polar divisor of x is $3P_1$ and that of y is $5P_1$ or $7P_1$. By Lemma 1 we have

$$y^3+A(x)y^2+B(x)y+C(x)=0\,, \tag{21}$$

where $\deg A\leqq 2$, $\deg B\leqq 4$ and $\deg C=5$ or 7. Noting that $x\circ\phi=x$ and $y\circ\phi=\omega y$ or $\omega^2 y$ we have

$$y^3+A(x)\omega^2y^2+B(x)\omega y+C(x)=0 \tag{22}$$

and

$$y^3+A(x)\omega y^2+B(x)\omega^2 y+C(x)=0\,. \tag{23}$$

Adding (21), (22) and (23) we have

$$y^3+C(x)=0\,, \tag{24}$$

that is, $A(x)\equiv 0$ and $B(x)\equiv 0$. Since $y(P_j)=y\circ\phi(P_j)=\omega y(P_j)$ or $\omega^2 y(P_j)$, $(j=2, \cdots, 6)$, we have $y(P_j)=0$. Hence, $x-x(P_j)$ divides $C(x)$. If $y(Q)=0$ for $Q\neq P_j$ $(j=1, \cdots, 6)$, then considering local expansions of x and y at Q, $\phi(Q)$, $\phi^2(Q)$ we know that $(x-x(Q))(x-x\circ\phi(Q))(x-x\circ\phi^2(Q))=(x-x(Q))^3$ divides $C(x)$. Thus,

$$C(x)=\prod_{j=2}^{6}(x-x(P_j))^{\nu_j}\,,$$

where $\sum \nu_j \geqq 1$ and $\sum \nu_j = 5$ or 7. Hence, applying suitable transformations we have that S is defined by

$$y^3 = x(x-1)(x-\alpha)(x-\beta)(x-\gamma), \quad \text{if} \quad N(P_1) = \{3, 5, 6, 8\} \tag{9}$$

and

$$y^3 = x^2(x-1)^2(x-\alpha)(x-\beta)(x-\gamma), \quad \text{if} \quad N(P_1) = \{3, 6, 7, 8\}, \tag{10}$$

where α, β, γ are mutually distinct complex numbers.

Case (V). Let P_1, P_2, P_3 be the fixed points of ϕ. By Lemma 2, we have holomorphic differentials θ_1, θ_2 whose divisors are $2P_1+2P_2+2P_3$ and $P_1+P_2+P_3+Q+\phi(Q)+\phi^2(Q)$, respectively. Furthermore, $\theta_1 \circ \phi = \theta_1$, $\theta_2 \circ \phi = \omega\theta_2$ or $\omega^2\theta_2$, where $\omega^2+\omega+1=0$.

If $Q=P_1$, say, then $2P_1 \equiv P_2+P_3$ and S is hyperelliptic. Thus S is defined by

$$y^2 = (x^3-1)(x^3-\alpha)(x^3-\beta). \tag{11}$$

Suppose that $Q \neq P_1$, P_2, P_3. Put $y = \theta_1/\theta_2$. Viewing y^3 as a function on $S/\langle\phi\rangle$, we have that the divisor of y^3 is $\pi(P_1)+\pi(P_2)+\pi(P_3)-3\pi(Q)$, where π is the natural projection of S onto $S/\langle\phi\rangle$. Let the torus $S/\langle\phi\rangle$ be defined by $Y^2 = x(x-1)(x-\alpha)$ so that $\pi(Q)$ corresponds to $(X, Y) = (\infty, \infty)$. Putting $y^3 = A(x)+B(x)Y$, since $(x)_\infty = 2\pi(Q)$ and $(Y)_\infty = 3\pi(Q)$, we have that A is a polynomial in x of degree at most 1 and B is a non-zero constant. Thus we have

$$y^3 = \beta x + \gamma + Y$$

and

$$y^6 - 2(\beta x+\gamma)y^3 + (\beta x+\gamma)^2 - x(x-1)(x-\alpha) = 0, \tag{12}$$

where β is a non-zero constant.

Case (VI). Let π be the natural projection of S onto $S/\langle\phi\rangle$ and let σ be the hyperelliptic involution of $S/\langle\phi\rangle$. Using Lemma 2 we can choose a basis $\theta_1, \theta_2, \theta_3, \theta_4$ of the space of holomorphic differentials on S such that $\theta_1 \circ \phi = \omega\theta_1$, $\theta_2 \circ \phi = \omega^2\theta_2$, $\theta_3 \circ \phi = \theta_3$ and $\theta_4 \circ \phi = \theta_4$. Let the divisor of θ_1 be $P+\phi(P)+\phi^2(P)+Q+\phi(Q)+\phi^2(Q)$. Since θ_3 and θ_4 are the liftings of holomorphic differentials on $S/\langle\phi\rangle$, we may assume that the divisor of θ_3 is $P+\varphi(P)+\phi^2(P)+P'+\phi(P')+\phi^2(P')$, where $\pi(P') = \sigma \circ \pi(P)$. If $Q=P$ or $Q=P'$, then we shall not need θ_4 in the proof and if $Q \neq P$ and $Q \neq P'$, then we assume that the divisor of θ_4 is $Q+\phi(Q)+\phi^2(Q)+Q'+\phi(Q')+\phi^2(Q')$, where $\pi(Q') = \sigma \circ \pi(Q)$.

Assuming that $Q=P'$ we have that θ_3/θ_1 is a constant, which is a contradiction.

Assume that $Q=P$. Put $y = \theta_3/\theta_1$. Viewing y^3 as a function on $S/\langle\phi\rangle$ we have $(y^3) = 3\pi(P') - 3\pi(P)$. Let $S/\langle\phi\rangle$ be defined by $Y^2 = C(x)$, where $C(x)$ is a polynomial of degree 6 and both $\pi(P)$ and $\pi(P')$ correspond to $x=\infty$. Put $y^3=A(x)+B(x)Y$, where $A(x)$ and $B(x)$ are rational functions in x. Then we have

$$y^3\circ\sigma=A(x)-B(x)Y\ ,$$

$$(A)_\infty=(y^3+y^3\circ\sigma)_\infty=3(\pi(P)+\pi(P'))\ ,$$

$$(BY)_\infty=(y^3-y^3\circ\sigma)_\infty=3(\pi(P)+\pi(P'))$$

and

$$(y^3\cdot y^3\circ\sigma)=(A^2-B^2C)=0\ .$$

Hence, A is a polynomial of degree 3 and both B and A^2-B^2C are non-zero constants. Therefore, applying suitable transformations we have

$$y^6+(\alpha x^3+\beta x^2+\gamma x+\delta)y^3+1=0\ . \tag{13}$$

Suppose that $Q\neq P$ and $Q\neq P'$. First, we avoid the case that $\pi(P)=\pi(P')$. Suppose that $P=P'$. The divisor of a function $f=\theta_3/\theta_1$ is $P+\phi(P)+\phi^2(P)-Q-\phi(Q)-\phi^2(Q)$ and $P+\phi(P)+\phi^2(P)$ is a half-canonical divisor. Put $g=\theta_3\theta_4/\theta_1^2$. Since $g\circ\phi=\omega g$, g is not a constant. Then the polar divisors of f and g are $Q+\phi(Q)+\phi^2(Q)$ both. Using Lemma 1 we have $f=ag+b$ for some constants a and b. Since $\omega^2 f=f\circ\phi=ag\circ\phi+b=a\omega g+b$, we have $(\omega^2-1)f=a(\omega-1)g$, which is a contradiction. Thus we have $P\neq P'$. Since ϕ is fixed point free, we have $\pi(P)\neq\pi(P')$. Put $y=1/f=\theta_1/\theta_3$. Viewing y^3 as a function on $S/\langle\phi\rangle$ we have $(y^3)=3\pi(Q)-3\pi(P')$ and $(y^3\circ\phi)=3\pi(Q')-3\pi(P)$. Let $S/\langle\phi\rangle$ be defined by $Y^2=C(x)$, where $C(x)$ is a polynomial in x of degree 6 and both $\pi(P)$ and $\pi(P')$ correspond to $x=\infty$. Put $y^3=A(x)+B(x)Y$. As above we have $A(x)$ is a polynomial of degree 3 and $B(x)$ is a non-zero constant. Viewing $(y^3\cdot y^3\circ\sigma)=3(\pi(Q)+\pi(Q')-\pi(P)-\pi(P'))$ and applying suitable transformations we have

$$A(x)^2-B(x)^2Y^2=x^3$$

and

$$y^6+(\alpha x^3+\beta x^2+\gamma x+\delta)y^3+x^3=0\ . \tag{14}$$

Case (VII). Let P_1, P_2, P_3, P_4 be the fixed points of ϕ. There are meromorphic functions x, y on S such that $(x)_\infty=5P_1$ and $(y)_\infty=3P_1$, $4P_1$ or $6P_1$. We may assume that $x\circ\phi=x$ and $y\circ\phi=\eta y$, where η is a primitive fifth root of unity. Then x has the multiplicity 5 at each of P_2, P_3 and P_4 and we have

$$y^5=(x-x(P_2))^\lambda(x-x(P_3))^\mu(x-x(P_4))^\nu\ ,$$

where $\lambda+\mu+\nu=\deg(y)_\infty$. Applying a suitable linear transformation we have

$$y^5=x(x-1)(x-\alpha)\ , \tag{15}$$

$$y^5=x^2(x-1)(x-\alpha)\ , \tag{25}$$

(16) $$y^5=x^3(x-1)^2(x-\alpha)$$

or

(17) $$y^5=x^4(x-1)(x-\alpha)\,.$$

Apply the birational transformation $X=1/x$, $Y=\alpha^{-1/5}yx^{-1}$ to (25). Then the surface defined by (25) is conformally equivalent to the surface defined by

$$y^5=x(x-1)\left(x-\frac{1}{\alpha}\right),$$

which is the same type as (15).

This completes the proof of Theorem 1.

§ 5. Details of Case (VII).

To prove Theorem 2 we shall discuss Cases (IV)-(VII) more closely. First we do with Case (VII).

Suppose that S admits an automorphism ϕ of order 5. Then $S/\langle\phi\rangle$ is the sphere, ϕ has 4 fixed points and S is defined by (15), (16) or (17). We have easily the following observation.

"If S is defined by (15), then the gap sequence of the point corresponding to $x=\infty$ is $\{3, 5, 6, 8\}$ and those to $x=0, 1, \alpha$ are $\{4, 5, 7, 8\}$. If S is defined by (16), then the points corresponding to $x=0, 1, \alpha, \infty$ i.e. all of the fixed points of ϕ are not Weierstrass points. If S is defined by (17), then S is hyperelliptic and the fixed points of ϕ are not Weierstrass points".

§ 6. Details of Case (IV).

Suppose that S admits an automorphism ϕ of order 3 such that $S/\langle\phi\rangle$ is of genus zero. Then S is defined by (9) or (10). It is noted that ϕ may be defined by $\phi(x, y)=(x, \omega y)$.

Suppose that S is defined by (9). Then the points $(x, y)=(0, 0), (1, 0), (\alpha, 0), (\beta, 0), (\gamma, 0)$ and (∞, ∞) are Weierstrass points of weight 4 and these are all of such points. Let ψ be another automorphism of S. Since ψ maps these 6 points onto themselves, $\Phi=\phi^{-1}\circ\psi^{-1}\circ\phi\circ\psi$ fixes these 6 points. Using the Riemann-Hurwitz formula we have that Φ is the identity, ϕ or ϕ^2. If $\Phi=\phi^2$, then $\phi\circ\psi=\psi$. This is a contradiction. Hence, $\phi\circ\psi=\psi\circ\phi$ or $\phi\circ\psi=\psi\circ\phi^2$. Put $\psi(x, y)=(X, Y)$. If $\phi\circ\psi=\psi\circ\phi$, then $X(x, y)=X(x, \omega y)$. Hence, $X(x, y)=X(x)$. If $\phi\circ\psi=\psi\circ\phi^2$, then $X(x, y)=X(x, \omega^2 y)$. Hence, $X(x, y)=X(x)$. Thus ψ induces an elliptic linear transformation of the x-sphere.

Suppose that S is defined by (10). Let ν be the number of Weierstrass points whose gap sequences are $\{1, 2, 4, 5\}$. Since all the fixed points of ϕ are Weierstrass points with the gap sequences $\{1, 2, 4, 5\}$, $\nu=12$, 9 or 6.

Suppose that $\nu=12$. These 12 Weierstrass points are divided into two groups $A=\{P_1, \cdots, P_6\}$ and $B=\{P_7, \cdots, P_{12}\}$ such that $3P_1\equiv \cdots \equiv 3P_6$ and $3P_7\equiv \cdots \equiv 3P_{12}$. This is shown from Lemma 4. There is an automorphism σ such that $\sigma(A)=B$, $\sigma(B)=A$. ([5] Theorem 4). Suppose that ψ is another automorphism of S. If $\psi(A)=A$, then as in the above discussion we have that ψ induces an elliptic linear transformation of the x-sphere. If $\psi(\{P_1, \cdots, P_k\})\subset A$ ($k=4$ or 5), then for $j=1, \cdots, k$, $\Phi(P_j)=\phi^{-1}\circ\psi^{-1}\circ\phi\circ\psi(P_j)=P_j$. Using the Riemann-Hurwitz formula we have that Φ is the identity or ϕ. Thus, as above, ψ induces an elliptic linear transformation of the x-sphere. Hence, $\psi(A)=A$. If $\psi(\{P_1, \cdots, P_k\})\subset B$ ($k=4$ or 5), then $\sigma\circ\psi(\{P_1, \cdots, P_k\})\subset A$. Hence, $\sigma\circ\psi(A)=A$. Thus $\psi(A)=B$, $\psi(B)=A$. If $\psi(\{P_1, P_2, P_3\})\subset A$, then $\Phi(P_j)=\phi^{-1}\circ\psi^{-1}\circ\phi\circ\psi(P_j)=P_j$ for $j=1, 2, 3$. Suppose that neither Φ is the identity nor ϕ. Then $\langle\Phi, \phi\rangle$, the group generated Φ and ϕ, is a cyclic group of order 3μ ($\mu\geqq 2$, integer). Since ϕ fixes $P_1, \cdots, P_6$ and no other point, P_4, P_5 and P_6 are projected into the same point on $S/\langle\Phi, \phi\rangle$. Hence, $\mu=3$. Using the Riemann-Hurwitz formula we have a contradiction. Thus Φ is the identity or ϕ. Hence, $\psi(A)=A$.

Suppose that $\nu=9$. These 9 Weierstrass points are divided into two groups $A=\{P_1, \cdots, P_6\}$ and $B=\{P_7, P_8, P_9\}$ such that $3P_1\equiv \cdots \equiv 3P_6$, $3P_7\equiv 3P_8\equiv 3P_9$. Suppose that ψ is another automorphism. As in the case of $\nu=12$, we have $\psi(A)=A$. Thus ψ induces an elliptic linear transformation of the x-sphere.

Suppose that $\nu=6$. These 6 Weierstrass points $P_1, \cdots, P_6$ satisfy $3P_1\equiv \cdots \equiv 3P_6$. Thus, as in the above discussion, every automorphism ψ of S induces an elliptic linear transformation of the x-sphere.

Thus a generic surface defined by (9) or (10) has the automorphisms group of order exactly three.

§ 7. Details of Case (V).

Suppose that S admits an automorphism ϕ of order 3 such that $S/\langle\phi\rangle$ is a torus. Then S is defined by (11) or (12).

If S is defined by (11), then the order of the automorphisms group of S is at least 6.

Discussing the automorphisms groups of surfaces defined by (12), we shall consider a particular surface. Let S be defined by

$$y^6+(\alpha x-1-\beta)y^3+x^3-\alpha x+\beta=0\,, \tag{26}$$

where $\alpha\neq 0$, $\beta\neq 1$ and $x^3-\alpha x+\beta-(\alpha x-1-\beta)^2/4$ has distinct three zeros. Let S' be defined by

$$\begin{aligned} &Y^3+3\alpha\omega(1-\omega)X(X-1)(X+\omega^2)Y \\ &\qquad +3\omega(1-\omega)X(X-1)((X+\omega)^3-\beta(X+\omega^2)^3)=0\,. \end{aligned} \tag{27}$$

Applying the birational transformation

$$X=-\frac{\omega^2(y-\omega^2)}{y-1}, \quad Y=-3(y-1)^2x\,,$$

we have that S is conformally equivalent to S'. The points P_1, P_2, P_3 on S corresponding to $y=1$, ω, ω^2, i.e. to $X=\infty$, 1, 0 on S', are Weierstrass points of weight 4. Since $3P_j$ $(j=1, 2, 3)$ are half-canonical divisors of dimension 2, by Lemma 4 there is no other Weierstrass point of weight 4. Let ϕ be the automorphism of S defined by $\phi(x, y)=(x, \omega y)$. Then, $\phi(P_1)=P_2$, $\phi(P_2)=P_3$ and $\phi(P_3)=P_1$ and $S/\langle\phi\rangle$ is a torus.

Let ψ be an automorphism of S. Using the fact that ψ preserves $\{P_1, P_2, P_3\}$ we shall show that ψ is the identity, ϕ or ϕ^2.

First, suppose that $\psi(P_j)=P_j$ $(j=1, 2, 3)$. We shall show that ψ is the identity. If the order of $\langle\psi\rangle$ is two, then the genus of $S/\langle\psi\rangle$ is zero, one or two. If it is zero, then S is hyperelliptic. If it is one, then 4 is a non-gap value of P_j. If it is two, then there are exactly two fixed points of ψ. All of these are absurd. If the order of $\langle\psi\rangle$ is three, then the genus of $S/\langle\psi\rangle$ is zero, one or two. If it is zero, then there are 6 Weierstrass points of weight 4 or there is no Weierstrass point of weight 4. If it is one, then P_j are not Weierstrass points of weight 4. If it is two, then there is no fixed point of ψ. All of these are absurd. If the order of $\langle\psi\rangle$ is five, then S is defined by (15), (16) or (17). By the result of the section 5 it cannot occur that there are exactly 3 Weierstrass points of weight 4. Suppose that the order of $\langle\psi\rangle$ is a composite number. Considering subgroups of $\langle\psi\rangle$ of prime orders and applying the above results successively we have that ψ is the identity.

Secondly, suppose that $\psi(P_1)=P_2$, $\psi(P_2)=P_3$ and $\psi(P_3)=P_1$. Then $\phi^2\circ\psi(P_j)=P_j$ $(j=1, 2, 3)$. Hence, $\phi^2\circ\psi$ is the identity and $\phi=\psi$. Similarly, if $\psi(P_1)=P_3$, $\psi(P_2)=P_1$, $\psi(P_3)=P_2$, then $\phi\circ\psi$ is the identity and $\psi=\phi^2$.

Lastly, suppose that $\psi(P_1)=P_1$, $\psi(P_2)=P_3$ and $\psi(P_3)=P_2$. Then $\psi^2(P_j)=P_j$ $(j=1, 2, 3)$ and ψ^2 is the identity. Put $\Phi=\phi^{-1}\circ\psi^{-1}\circ\phi\circ\psi=\phi^2\circ\psi\circ\phi\circ\psi$. Since $\Phi(P_1)=P_2$, $\Phi(P_2)=P_3$ and $\Phi(P_3)=P_1$, we have $\Phi=\phi$ and $\psi\circ\phi=\phi^2\circ\psi$. Let Q be a fixed point of ϕ. Since $\phi^2\circ\psi(Q)=\psi\circ\phi(Q)=\psi(Q)$, $\psi(Q)$ is also a fixed point of ϕ. Since there are 3 fixed points of ϕ and the order of $\langle\psi\rangle$ is two, there must be a common fixed point of ϕ and ψ. Thus $\langle\phi, \psi\rangle$ is a cyclic group. This contradicts $\Phi=\phi$.

Summing up we have that the automorphisms group of S is $\langle\phi\rangle$.

The equation (26) is a specialization of (12). Hence, by Lemma 3 a generic surface defined by (12) has the automorphisms group of order three.

§8. Details of Case (VI).

Suppose that S admits an automorphism ϕ of order 3 such that $S/\langle\phi\rangle$ is of genus two. Then S is defined by (13) or (14).

Consider the birational transformations $(X, Y)=(x, 1/y)$, if S is defined by (15) and $(X, Y)=(x, x/y)$, if S is defined by (16). Then we have that the order of the automorphisms group of S is at least 6.

Conclusion. Summing up the results of the sections 6 and 7 and this section, we can conclude Theorem 2.

Added in proof. Recently, R. TSUJI (Thesis, Nihon Univ. 1981) studied related problems. He proved several results which overlap a part of Theorem 1.

REFERENCES

[1] BAILY, Jr. W.L., On the automorphism group of a generic curve of genus>2, J. Math. Kyoto Univ. 1-1, 101-108 (1961).
[2] BAKER, H.F., Abel's theorem and the allied theory, Cambridge Univ. Press 1879.
[3] HENSEL, K. AND G. LANDSBERG, Theorie der algebraiscen Funktionen einer Variablen, Chelsea reprint 1965.
[4] IGUSA, J., Arithmetic variety of moduli for genus two, Ann. Math., **72**, 612-649 (1960).
[5] KATO, T., On Weierstrass points whose first non-gaps are three, J. Reine Angew. Math. **316**, 99-106 (1980).
[6] KURIBAYASHI, A. AND K. KOMIYA, On Weierstrass points and automorphisms of curves of genus three, Lecture Notes in Math., **732**, Berlin, Heidelberg, New York: Springer 1979.
[7] LEWITTES, J., Automorphisms of compact Riemann surfaces, Amer. J. Math., **85**, 734-752 (1963).
[8] MARTENS, H.H., Varieties of special divisors on a curve, II. J. Reine Angew. Math., **233**, 89-100 (1968).
[9] WEIERSTRASS, K., Über Normalformen algebraischer Gebilde, Math. Werke III, 297-307, Georg Olms and Johnson reprint, 1967.

DEPARTMENT OF MATHEMATICS,
YAMAGUCHI UNIVERSITY,
YAMAGUCHI, JAPAN