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WEAKLY NULL SEQUENCES IN JAMES SPACES ON TREES

Dedecated to Professor Goro Azumaya on his sixtieth birthday

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Introduction. R. C. James [2] and J. Lindenstrauss and C. Stegall [3] gave the examples of separable Banach spaces having no subspace isomorphic to l^1 whose duals are non-separable. We are concerned here with James' example. In [2], he constructed a Banach space having properties a) it is separable and its dual is non-separable and b) every infinite dimensional subspace contains a subspace isomorphic to l^2. Property a) is a direct consequence of his construction, but to see property b) requires a rather deep observation. Property b) is equivalent to

b′) for any weakly null normalized sequence $\{x_n\,;\,n=1, 2, \cdots\}$ there is a sequence $\{y_n\,;\,n=1, 2, \cdots\}$ equivalent to an l^2-basis for which each y_n is a linear combination of x_n's together with

b″) every infinite dimensional subspace contains a weakly null normalized sequence.

In this paper we will prove a stronger property than b′), namely that there is a *subsequence*, instead of linear combinations, of $\{x_n\,;\,n=1, 2, \cdots\}$ which is equivalent to an l^2-basis. In fact, we will show this under an (apparently) weaker assumption than being weakly null. It should be mentioned here that if we use H. P. Rosenthal's characterization of Banach spaces containing l^1 [5], property b″) is equivalent to saying that there is no subspace isomorphic to l^1.

In section 1, we give a definition of James spaces on trees, which are slightly more general than James' example, and we formulate our main result in Theorem. In section 2 we prove our main result.

§ 1. James Spaces and the Main Result.

Let T be a union of a countable family of pairwise disjoint non-empty finite sets P_n, $n=0, 1, 2, \cdots$. We call a point t of P_n a point of *level* n, and write $l(t)=n$. We assume there is a binary relation between points of P_n and points of P_{n+1}, which we call a *connection*, such that for every $n=0, 1, 2, \cdots$, each point of level n is connected to at least one point of level $n+1$ and each point of level $n+1$ is connected to only one point of level n. The following illustrates an example of connections between points of the first three levels

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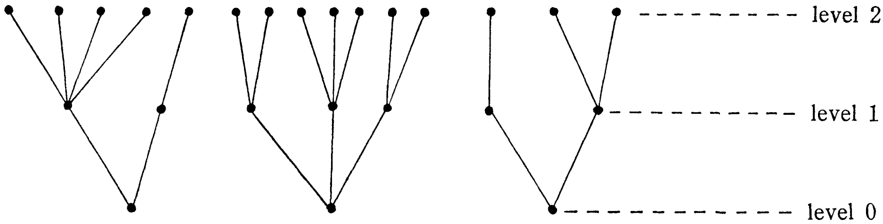

We call T with connection as above an infinite *tree*. A finite sequence $S=\{t_0, t_1, \cdots, t_n\}$ in T is called a *segment* if $t_k \in P_{k+n_0}$ for all $k=0, 1, \cdots, n$, where $n_0=l(t_0)$, and t_k is connected to t_{k+1} for all $k=0, 1, \cdots, n-1$. Any two points s and t of a tree T are called *connected* if there is a segment which initiates with either s or t and terminates with the other. An infinite sequence $B=\{t_0, t_1, t_2, \cdots\}$ in T is called a *branch* if $t_k \in P_{k+n_0}$ for all $k=0, 1, 2, \cdots$ with $n_0=l(t_0)$ and t_k is connected to t_{k+1} for all $k=0, 1, 2, \cdots$. The starting point t_0 of B is called the *initial point* of B and $l(t_0)=n_0$ is called the *initial level* of B.

The James space $J(T)$ on a tree T is defined to be the space consisting of all complex valued functions x on T such that

$$\|x\|=\sup\Big(\sum_{j=1}^{k}\Big|\sum_{t\in S_j}x(t)\Big|^2\Big)^{1/2}<+\infty,$$

where the supremum is taken over all choices of mutually disjoint segments $S_1, S_2, \cdots, S_k$ in T. It is not hard to see that $J(T)$ is a Banach space with respect to this norm. In particular, if every P_n consists of one point, than $J(T)$ is identical with the well known classical James space given in [1].

The space $J(T)$ has the natural basis $\{e_t\,;\, t\in T\}$,

$$x=\sum_{t\in T}x(t)e_t=\sum_{n=0}^{\infty}\sum_{t\in P_n}x(t)e_t$$

for all $x\in J(T)$, where e_t is the characteristic function of $\{t\}$ and the order in the summation $\sum_{t\in T}$ follows the order of the level of t and any fixed ordering among points on the same level. It is easy to see that $\{e_t\,;\, t\in T\}$ is a normalized, monotone and boundedly complete basis.

Any segment S or branch B in T gives a linear functional with norm 1 on $J(T)$ by

$$S(x)=\sum_{t\in S}x(t) \quad\text{or}\quad B(x)=\sum_{t\in B}x(t) \qquad \text{for } x\in J(T),$$

where the order in the summation $\sum_{t\in B}$ follows the order of the level of $t\in B$.

We recall that a sequence $\{x_n\,;\, n=1, 2, \cdots\}$ in a Banach space is said to be equivalent to an l^2-basis if for any linear combination $\sum_n \alpha_n x_n$ of x_n's we have

$$a\Big(\sum_n|\alpha_n|^2\Big)^{1/2}\leqq\Big\|\sum_n\alpha_n x_n\Big\|\leqq b\Big(\sum_n|\alpha_n|^2\Big)^{1/2},$$

where a and b are fixed positive numbers.

Our main result can be stated as follows.

THEOREM. *Suppose* $\{x_n ; n=1, 2, \cdots\}$ *is a normalized sequence in* $J(T)$ *satisfying* $\lim_{n\to\infty} B(x_n)=0$ *for all branches* B *in* T. *Then there is a subsequence of* $\{x_n ; n=1, 2, \cdots\}$ *which is equivalent to an* l^2*-basis. More precisely, for any* $\varepsilon>0$ *we can choose a subsequence* $\{x_{n_k} ; k=1, 2, \cdots\}$ *such that for any linear combination* $\sum_k \alpha_k x_{n_k}$ *of* x_{n_k}*'s with* $\sum_k |\alpha_k|^2=1$ *we have*

$$1-\varepsilon \leqq \|\sum_k \alpha_k x_{n_k}\| \leqq 2+\varepsilon .$$

The constant 2 may not be best possible. We do not know the best possible constant.

§ 2. Lemma and Proof of Theorem.

A sequence $\{x_n, a_n ; n=1, 2, \cdots\}$, where $\{x_n ; n=1, 2, \cdots\}$ is a sequence in $J(T)$ and $\{a_n ; n=1, 2, \cdots\}$ is an increasing sequence of levels, is called a *block* sequence if the support of x_n is located between the level a_n, including a_n, and the level a_{n+1}, excluding a_{n+1}, for all $n=1, 2, \cdots$. We call it bounded or normalized if $\{x_n ; n=1, 2, \cdots\}$ is bounded or normalized. The following Lemma is a key to the proof of our theorem. We wish to thank Tom Starbird for simplifying the original proof by suggesting the use of Ramsey's theorem. Our original proof involved more combinatorial arguments.

LEMMA. *Let* $\{x_n, a_n ; n=1, 2, \cdots\}$ *be a bounded block sequence satisfying* $\lim_{n\to\infty} B(x_n)=0$ *for all branches* B *in* T. *Then for given* $\varepsilon>0$ *there is a subsequence* $\{x_n, a_n ; n\in M\}$ *of* $\{x_n, a_n ; n=1, 2, \cdots\}$ *such that for any segment* S *initiating with the level* 0, $|S(x_n)|\leqq\varepsilon$ *for all* $n\in M$ *except at most one* $n=n(S)$ *in* M.

Proof. For given $\varepsilon>0$, let Q_n be the set of all points t with $l(t)=a_n$ such that $|S(x_n)|>\varepsilon$ for some segment S initiating with t. By our definition of the norm $\|x_n\|$, it is clear that the number of points of Q_n is dominated by $\|x_n\|^2/\varepsilon^2 \leqq K^2/\varepsilon^2$, where $K=\sup_{n\geqq 1}\|x_n\|$. Thus we may assume, by passing to a subsequence if necessary, that each Q_n consists of a points for all $n=1, 2, \cdots$. There is nothing to prove if $a=0$, so assume $a\geqq 1$. Ramsey's theorem is applicable in the following way for choosing a subsequence $\{x_n, a_n ; n\in M\}$ of $\{x_n, a_n ; n=1, 2, \cdots\}$ with property we desire. Let $t_{i,n}$, $1\leqq i\leqq a$, be all points of Q_n. For $1\leqq i, j\leqq a$, let $A_{i,j}$ be the set of all pairs $\{n, m\}$ of positive integers n and m with $n<m$ such that there is a segment S initiating at $t_{i,n}$ and terminating at $t_{j,m}$ with $|S(x_n)|>\varepsilon$. Finally, let A be the set of all pairs $\{n, m\}$ of positive integers n and m with $n<m$ which are not in any $A_{i,j}$ for $1\leqq i, j\leqq a$. It is

clear that $A_{i,j}$ for $1\leqq i, j\leqq a$ together with A give a finite cover of the space of all pairs $\{n, m\}$ of positive integers n and m with $n<m$. By Ramsey's theorem [4], there is an infinite subset M of positive integers such that $M^{(2)}$ is contained in $A_{i,j}$ for some i and j or otherwise contained in A, where $M^{(2)}$ denote the set of all pairs $\{n, m\}$ with n and m in M and $n<m$. We claim $M^{(2)}\subset A$. Suppose $M^{(2)}\subset A_{i,j}$ for some i and j, then we will see that for any n and m in M with $n<m$ there is a segment $S_{n,m}$ connecting $t_{i,n}$ to $t_{i,m}$ such that $|S_{n,m}(x_n)|>\varepsilon$. For given n and m in M with $n<m$, choose k in M with $n<m<k$. Since $\{n, k\}$ and $\{m, k\}$ are in $A_{i,j}$, there are segments S_1 and S_2 initiating $t_{i,n}$ and $t_{i,m}$ respectively and terminating at $t_{j,k}$ with $|S_1(x_n)|>\varepsilon$ and $|S_2(x_m)|>\varepsilon$. Since S_1 and S_2 are terminating at the same point $t_{j,k}$, S_2 must be a subsegment of S_1, thus S_1 must contain the point $t_{i,m}$ which is the initial point of S_2. Let $S_{n,m}$ be the part of S_1 between $t_{i,n}$ and $t_{i,m}$, then we have $|S_{n,m}(x_n)|=|S_1(x_n)|>\varepsilon$. It is clear that this property of M we have just shown implies that there is a branch B_0 which connects all points $t_{i,n}$, $n\in M$, and that $|B_0(x_n)|>\varepsilon$ for all $n\in M$. However this contradicts our assumption $\lim_{n\to\infty} B(x_n)=0$ for all branches B. Thus we have shown $M^{(2)}\subset A$.

Now we can see that this subsequence $\{x_n, a_n ; n\in M\}$ has the property we desire. Suppose there is a segment S initiating with level 0 such that $|S(x_n)|>\varepsilon$ and $|S(x_m)|>\varepsilon$ for some n and m in M with $n<m$. Let t_1 and t_2 be points of S with level a_n and level a_m respectively, then we have $t_1=t_{i,n}$ and $t_2=t_{j,m}$ for some i and j because $|S(x_n)|>\varepsilon$ and $|S(x_m)|>\varepsilon$, thus $\{n, m\}$ belongs to $A_{i,j}$, which contradicts $\{n, m\}\in M^{(2)}\subset A$ and $A\cap A_{i,j}=\emptyset$. This completes the proof.

A block sequence $\{x_n, a_n ; n\in M\}$ which satisfies the conclusion of the lemma will be called *ε-separated.*

Proof of Theorem. We are given an $\varepsilon>0$ and a normalized sequence $\{x_n ; n=1, 2, \cdots\}$ in $J(T)$ satisfying

1) $$\lim_{n\to\infty} B(x_n)=0 \qquad \text{for all branches } B \text{ in } T.$$

Since this assumption implies that $\lim_{n\to\infty} x_n(t)=0$ for all $t\in T$, by the use of the standard "gliding hump" argument, we first choose a subsequence $\{x_n' ; n=1, 2, \cdots\}$ of $\{x_n ; n=1, 2, \cdots\}$ and a normalized block sequence $\{y_n, a_n ; n=1, 2, \cdots\}$ such that

2) $$\sum_{n=1}^{\infty} \|x_n'-y_n\|^2<\varepsilon^2.$$

It is clear that the normalized block sequence $\{y_n, a_n ; n=1, 2, \cdots\}$ also satisfies 1). Using the lemma, choose a decreasing sequence of infinite subsets M_k of positive integers associated with a sequence of positive numbers ε_k tending to 0 such that

3) $\{y_n, a_n ; n \in M_k\}$ is ε_k-separated for all $k=1, 2, \cdots$.

That is, for each $k=1, 2, \cdots$ and any segment S initiating at level 0 we have $|S(y_n)| \leqq \varepsilon_k$ for all n in M_k except at most one $n=n(S)$ in M_k.

We now apply the diagonal process and choose sequence $\rho_1 < \rho_2 < \cdots$ and $k_1 < k_2 < \cdots$ such that

4) $\rho_n \in M_{k_n}$ for all $n=1, 2, \cdots$,

and

5) $$\sum_{n=1}^{\infty} m_n \Big(\sum_{l=n+1}^{\infty} \varepsilon_{k_l}^2 \Big) < \varepsilon^2,$$

where m_n is the number of all points of the level a_{ρ_n} for $n=1, 2, \cdots$. Setting $z_n = y_{\rho_n}$, $b_n = a_{\rho_n}$ and $\delta_n = \varepsilon_{k_n}$ for all $n=1, 2, \cdots$, we have, from 3), 4) and 5),

6) $\{z_n, b_n ; n=k, k+1, \cdots\}$ is δ_k-separated for all $k=1, 2, \cdots$,

and

7) $$\sum_{n=1}^{\infty} m_n \Big(\sum_{l=n+1}^{\infty} \delta_l^2 \Big) < \varepsilon^2,$$

where m_n is the cardinality of P_{b_n} for all $n=1, 2, \cdots$. Property 6) tells us that for any $k=1, 2, \cdots$ and any segment S initiating at level b_k we have $|S(z_n)| \leqq \delta_k$ for all $n=1, 2, \cdots$ except at most one $n=n(S) \geqq k$.

For a segment S, we denote by $i(S)$ the smallest positive integer n with the initial level of $S \leqq b_n$. We call S *regular* if S initiates with a point of level $b_{i(S)}$. For a regular segment S we denote by $\lambda(S)$ the smallest positive integer n with $|S(z_n)| > \delta_n$, and we put $\lambda(S) = +\infty$ if there is no such n. The following inequality will be used to estimate the norm of a linear combination of z_n's. For any regular segment S we have

8) $$\sum_{n \neq \lambda(S)} |S(z_n)|^2 \leqq \sum_{n \geqq i(S)} \delta_n^2 .$$

In fact, putting $\{n ; |S(z_n)| > \delta_n\} = \{n_1 < n_2 < \cdots\}$, we see that $i(S) \leqq n_1 = \lambda(S)$. By 6), the block sequence $\{z_n, b_n ; n=n_1, n_1+1, \cdots\}$ is δ_{n_1}-separated. Since $|S(z_{n_1})| > \delta_{n_1}$, $|S(z_n)| \leqq \delta_{n_1}$ holds for all $n > n_1$, so $|S(z_{n_2})| \leqq \delta_{n_1}$. Similarly, we see that $|S(z_{\nu+1})| \leqq \delta_{n_\nu}$ for all $\nu = 1, 2, \cdots$. Thus we have

$$\sum_{n \neq \lambda(S)} |S(z_n)|^2 = \sum_{\substack{n \geqq i(S) \\ n \neq n_1}} |S(z_n)|^2 = \sum_{\nu=2}^{\infty} |S(z_{n_\nu})|^2 + \sum_{\substack{n \geqq i(S) \\ n \neq n_\nu}} |S(z_n)|^2$$

$$\leqq \sum_{\nu=1}^{\infty} \delta_{n_\nu}^2 + \sum_{\substack{n \geqq i(S) \\ n \neq n_\nu}} \delta_n^2 = \sum_{n \geqq i(S)} \delta_n^2 .$$

Now we will estimate the norm of a linear combination $\sum_n \alpha_n z_n$ with $\sum_n |\alpha_n|^2 = 1$. We claim

9) $$1\leqq\|\sum_n\alpha_n z_n\|\leqq 2+2\varepsilon\,.$$

The first inequality is clear because $\{z_n, b_n\,;\ n=1, 2, \cdots\}$ is a normalized block sequence. To see the other inequality, suppose $\mathcal{J}=\{S\}$ is a finite family of mutually disjoint segments S. We decompose S into its *initial end* S_0 and its *regular part* S', where $S_0=\{t\in S\,;\ l(t)<b_{\iota(S)}\}$ and $S'=\{t\in S\,;\ l(t)\geqq b_{\iota(S)}\}$. S_0 is empty if S is regular. We decompose S' into S_1 and S''; $S'=S_1\cup S''$, where $S_1=\{t\in S\,;\ b_{\iota(S)}\leqq l(t)<b_{\iota(S)+1}\}$ and $S''=\{t\in S\,;\ b_{\iota(S)+1}\leqq l(t)\}$, S'' possibly being empty. Furthermore, we decompose S'' into (possibly) three segments S_1'', S_2'' and S_3'' as follows

$$S_1''=\{t\in S\,;\ b_{\iota(S)+1}\leqq l(t)<b_{\lambda(S'')}\}\,,$$

$$S_2''=\{t\in S\,;\ b_{\lambda(S'')}\leqq l(t)<b_{\lambda(S'')+1}\}\,,$$

$$S_3''=\{t\in S\,;\ b_{\lambda(S'')+1}\leqq l(t)\}\,.$$

We have

$$S=S_0\cup S'=S_0\cup S_1\cup S''=S_0\cup S_1\cup S_1''\cup S_2''\cup S_3''\,.$$

Let $x=\sum_n\alpha_n z_n$ with $\sum_n|\alpha_n|^2=1$, and observe that

$$\sum_{S\in\mathcal{J}}|S(x)|^2=\sum_{S\in\mathcal{J}}|S_0(x)+S_1(x)+S_1''(x)+S_2''(x)+S_3''(x)|^2$$

$$\leqq 4\sum_{S\in\mathcal{J}}\{|S_0(x)|^2+|S_1(x)|^2+|S_2''(x)|^2+|S_1''(x)+S_3''(x)|^2\}$$

$$=4\sum_{S\in\mathcal{J}}\{|S_0(x)|^2+|S_1(x)|^2+|S_2''(x)|^2\}+4\sum_{S\in\mathcal{J}}|S_1''(x)+S_3''(x)|^2\,.$$

To estimate the first summation, note that $|R(x)|^2=\sum_n|\alpha_n|^2|R(z_n)|^2$ if $R=S_0$, S_1 or S_2'', because we have only one non-zero term in the summation $\sum_n\alpha_n R(z_n)$. Thus we have

$$\sum_{S\in\mathcal{J}}\{|S_0(x)|^2+|S_1(x)|^2+|S_2''(x)|^2\}$$

$$=\sum_{S\in\mathcal{J}}\sum_n|\alpha_n|^2\{|S_0(z_n)|^2+|S_1(z_n)|^2+|S_2''(z_n)|^2\}$$

$$=\sum_n|\alpha_n|^2\sum_{S\in\mathcal{J}}\{|S_0(z_n)|^2+|S_1(z_n)|^2+|S_2''(z_n)|^2\}$$

$$\leqq\sum_n|\alpha_n|^2\|z_n\|^2=\sum_n|\alpha_n|^2=1\,.$$

To estimate $\sum_{S\in\mathcal{J}}|S_1''(x)+S_3''(x)|^2$, we note from 8) that

$$|S_1''(x)+S_3''(x)|^2=|\sum_{n\neq\lambda(S'')}\alpha_n S''(z_n)|^2$$

$$\leqq(\sum_{n\neq\lambda(S'')}|\alpha_n|^2)(\sum_{n\neq\lambda(S'')}|S''(z_n)|^2)\leqq\sum_{n\geqq\iota(S'')}\delta_n^2\,.$$

Let $\mathcal{J}_k$ be the set of all $S\in\mathcal{J}$ whose regular part S' initiates with a point of the level b_k, that is $\mathcal{J}_k=\{S\in\mathcal{J};\ i(S)=k\}$ for $k=1, 2, \cdots$. Then we have

$$\sum_{S\in\mathcal{J}} |S_1''(x)+S_3''(x)|^2=\sum_{k=1}^{\infty}\sum_{S\in\mathcal{J}_k} |S_1''(x)+S_3''(x)|^2$$

$$\leqq \sum_{k=1}^{\infty}\sum_{S\in\mathcal{J}_k}\sum_{n\geqq i(S'')}\delta_n^2=\sum_{k=1}^{\infty}\sum_{S\in\mathcal{J}_k}\sum_{n\geqq k+1}\delta_n^2\leqq\sum_{k=1}^{\infty} m_k(\sum_{n\geqq k+1}\delta_n^2)\leqq\varepsilon^2,$$

where we used 7) for the last inequality above. Finally, we have

$$\sum_{S\in\mathcal{J}}|S(x)|^2\leqq 4+4\varepsilon^2\leqq 4(1+\varepsilon)^2.$$

Thus

$$\|\sum_n \alpha_n z_n\|=\sup_{\mathcal{J}}(\sum_{S\in\mathcal{J}}|S(x)|^2)^{1/2}\leqq 2(1+\varepsilon).$$

This establishes 9).

Since $\{z_n;\ n=1, 2, \cdots\}$ is a subsequence of $\{y_n;\ n=1, 2, \cdots\}$ and $\{y_n;\ n=1, 2, \cdots\}$ satisfies 2), there is a subsequence $\{x_n'';\ n=1, 2, \cdots\}$ of the originally given sequence $\{x_n;\ n=1, 2, \cdots\}$ such that

$$\text{10)} \qquad \sum_{n=1}^{\infty}\|x_n''-z_n\|^2<\varepsilon^2.$$

Now we can see that $\{x_n'';\ n=1, 2, \cdots\}$ is equivalent to an l^2-basis. In fact, for any linear combination $\sum_n\alpha_n x_n''$ with $\sum_n|\alpha_n|^2=1$ properties 9) and 10) yield

$$\|\sum_n\alpha_n x_n''\|\geqq\|\sum_n\alpha_n z_n\|-\sum_n|\alpha_n|\,\|x_n''-z_n\|$$

$$\geqq\|\sum_n\alpha_n z_n\|-(\sum_n|\alpha_n|^2)^{1/2}(\sum_n\|x_n''-z_n\|^2)^{1/2}\geqq 1-\varepsilon,$$

and

$$\|\sum_n\alpha_n x_n''\|\leqq\|\sum_n\alpha_n z_n\|+\sum_n|\alpha_n|\,\|x_n''-z_n\|\leqq 2+2\varepsilon+\varepsilon=2+3\varepsilon.$$

The proof of the theorem is complete.

Finally we would like to show that our theorem implies property b) mentioned in the introduction.

The following fact was proved by J. Lindenstrauss and C. Stegall (see the proof of Corollary 3 in [3]).

For any bounded sequence in $J(T)$ we can choose a subsequence $\{x_n;\ n=1, 2, \cdots\}$ such that $\lim_{n\to\infty}B(x_n)$ exists for all branches B in T.

From this, it is easy to see that every infinite dimensional subspace of $J(T)$ contains a normalized sequence $\{x_n;\ n=1, 2, \cdots\}$ with the property that $\lim_{n\to\infty}B(x_n)=0$ for all branches B in T. Thus we have the following result.

COROLLARY. *Every infinite dimensional subspace of $J(T)$ contains a subspace isomorphic to l^2.*

REFERENCES

[1] R.C. JAMES, A non-reflexive Banach space isometric with its second conjugate, Proc. Nat. Acad. Sci. USA, **37** (1951), 174-177.

[2] R.C. JAMES, A separable somewhat reflexive Banach space with non-separable dual, Bull. Amer. Math. Soc., **80** (1974), 738-743.

[3] J. LINDENSTRAUSS AND C. STEGALL, Examples of separable spaces which do not contain l^1 and whose duals are non-separable, Studia Math., **54** (1975), 81-105.

[4] H.P. RAMSEY, On a problem of formal logic, Proc. London Math. Soc., **30** (1929), 264-286.

[5] H.P. ROSENTHAL, A characterization of Banach spaces containing l^1, Proc. Nat. Acad. Sci. USA, **71** (1974), 2411-2413.

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