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AFFINE KILLING VECTORS IN THE TANGENT BUNDLES

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Recently, by defining the (*)-Lie derivative, the present author [12] studied vector fields in fibred Riemannian spaces with higher dimensional fibre.

Let M be a Riemannian manifold and $T(M)$ its tangent bundle. For an arbitrary vector field X in M, we denote by X^H, X^V and X^C the horizontal lift of X, the vertical lift of X and the complete lift of X respectively.

The theory of these lifts in $T(M)$ has been studied by many authors and has been investigated systematically by Yano-Ishihara [8].

The main purpose of the present paper is to study the conditions that these lifts are affine Killing, by applying the (*)-Lie derivative to $T(M)$.

In Section 1, we first recall definitions and properties concerning $T(M)$ following [8]. In Section 2 and Section 3, we obtain in $T(M)$ the results corresponding to those obtained in [12]. Section 4 and Section 5 are devoted to the study of these lifts, affine Killing and projective Killing respectively.

1. Preliminaries in tangent bundles.

In this section, we shall recall definitions and properties concerning tangent bundles following Yano-Ishihara [8].

Let M be an n-dimensional differentiable manifold and $T_P(M)$ the tangent space at a point P of M. Then the set

$$T(M)=\bigcup_{P\in M} T_P(M)$$

is, by definition, the *tangent bundle* over M. Throughout the paper, the differentiability of manifolds, mappings and geometric objects we discuss are assumed to be of C^∞. For any point $\tilde{P}$ of $T(M)$ such that $\tilde{P}\in T_P(M)$, there exists a mapping $\pi : T(M)\ni\tilde{P}\rightarrow P\in M$, which is onto and maximal rank n everywhere. The set $\pi^{-1}(P)$, that is, $T_P(M)$ is called the *fibre* over P and M the *base space*.

Let $\{U, x^i\}$ be a coordinate neighborhood of the base space M[1], where (x^i) is a system of local coordinates defined in the neighborhood U. Let (y^i) be the system of cartesian coordinates in each tangent space $T_P(M)$ of M at P with respect to the natural frame $\partial_i=\partial/\partial x^i$, where P is an arbitrary point belonging

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1) The indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$. The summation convention will be used with respect to this system of indices.

to U. Then, in the open set $\pi^{-1}(U)$ of $T(M)$ we can introduce local coordinates (x^i, y^i), which are called the coordinates induced in $\pi^{-1}(U)$ from (x^i) or, simply, the *induced coordinates* in $\pi^{-1}(U)$. From now on, we denote by $(\tilde{x}^I)$ the induced coordinates (x^i, y^i).[1] We also denote by $\partial_I=\partial/\partial\tilde{x}^I$ the natural frame $(\partial_i, \partial_{\bar{i}})$ in $\pi^{-1}(U)$, and by $d\tilde{x}^J$ the coframe dual to the frame ∂_I in $\pi^{-1}(U)$, where $\partial_i=\partial/\partial x^i$ and $\partial_{\bar{i}}=\partial/\partial y^i$.

Let there be given a function $\phi=\phi(x)$ in U. Then its *vertical lift* ϕ^V and its *complete lift* ϕ^C are respectively represented by

$$\phi^V=\phi \quad\text{and}\quad \phi^C=\partial\phi \tag{1.1}$$

in $\pi^{-1}(U)$ with respect to the induced coordinates $(\tilde{x}^I)=(x^i, y^i)$, where the symbol ∂ denotes the operator

$$\partial=y^i\partial_i=y^i\partial/\partial x^i\,.$$

If a vector field X has components X^i in U, then its *vertical lift* X^V and its *complete lift* X^C have respectively components of the form

$$X^V:\begin{pmatrix}0\\ X^i\end{pmatrix} \quad\text{and}\quad X^C:\begin{pmatrix}X^i\\ \partial X^i\end{pmatrix} \tag{1.2}$$

in $\pi^{-1}(U)$ with respect to the induced coordinates $(\tilde{x}^I)=(x^i, y^i)$.

Let there be given a Riemannian metric g in M which has components g_{ji} in U. We denote by $\{{}_j{}^h{}_i\}$ the Christoffel symbols constructed with g_{ji}. If we put

$$\Gamma_j^h=y^k\{{}_k{}^h{}_j\}, \quad \Lambda_{ji}=\Gamma_j^k g_{ki}, \tag{1.3}$$

then we see that the tensor $\tilde{g}$ having components

$$(\tilde{g}_{JI})=\begin{pmatrix}g_{ji}+g_{kh}\Gamma_j^k\Gamma_i^h, & \Lambda_{ij}\\ \Lambda_{ji} & , \; g_{ji}\end{pmatrix} \tag{1.4}$$

with respect to the induced coordinates $(\tilde{x}^I)=(x^i, y^i)$ in $T(M)$, defines a Riemannian metric in $T(M)$ (see Sasaki [3]). Then contravariant components of $\tilde{g}$ are given by

$$(\tilde{g}^{JI})=\begin{pmatrix}g^{ji} & , \quad -\Lambda^{ji}\\ -\Lambda^{ji}, & g^{ji}+g^{kh}\Gamma_k^j\Gamma_h^i\end{pmatrix} \tag{1.5}$$

with respect to $(\tilde{x}^I)=(x^i, y^i)$, where g^{ji} denote contravariant components of g in M, and

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1) Putting $x^{\bar{i}}=y^i$, the indices $H, I, J, K, \cdots$ run over the range $\{1, 2, \cdots, n\,;\, \bar{1}, \bar{2}, \cdots, \bar{n}\}$ and the indices $\bar{h}, \bar{i}, \bar{j}, \bar{k}, \cdots$ the range $\{\bar{1}, \bar{2}, \cdots, \bar{n}\}$ The summation convention will be used in relation to this system of indices.

(1.6) $$\Lambda^{ji}=g^{jk}\Gamma_{k}^{i}.$$

From now on, we restrict ourselve to a Riemannian manifold M. The *vertical distribution,* i.e., the distribution formed of the tangent spaces of fibres is locally spanned by n independent local vector fields $C_{\bar{b}}$ with components[1)]

(1.7) $$(C^{H}{}_{\bar{b}})=\begin{pmatrix} 0 \\ \delta_{b}^{h} \end{pmatrix}$$

in $\pi^{-1}(U)$.

Let us consider n local vector fields E_b with components

(1.8) $$(E^{H}{}_{b})=\begin{pmatrix} \delta_{b}^{h} \\ -\Gamma_{b}^{h} \end{pmatrix}$$

in $\pi^{-1}(U)$. Then E_b span a distribution defined globally in $T(M)$, which is called the *horizontal distribution.* The vertical and horizontal distributions are complementary and orthogonal with respect to the metric $\tilde{g}$. The set $(E_b, C_{\bar{b}})$ forms a frame in $\pi^{-1}(U)$, which is called the *adapted frame,* and has components

(1.9) $$(B^{H}{}_{B})=\begin{pmatrix} E^{H}{}_{b} \\ C^{H}{}_{\bar{b}} \end{pmatrix}^{2)}.$$

The inverse of the matrix $(B^{H}{}_{B})$ is given by

(1.10) $$(B_{I}{}^{A})=(E_{I}{}^{a},\ C_{I}{}^{\bar{a}}),$$

$E_{I}{}^{a}$ and $C_{I}{}^{\bar{a}}$ being defined by

(1.11) $$(E_{I}{}^{a})=(\delta_{i}^{a},\ 0),\quad (C_{I}{}^{\bar{a}})=(\Gamma_{i}^{a},\ \delta_{i}^{a}).$$

If we put

(1.12) $$E^{a}=E_{I}{}^{a}d\tilde{x}^{I},\quad C^{\bar{a}}=C_{I}{}^{\bar{a}}d\tilde{x}^{I},$$

then the set $(E^{a}, C^{\bar{a}})$ forms a coframe dual to the adapted frame $(E_b, C_{\bar{b}})$ in $\pi^{-1}(U)$.

We now consider local vector fields D_B and local 1-forms θ^A defined in $\pi^{-1}(U)$ by

(1.13) $$D_{B}=B^{I}{}_{B}\partial_{I},\quad \theta^{A}=B_{J}{}^{A}d\tilde{x}^{J},$$

where $\partial_{I}=\partial/\partial\tilde{x}^{I}$. Then we see that

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1) The indices $a, b, c, d, e, \cdots$ run over the range $\{1, 2, \cdots, n\}$ and the indices $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \cdots$ the range $\{\bar{1}, \bar{2}, \cdots, \bar{n}\}$, and indicate the indices with respect to the adapted frame. The summation convention will be used with respect to this system of indices.

2) The indices $A, B, C, D, E, \cdots$ run over the range $\{1, 2, \cdots, n; \bar{1}, \bar{2}, \cdots, \bar{n}\}$ and indicate the indices with respect to the adapted frame. The summation convention will be used in relation to this system of indices.

$$D_b=\partial_b-\Gamma_b^h\partial_{\bar h}=E_b\,,\quad D_{\bar b}=\partial_{\bar b}=C_{\bar b}\,,$$
(1.14)
$$\theta^a=dx^a=E^a\,,\quad \theta^{\bar a}=\Gamma_i^a dx^i+dy^a=C^{\bar a}\,.$$

We often use D_B as differential operators in $\pi^{-1}(U)$ if there is no fear of confusion.

Let there be given an arbitrary tensor field in $T(M)$, say $\tilde{T}$ of type (1, 2) with local expression

$$\tilde{T}=\tilde{T}_{JI}{}^H d\tilde{x}^J\otimes d\tilde{x}^I\otimes\partial_H \tag{1.15}$$

in $\pi^{-1}(U)$. Taking account of (1.11) and (1.12), we see that $\tilde{T}$ is also represented as followings:

$$\tilde{T}=T_{cb}{}^a E^c\otimes E^b\otimes E_a+T_{cb}{}^{\bar a}E^c\otimes E^b\otimes C_{\bar a}+\cdots$$
(1.16)
$$+T_{\bar c\bar b}{}^a C^{\bar c}\otimes C^{\bar b}\otimes E_a+T_{\bar c\bar b}{}^{\bar a}C^{\bar c}\otimes C^{\bar b}\otimes C_{\bar a}\,,$$

where

$$T_{cb}{}^a=E^J{}_c E^I{}_b E_H{}^a\tilde{T}_{JI}{}^H\,,\quad T_{cb}{}^{\bar a}=E^J{}_c E^I{}_b C_H{}^{\bar a}\tilde{T}_{JI}{}^H\,,$$

$$T_{\bar c\bar b}{}^a=C^J{}_{\bar c} C^I{}_{\bar b} E_H{}^a\tilde{T}_{JI}{}^H\,,\quad T_{\bar c\bar b}{}^{\bar a}=C^J{}_{\bar c} C^I{}_{\bar b} C_H{}^{\bar a}\tilde{T}_{JI}{}^H\,.$$

In the right-hand side of (1.16), the first term $T_{cb}{}^a E^c\otimes E^b\otimes E_a$ and the last term $T_{\bar c\bar b}{}^{\bar a}C^{\bar c}\otimes C^{\bar b}\otimes C_{\bar a}$ are called the *horizontal part* of $\tilde{T}$ and the *vertical part* of $\tilde{T}$, respectively. A tensor field, say $\tilde{T}$ of type (1, 2) with local expression (1.15), in $T(M)$ is said to be *projectable* if $T_{cb}{}^a$ satisfy

$$D_{\bar d}T_{cb}{}^a=0\,. \tag{1.17}$$

Then, for a projectable tensor field $\tilde{T}$ of this type, we can define a tensor T in M having components $T_{cb}{}^a$ with respect to $\{U,\,x^i\}$. T is called the *projection* of $\tilde{T}$ and denoted by $T=p\tilde{T}$.

Conversely, given a tensor field T in M, there is a unique horizontal and projectable tensor field $\hat{T}$ in $T(M)$ such that $p\hat{T}=T$. This $\hat{T}$ is called the *horizontal lift* of T and denoted by $\hat{T}=T^H$.

If a vector field X has components X^i in U, then its *horizontal lift* X^H has components

$$X^H:\begin{pmatrix} X^i \\ -\Gamma_j^i X^j\end{pmatrix} \tag{1.18}$$

with respect to the induced coordinates $(\tilde{x}^I)=(x^i,\,y^i)$ in $\pi^{-1}(U)$.

Let ∇ and $\tilde{\nabla}$ be the Riemannian connections determined by the metric g in M and the metric $\tilde{g}$ in $T(M)$ respectively, and denote by $\widetilde{\left\{ {H \atop J\ I} \right\}}$ the Christoffel symbols constructed with $\tilde{g}_{JI}$ in $\pi^{-1}(U)$.

If we put

$$\tilde{\nabla}_J B^H{}_B=\Gamma_C{}^A{}_B B_J{}^C B^H{}_A \tag{1.19}$$

in $\pi^{-1}(U)$, where $\Gamma_C{}^A{}_B$ are local functions defined in $\pi^{-1}(U)$, then we have

$$(1.20) \qquad \Gamma_C{}^A{}_B=(D_C B^H{}_B+\widetilde{\left\{ {H \atop J\ K} \right\}} B^J{}_C B^K{}_B)B_H{}^A .$$

Rewriting $\Gamma_c{}^{\bar a}{}_b$ and $\Gamma_c{}^a{}_{\bar b}(=\Gamma_{\bar b}{}^a{}_c)$ into $h_{cb}{}^{\bar a}$ and $h^a{}_{c\bar b}$ respectively, we see that

$$\Gamma_c{}^a{}_b=\left\{ {a \atop c\ b} \right\}, \quad \Gamma_c{}^a{}_{\bar b}=\Gamma_{\bar b}{}^a{}_c=h^a{}_{c\bar b}=\frac{1}{2}y^d K_{dbc}{}^a ,$$

$$(1.21) \qquad \Gamma_{\bar c}{}^a{}_{\bar b}=0 , \quad \Gamma_c{}^{\bar a}{}_b=h_{cb}{}^{\bar a}=-\frac{1}{2}y^d K_{cbd}{}^a ,$$

$$\Gamma_c{}^{\bar a}{}_{\bar b}=\left\{ {a \atop c\ b} \right\}, \quad \Gamma_{\bar c}{}^{\bar a}{}_b=0 , \quad \Gamma_{\bar c}{}^{\bar a}{}_{\bar b}=0 ,$$

where $K_{kji}{}^h$ denote components of the curvature tensor of M with the metric g. The elements

$$(1.22) \qquad h_{cb}{}^{\bar a}=-\frac{1}{2}y^d K_{cbd}{}^a , \quad h^a{}_{c\bar b}=\frac{1}{2}y^d K_{dbc}{}^a$$

play an important rôle in later sections, and we see that

$$(1.23) \qquad h_{cb}{}^{\bar a}+h_{bc}{}^{\bar a}=0 , \quad h^a{}_{c\bar b}=g^{ae}h_{ec}{}^{\bar d}g_{db} .$$

2. Operators $'\nabla$ and $''\nabla$ of $T(M)$.

In this section, we shall recall definitions and properties concerning two covariant derivative operators $'\nabla$ and $''\nabla$ of $T(M)$, following our previous results (see the section 2, [12]).

Let $\mathcal{T}^p_q(TM)$ be the space of all tensor fields of type (p, q) in $T(M)$. Let $\mathcal{T}^r_s(hTM)$ (resp. $\mathcal{T}^t_u(vTM)$) be the space of all horizontal (resp. vertical) tensor fields of type (r, s) (resp. type (t, u)) in $T(M)$. We now consider the formal tensor product in $T(M)$ such as $\mathcal{T}^p_q(TM) \sharp \mathcal{T}^r_s(hTM) \sharp \mathcal{T}^t_u(vTM)$. We call an element $\tilde T$ of this space a $\binom{prt}{qsu}$-*partial tensor* in $T(M)$ and denote by $\mathcal{T}^{prt}_{qsu}(TM)$ the space of all $\binom{prt}{qsu}$-partial tensors in $T(M)$. We may identify $\mathcal{T}^{p00}_{q00}(TM)$, $\mathcal{T}^{0r0}_{0s0}(TM)$ and $\mathcal{T}^{00t}_{00u}(TM)$ with $T^p_q(TM)$, $\mathcal{T}^r_s(hTM)$ and $\mathcal{T}^t_u(vTM)$, respectively. For any element of $\mathcal{T}^{prt}_{qsu}(TM)$, say an element $\tilde T$ of $\mathcal{T}^{111}_{111}(TM)$ with components $T_J{}^I{}_b{}^a{}_{\bar d}{}^{\bar c}$ in $\pi^{-1}(U)$, we define the (*)-*covariant derivative* $\nabla^* \tilde T$ of $\tilde T$ as a partial tensor with components of the form

$$(2.1) \qquad \nabla^*_K T_J{}^I{}_b{}^a{}_{\bar d}{}^{\bar c}=\partial_K T^{::::}_{::::}+\left\{ {I \atop K\ H} \right\} T^{H::}_{::::}-T^{::::}_{H::}\left\{ {H \atop K\ J} \right\}$$
$$+(\Gamma_C{}^a{}_e T^{\cdot e \cdot}_{::::}+\Gamma_C{}^{\bar c}{}_{\bar e} T^{::\bar e}_{::::}-T^{::::}_{\cdot e \cdot}\Gamma_C{}^e{}_b-T^{::::}_{::\bar e}\Gamma_C{}^{\bar e}{}_{\bar d})B_K{}^C$$

in $\pi^{-1}(U)$, where Γ'_s are given by (1.21).

If we define two covariant derivations $'\nabla$ and $''\nabla$ acting on elements of

$\mathcal{T}^{prt}_{qsu}(TM)$ by

$$'\nabla_e = E^K{}_e \nabla^*_K , \quad ''\nabla_{\bar{e}} = C^K{}_{\bar{e}} \nabla^*_K , \tag{2.2}$$

respectively, then we have the following results:

(a) For any elements of $\mathcal{T}^{prt}_{qsu}(TM)$, say an element $\tilde{T}$ of $\mathcal{T}^{111}_{111}(TM)$ with components $T_J{}^I{}_b{}^a{}_{\bar{d}}{}^{\bar{c}}$ in $\pi^{-1}(U)$, $'\nabla\tilde{T}$ and $''\nabla\tilde{T}$ are respectively elements of $\mathcal{T}^{111}_{121}(TM)$ and $\mathcal{T}^{111}_{112}(TM)$, and have respectively components of the forms

$$\begin{aligned} '\nabla_e T_J{}^I{}_b{}^a{}_{\bar{d}}{}^{\bar{c}} = {} & D_e T^{\cdots}_{\cdots} + \left(\widetilde{\begin{Bmatrix} I \\ K\ H \end{Bmatrix}} T^{H\cdots}_{\cdots\cdot} - T^{\cdots\cdot}_{H\cdots} \widetilde{\begin{Bmatrix} H \\ K\ J \end{Bmatrix}}\right) E^K{}_e \\ & + \Gamma_e{}^a{}_f T^{\cdot f\cdot}_{\cdots\cdot} + \Gamma_e{}^{\bar{c}}{}_{\bar{f}} T^{\cdot\cdot\cdot\bar{f}}_{\cdots\cdot} - T^{\cdots\cdot}_{\cdot f\cdot\cdot} \Gamma_e{}^f{}_b - T^{\cdots}_{\cdot\cdot\bar{f}} \Gamma_e{}^{\bar{f}}{}_{\bar{d}} , \end{aligned} \tag{2.3}$$

$$\begin{aligned} ''\nabla_{\bar{e}} T_J{}^I{}_b{}^a{}_{\bar{d}}{}^{\bar{c}} = {} & D_{\bar{e}} T^{\cdots}_{\cdots} + \left(\widetilde{\begin{Bmatrix} I \\ K\ H \end{Bmatrix}} T^{H\cdots}_{\cdots\cdot} - T^{\cdots\cdot}_{H\cdots} \widetilde{\begin{Bmatrix} H \\ K\ J \end{Bmatrix}}\right) C^K{}_{\bar{e}} \\ & + \Gamma_{\bar{e}}{}^a{}_f T^{\cdot f\cdot}_{\cdots\cdot} + \Gamma_{\bar{e}}{}^{\bar{c}}{}_{\bar{f}} T^{\cdot\cdot\cdot\bar{f}}_{\cdots\cdot} - T^{\cdots\cdot}_{\cdot f\cdot\cdot} \Gamma_{\bar{e}}{}^f{}_b - T^{\cdots}_{\cdot\cdot\bar{f}} \Gamma_{\bar{e}}{}^{\bar{f}}{}_{\bar{d}} . \end{aligned} \tag{2.4}$$

(b) For any projectable element $\hat{T}$ of $\mathcal{T}^r_s(hTM)$, we have

$$\nabla T = p('\nabla\hat{T}) , \tag{2.5}$$

where $T = p\hat{T}$.

We call $'\nabla$ and $''\nabla$ the *van der Waerden-Bortolotti covariant derivations of* $T(M)$ *for the base space* M *and for the fibre* respectively. From the definition, we easily obtain the following two results.

PROPOSITION 2.1. *The equations*

$$\nabla^*_K \tilde{g}_{JI} = 0 , \quad \nabla^*_K g_{cb} = 0 , \quad '\nabla_a \tilde{g}_{JI} = 0 , \quad '\nabla_a g_{cb} = 0 ,$$

$$''\nabla_{\bar{a}} \tilde{g}_{JI} = 0 \quad \text{and} \quad ''\nabla_{\bar{a}} g_{cb} = 0$$

hold in $T(M)$.

PROPOSITION 2.2. *The equations*

$$'\nabla_d h_{cb}{}^{\bar{a}} = -\frac{1}{2} y^e \nabla_d K_{cbe}{}^a , \tag{2.10}$$

$$'\nabla_d h^a{}_{b\bar{c}} = \frac{1}{2} y^e \nabla_d K_{ecb}{}^a , \tag{2.11}$$

$$''\nabla_{\bar{d}} h_{cb}{}^{\bar{a}} = -\frac{1}{2} K_{dcb}{}^a + \frac{1}{4} (K_{fdc}{}^e K_{ebg}{}^a - K_{fdb}{}^e K_{ecg}{}^a) y^f y^g , \tag{2.12}$$

and

$$''\nabla_{\bar{d}} h^a{}_{b\bar{c}} = \frac{1}{2} K_{dcb}{}^a + \frac{1}{4} (K_{fde}{}^a K_{gcb}{}^e - K_{fce}{}^a K_{gdb}{}^e) y^f y^g \tag{2.13}$$

hold in $T(M)$.

3. The (*)-Lie derivative.

In this section, we shall recall definitions and properties concerning the (*)-Lie derivative, following to our previous results (see the section 3, [12]).

Let there be given a projectable vector field $\tilde{X}$ in $T(M)$, which has components $\tilde{X}^I$ with respect to the induced coordinates $(\tilde{x}^I)=(x^i, y^i)$ in $\pi^{-1}(U)$. Then we have an expression of the form

$$\tilde{X}^I=B^I{}_A X^A=E^I{}_a X^a+C^I{}_{\bar{a}} X^{\bar{a}}, \quad D_{\bar{b}} X^a=0, \tag{3.1}$$

where $X^a=E_J{}^a\tilde{X}^J$ and $X^{\bar{a}}=C_J{}^{\bar{a}}\tilde{X}^J$.

For an arbitrary element of $\mathcal{T}^{0rt}_{0su}(TM)$, say an element $\tilde{T}$ of $\mathcal{T}^{011}_{011}(TM)$ with components $T_b{}^a{}_{\bar{d}}{}^{\bar{c}}$ in $\pi^{-1}(U)$, we define the *(*)-Lie derivative* $\overset{*}{\mathcal{L}}_{\tilde{X}}\tilde{T}$ *of* $\tilde{T}$ *with respect to* $\tilde{X}$ as a partial tensor with components of the form

$$\begin{aligned}\overset{*}{\mathcal{L}}_{\tilde{X}} T_b{}^a{}_{\bar{d}}{}^{\bar{c}}=&X^E D_E T_b{}^a{}_{\bar{d}}{}^{\bar{c}}-T_b{}^e{}_{\bar{d}}{}^{\bar{c}} D_e X^a+T_e{}^a{}_{\bar{d}}{}^{\bar{c}} D_b X^e\\ &-T_b{}^a{}_{\bar{d}}{}^{\bar{e}}\Big(D_{\bar{e}} X^{\bar{c}}-\Big\{{}_e{}^c{}_f\Big\}X^f\Big)+T_b{}^a{}_{\bar{e}}{}^{\bar{c}}\Big(D_{\bar{d}} X^{\bar{e}}-\Big\{{}_d{}^e{}_f\Big\}X^f\Big).\end{aligned} \tag{3.2}$$

Taking account of (2.3) and (2.4), we see that the relation (3.2) is equivalent to

$$\begin{aligned}\overset{*}{\mathcal{L}}_{\tilde{X}} T_b{}^a{}_{\bar{d}}{}^{\bar{c}}=&X^{e\prime}\nabla_e T_b{}^a{}_{\bar{d}}{}^{\bar{c}}+X^{\bar{e}\prime\prime}\nabla_{\bar{e}} T_b{}^a{}_{\bar{d}}{}^{\bar{c}}-T_b{}^e{}_{\bar{d}}{}^{\bar{c}}({}^\prime\nabla_e X^a+h^a{}_{e\bar{f}} X^{\bar{f}})\\ &+T_e{}^a{}_{\bar{d}}{}^{\bar{c}}({}^\prime\nabla_b X^e+h^e{}_{b\bar{f}} X^{\bar{f}})-T_b{}^a{}_{\bar{d}}{}^{\bar{e}\prime\prime}\nabla_{\bar{e}} X^{\bar{c}}+T_b{}^a{}_{\bar{e}}{}^{\bar{c}\prime\prime}\nabla_{\bar{d}} X^{\bar{e}}.\end{aligned} \tag{3.3}$$

From this definition, we see the following results:

(a) Denoting by X and by $\bar{X}$ the horizontal part of $\tilde{X}$ and the vertical part of $\tilde{X}$ respectively, we have

$$\overset{*}{\mathcal{L}}_{\tilde{X}}=\overset{*}{\mathcal{L}}_X+\overset{*}{\mathcal{L}}_{\bar{X}}.$$

(b) Denoting by $\mathcal{L}_Y$ the Lie derivation with respect to the vector field Y in M, we have for an arbitrary projectable element $\hat{T}$ of $\mathcal{T}^r_s(hTM)$

$$\mathcal{L}_X T=p(\overset{*}{\mathcal{L}}_{\tilde{X}}\hat{T})$$

in M, where $X=p\tilde{X}$ and $T=p\hat{T}$.

For an arbitrary element $\tilde{T}$ of $\mathcal{T}^{0rt}_{0su}(TM)$, we say that $\tilde{X}$ *leaves* $\tilde{T}$ *(*)-invariant* if the equation

$$\overset{*}{\mathcal{L}}_{\tilde{X}}\tilde{T}=0$$

holds in $T(M)$.

We shall now give some identities obtained from (3.3) for later use. In the first, for the elements $h_{cb}{}^{\bar{a}}$ and $h^a{}_{b\bar{c}}$, we have

$$\begin{aligned}\overset{*}{\mathcal{L}}_{\tilde{X}} h_{cb}{}^{\bar{a}}=&X^{e\prime}\nabla_e h_{cb}{}^{\bar{a}}+X^{\bar{e}\prime\prime}\nabla_{\bar{e}} h_{cb}{}^{\bar{a}}+h_{ce}{}^{\bar{a}}({}^\prime\nabla_b X^e+h^e{}_{b\bar{d}} X^{\bar{d}})\\ &+h_{eb}{}^{\bar{a}}({}^\prime\nabla_c X^e+h^e{}_{c\bar{d}} X^{\bar{d}})-h_{cb}{}^{\bar{e}\prime\prime}\nabla_{\bar{e}} X^{\bar{a}},\end{aligned} \tag{3.4}$$

$$\overset{*}{\mathcal{L}}_{\tilde{X}} h^a{}_{b\bar{c}} = X^{e\prime}\nabla_e h^a{}_{b\bar{c}} + X^{\bar{e}\prime\prime}\nabla_{\bar{e}} h^a{}_{b\bar{c}} - h^e{}_{b\bar{c}}({}'\nabla_e X^a + h^a{}_{e\bar{d}} X^{\bar{d}}) + h^a{}_{e\bar{c}}({}'\nabla_b X^e + h^e{}_{b\bar{d}} X^{\bar{d}}) + h^a{}_{b\bar{e}}{}''\nabla_{\bar{c}} X^{\bar{e}} \tag{3.5}$$

If we put

$$Z_b{}^{\bar{a}} = {}'\nabla_b X^{\bar{a}} + 2h_{bc}{}^{\bar{a}} X^c, \tag{3.6}$$

then we have

$${}'\nabla_b Z_c{}^{\bar{a}} - {}'\nabla_c Z_b{}^{\bar{a}} = 2\overset{*}{\mathcal{L}}_{\tilde{X}} h_{cb}{}^{\bar{a}}. \tag{3.7}$$

The elements $Z_b{}^{\bar{a}}$ play an important rôle in the following sections.

Let there be given a vector field X in M, which has components X^i with respect to $\{U, x^i\}$. Then its horizontal lift X^H, vertical lift X^V and complete lift X^C have respectively components

$$({}'X^I) = \begin{pmatrix} X^i \\ -\Gamma^i_j X^j \end{pmatrix}, \quad ({}''X^I) = \begin{pmatrix} 0 \\ X^i \end{pmatrix}, \quad (\mathring{X}^I) = \begin{pmatrix} X^i \\ \partial X^i \end{pmatrix} \tag{3.8}$$

with respect to the induced coordinates $(\tilde{x}^I) = (x^i, y^i)$ in $\pi^{-1}(U)$. Then their components

$${}'X^A = B_J{}^{A\prime} X^J, \quad {}'X^A = B_J{}^{A\prime\prime} X^J, \quad \mathring{X}^A = B_J{}^A \mathring{X}^J$$

with respect to the adapted frame are given respectively by

$$({}'X^A) = \begin{pmatrix} X^a \\ 0 \end{pmatrix}, \quad ({}''X^A) = \begin{pmatrix} 0 \\ X^a \end{pmatrix}, \quad (\mathring{X}^A) = \begin{pmatrix} X^a \\ \nabla X^a \end{pmatrix}, \tag{3.9}$$

where $\nabla X^a = y^b \nabla_b X^a$. Thus, we see that these three lifts are all projectable. Now let us consider the (*)-Lie derivative with respect to these lifts.

In the first, if $\tilde{X} = X^H$, then, taking account of (1.22), (2.3), (2.4), (2.10), (2.11), (2.12), (2.13), (3.4), (3.5), (3.6) and (3.9), we have

$$\overset{*}{\mathcal{L}}_{X^H} h_{cb}{}^{\bar{a}} = -\frac{1}{2} y^d (X^e \nabla_e K_{cbd}{}^a + K_{ebd}{}^a \nabla_c X^e + K_{ced}{}^a \nabla_b X^e) = -\frac{1}{2} y^d (\mathcal{L}_X K_{cbd}{}^a - K_{cbe}{}^a \nabla_d X^e + K_{cbd}{}^e \nabla_e X^a), \tag{3.10}$$

$$\overset{*}{\mathcal{L}}_{X^H} h^a{}_{b\bar{c}} = \frac{1}{2} y^d (X^e \nabla_e K_{dcb}{}^a - K_{dcb}{}^e \nabla_e X^a + K_{dce}{}^a \nabla_b X^e) = \frac{1}{2} y^d (\mathcal{L}_X K_{dcb}{}^a - K_{ecb}{}^a \nabla_d X^e - K_{deb}{}^a \nabla_c X^e), \tag{3.11}$$

$$Z_b{}^{\bar{a}} = -K_{bcd}{}^a X^c y^d, \tag{3.12}$$

$${}'\nabla_b Z_c{}^{\bar{a}} = y^d \nabla_b (K_{ecd}{}^a X^e), \tag{3.13}$$

(3.14) $$''\nabla_{\bar b} Z_c{}^{\bar a} = \left(K_{dcb}{}^a + \frac{1}{2} K_{fbc}{}^e K_{edg}{}^a y^f y^g\right) X^d .$$

Similarly, if $\tilde X = X^V$, then we have

(3.15) $$\overset{*}{\mathcal{L}}_{X^V} h_{cb}{}^{\bar a} = -\frac{1}{2} K_{cbd}{}^a X^d ,$$

(3.16) $$\overset{*}{\mathcal{L}}_{X^V} h^a{}_{b\bar c} = \frac{1}{2} X^d K_{dcb}{}^a ,$$

(3.17) $$Z_b{}^{\bar a} = \nabla_b X^a ,$$

(3.18) $$'\nabla_b Z_c{}^{\bar a} = \nabla_b \nabla_c X^a ,$$

(3.19) $$''\nabla_{\bar b} Z_c{}^{\bar a} = -\frac{1}{2} y^d K_{dbc}{}^e \nabla_e X^a ,$$

and if $\tilde X = X^C$, then we have

(3.20) $$\overset{*}{\mathcal{L}}_{X^C} h_{cb}{}^{\bar a} = -\frac{1}{2} y^d \mathcal{L}_X K_{cbd}{}^a ,$$

(3.21) $$\overset{*}{\mathcal{L}}_{X^C} h^a{}_{b\bar c} = \frac{1}{2} y^d \mathcal{L}_X K_{dcb}{}^a ,$$

(3.22) $$Z_b{}^{\bar a} = y^c \mathcal{L}_X \left\{ {a \atop c\ \ b} \right\} ,$$

(3.23) $$'\nabla_c Z_b{}^{\bar a} = y^d \nabla_c \mathcal{L}_X \left\{ {a \atop d\ \ b} \right\} ,$$

(3.24) $$''\nabla_{\bar c} Z_b{}^{\bar a} = \mathcal{L}_X \left\{ {a \atop c\ \ b} \right\} - \frac{1}{2} y^f y^d K_{dcb}{}^e \mathcal{L}_X \left\{ {a \atop f\ \ e} \right\} .$$

4. Affine Killing vectors in $T(M)$.

Let $\tilde X$ be a projectable affine Killing vector field in $T(M)$ such that $\tilde X$ has components $\tilde X^I$ of the form (3.1). Then we see that the condition

(4.1) $$\mathcal{L}_{\tilde X} \widetilde{\left\{ {H \atop J\ \ I} \right\}} = \tilde\nabla_J \tilde\nabla_I \tilde X^H + \tilde K_{KJI}{}^H \tilde X^K = 0$$

holds in $\pi^{-1}(U)$, where $\tilde K_{KJI}{}^H$ denote components of the curvature tensor of $T(M)$ with the metric $\tilde g$.

We find that the condition (4.1) is equivalent to the following equations (see the section 5, [12]),

(4.2) $$\mathcal{L}_X \left\{ {a \atop c\ \ b} \right\} + h^a{}_{b\bar e} Z_c{}^{\bar e} + h^a{}_{c\bar e} Z_b{}^{\bar e} = 0 ,$$

(4.3) $$\overset{*}{\mathcal{L}}_{\tilde X} h^a{}_{c\bar b} = 0 ,$$

(4.4) $$'\nabla_c Z_b{}^{\bar a} + {}'\nabla_b Z_c{}^{\bar a} = 0 ,$$

(4.5) $$''\nabla_{\bar b} Z_c{}^{\bar a} = 0 ,$$

where $X=pX$ and

$$Z_b{}^{\bar a}='\nabla_b X^{\bar a}+2h_{bc}{}^{\bar a}X^c .$$

Let there be given a vector field X in M, which has components X^i with respect to $\{U, x^i\}$.

We now consider the conditions in order that the horizontal lift X^H, the vertical lift X^V and the complete lift X^C to $T(M)$ of a vector field X in M be an affine Killing vector field in $T(M)$, respectively.

In the first, suppose that $\tilde{X}=X^H$ is an affine Killing vector field in $T(M)$.

Taking account of (3.14) and (4.5), we easily obtain

$$X^d K_{dcb}{}^a=0 . \tag{4.6}$$

Taking account of (3.12) and (4.6), we find

$$Z_b{}^{\bar a}=0 . \tag{4.7}$$

Taking account of (4.2) and (4.7), we have

$$\mathcal{L}_X\left\{ {a \atop c\ \ b} \right\}=0 , \tag{4.8}$$

which implies that X is an affine Killing vector field in M. Moreover, we have

$$\mathcal{L}_X K_{dcb}{}^a=0, \tag{4.9}$$

because of (4.8). Taking account of (3.11), (4.3), (4.6) and (4.9), we have

$$\overset{*}{\mathcal{L}}_{X^H} h^a{}_{b\bar c}=-\frac{1}{2} y^d (K_{ecb}{}^a\nabla_d X^e+K_{deb}{}^a\nabla_c X^e)$$

$$=\frac{1}{2} y^d X^e(\nabla_d K_{ecb}{}^a+\nabla_c K_{deb}{}^a)=\frac{1}{2} y^d X^e\nabla_e K_{dcb}{}^a=0 ,$$

from which we have

$$X^e\nabla_e K_{dcb}{}^a=0 . \tag{4.10}$$

Conversely, suppose that the relations (4.6), (4.8) and (4.10) hold in M. Taking account of (3.12) and (4.6), we have the equation (4.7), i.e.,

$$Z_b{}^{\bar a}=0 ,$$

from which we see that the equations (4.4) and (4.5) hold. Furthermore, taking account of (4.7) and (4.8), we see that the equation (4.2) holds. Similarly, taking account of (4.9) and (4.10), we see that the equation (4.3) holds. Thus, X^H is an affine Killing vector field in $T(M)$. Thus we have

LEMMA 4.1. *The horizontal lift X^H to $T(M)$ of a vector field X in M is an affine Killing vector field in $T(M)$ if and only if the following three relations hold in M.*

(4.6) $$X^d K_{dcb}{}^a = 0 .$$

(4.8) $\mathcal{L}_X \{c{}^a{}_b\} = 0$, *that is, X is an affine Killing vector field in M.*

(4.10) $$X^e \nabla_e K_{dcb}{}^a = 0 .$$

Applying Bianchi identities, we easily have

LEMMA 4.2. *If X is a parallel vector field in M which satisfies* (4.6), *then X satisfies* (4.10).

The following result is well-known (see Yano [5]):

LEMMA 4.3. *In a compact Riemannian manifold M, an affine Killing vector field in M is a Killing vector field.*

Applying Green's theorem, we easily have

LEMMA 4.4. *In a compact Riemannian manifold M, if X satisfies* (4.6), *that is $X^d K_{dcb}{}^a = 0$, then X is a parallel vector field in M.*

Taking account of lemma 4.1, lemma 4.2, lemma 4.3 and lemma 4.4, we have

THEOREM 4.5. *Let M be a compact Riemannian manifold. The horizontal lift X^H to $T(M)$ of a vector field X in M is an affine Killing vector field in $T(M)$ if and only if X is a parallel Killing vector field in M. Moreover, if M is irreducible, then an affine Killing vector field X^H other than zero does not exist in $T(M)$.*

In the next, suppose that $\tilde{X} = X^V$ is an affine Killing vector field in $T(M)$. Taking account of (3.16), (3.19), (4.3) and (4.5), we have

(4.11) $$X^d K_{dcb}{}^a = 0 ,$$

(4.12) $$K_{dcb}{}^e \nabla_e X^a = 0 .$$

The equation (4.11) is the same one as (4.6). Taking account of (1.22), (3.17) and (4.2), we have

$$\mathcal{L}_X \left\{ {}_c{}^a{}_b \right\} + \frac{1}{2} y^e (K_{edb}{}^a \nabla_c X^d + K_{edc}{}^a \nabla_b X^d) = 0 ,$$

from which we have

(4.13) $$\mathcal{L}_X \left\{ {}_c{}^a{}_b \right\} = 0 ,$$

(4.14) $$K_{edb}{}^a \nabla_c X^d + K_{edc}{}^a \nabla_b X^d = 0 .$$

The equation (4.13) implies that X is an affine Killing vector field in M, and moreover, taking account of (4.11), we see that (4.14) reduces to

$$(4.15)\qquad (\nabla_c K_{deb}{}^a + \nabla_b K_{dec}{}^a) X^d = 0 .$$

If (4.11) and (4.13) hold, then, from (3.18) we have

$$(4.16)\qquad '\nabla_c Z_b{}^{\bar a} + '\nabla_b Z_c{}^{\bar a} = \nabla_c \nabla_b X^a + \nabla_b \nabla_c X^a = -X^d (K_{dcb}{}^a + K_{dbc}{}^a) = 0 ,$$

which implies that the equation (4.4) holds.

Conversely, if the relations (4.11), (4.12), (4.13) and (4.15) hold in M, then, we easily see that the relations (4.2), (4.3), (4.4) and (4.5) hold. Therefore, X^V is an affine Killing vector field in $T(M)$. Thus we have

LEMMA 4.6. *The vertical lift X^V to $T(M)$ of a vector field X in M is an affine Killing vector field in $T(M)$ if and only if the following four relations hold in M.*

$$(4.11)\qquad X^d K_{dcb}{}^a = 0 .$$

$$(4.12)\qquad K_{dcb}{}^e \nabla_e X^a = 0 .$$

(4.13) $\mathcal{L}_X \{_c{}^a{}_b\} = 0$, *that is, X is an affine Killing vector field in M.*

$$(4.15)\qquad X^d (\nabla_c K_{deb}{}^a + \nabla_b K_{dec}{}^a) = 0 .$$

Applying Bianchi identities, we easily have

LEMMA 4.7. *If X is a parallel vector field in M which satisfies* (4.11), *then X satisfies* (4.12) *and* (4.15).

Taking account of lemma 4.3, lemma 4.4, lemma 4.6 and lemma 4.7, we have

THEOREM 4.8. *Let M be a compact Riemannian manifold. The vertical lift X^V to $T(M)$ of a vector field X in M is an affine Killing vector field in $T(M)$ if and only if X is a parallel Killing vector field in M. Moreover, if M is irreducible, then an affine Killing vector field X^V other than zero does not exist in $T(M)$.*

In the last, suppose that $\tilde{X} = X^C$ is an affine Killing vector field in $T(M)$. Taking account of (1.22), (3.22) and (4.2), we have

$$\mathcal{L}_X \left\{ {}_c{}^a{}_b \right\} + \frac{1}{2} y^d y^f \left(K_{deb}{}^a \mathcal{L}_X \left\{ {}_f{}^e{}_c \right\} + K_{dec}{}^a \mathcal{L}_X \left\{ {}_f{}^e{}_b \right\} \right) = 0 ,$$

from which we have

$$(4.17)\qquad \mathcal{L}_X \left\{ {}_c{}^a{}_b \right\} = 0 ,$$

which implies that X is an affine Killing vector field in M.

Conversely, if (4.17) holds, then, taking account of (3.21) and (3.22), we easily see that the relations (4.2), (4.3), (4.4) and (4.5) hold. Therefore, X^C is an affine Killing vector field in $T(M)$. Thus we have

THEOREM 4.9. *The complete lift X^C to $T(M)$ of a vector field X in M is an affine Killing vector field in $T(M)$ if and only if X is an affine Killing vector field in M.*

5. Projective Killing vectors in $T(M)$.

Let $\tilde{X}$ be a projectable projective Killing vector field in $T(M)$ such that $\tilde{X}$ has components $\tilde{X}^I$ of the form (3.1). Then we see that the condition

$$\tilde{\mathcal{L}}_{\tilde{X}}\widetilde{\left\{ {H \atop J\ I} \right\}}=\tilde{\nabla}_J\tilde{\nabla}_I\tilde{X}^H+\tilde{K}_{KJI}{}^H\tilde{X}^K=\tilde{\phi}_J\delta_I^H+\tilde{\phi}_I\delta_J^H \tag{5.1}$$

holds in $\pi^{-1}(U)$, $\tilde{\phi}_I$ being the components of a certain 1-form $\tilde{\phi}$ in $T(M)$. Then we have an expression of the form

$$\tilde{\phi}_I=B_I{}^A\phi_A=E_I{}^a\phi_a+C_I{}^{\bar{a}}\phi_{\bar{a}}\,, \tag{5.2}$$

where $\phi_a=E^I{}_a\tilde{\phi}_I$ and $\phi_{\bar{a}}=C^I{}_{\bar{a}}\tilde{\phi}_I$.

We find that the condition (5.1) is equivalent to the following equations (see the section 6, [12])

$$\mathcal{L}_X\left\{ {a \atop c\ b} \right\}+h^a{}_{b\bar{e}}Z_c{}^{\bar{e}}+h^a{}_{c\bar{e}}Z_b{}^{\bar{e}}=\delta_c^a\phi_b+\delta_b^a\phi_c\,, \tag{5.3}$$

$$\overset{*}{\mathcal{L}}_{\tilde{X}}h^a{}_{c\bar{b}}=\delta_c^a\phi_{\bar{b}}\,, \tag{5.4}$$

$$'\nabla_cZ_b{}^{\bar{a}}+'\nabla_bZ_c{}^{\bar{a}}=0\,, \tag{5.5}$$

$$''\nabla_{\bar{b}}Z_c{}^{\bar{a}}=\delta_{\bar{b}}^{\bar{a}}\phi_c\,, \tag{5.6}$$

$$\delta_{\bar{c}}^{\bar{a}}\phi_{\bar{b}}+\delta_{\bar{b}}^{\bar{a}}\phi_{\bar{c}}=0\,, \tag{5.7}$$

where $X=p\tilde{X}$ and

$$Z_b{}^{\bar{a}}='\nabla_bX^{\bar{a}}+2h_{bc}{}^{\bar{a}}X^c\,.$$

Contracting with respect to the indices $\bar{a}$ and $\bar{c}$ in (5.7), we have

$$\phi_{\bar{b}}=0\,. \tag{5.8}$$

Therefore, taking account of (5.4) and (5.8), we have

$$\overset{*}{\mathcal{L}}_{\tilde{X}}h^a{}_{c\bar{b}}=0\,. \tag{5.4)'$$

Let there be given a vector field X in M, which has components X^i with respect to $\{U, x^i\}$.

We now consider the conditions in order that the horizontal lift X^H, the vertical lift X^V and the complete lift X^C to $T(M)$ of a vector field X in M be a projective Killing vector field in $T(M)$, respectively.

In the first, suppose that $\tilde{X}=X^H$ is a projective Killing vector field in $T(M)$. Contracting with respect to the indices $\bar{a}$ and $\bar{b}$ in (5.6), and taking account of (3.14), we have

$$\phi_c=\frac{1}{n}{}''\nabla_{\bar{a}}Z_{c}^{\bar{a}}=\frac{1}{2n}K_{fac}{}^{e}K_{edg}{}^{a}X^{d}y^{f}y^{g}\,. \tag{5.9}$$

Taking account of (3.14) and substituting (5.9) into (5.6), we have

$$X^{d}K_{dcb}{}^{a}+\frac{1}{2n}y^{f}y^{g}X^{d}(nK_{fbc}{}^{e}K_{edg}{}^{a}-\delta_{\bar{b}}^{\bar{a}}K_{fhc}{}^{e}K_{edg}{}^{h})=0\,,$$

from which we have

$$X^{d}K_{dcb}{}^{a}=0\,. \tag{5.10}$$

Substituting (5.10) into (5.9), we have

$$\phi_c=0\,. \tag{5.11}$$

From (5.8) and (5.11), we see that $\tilde{\phi}=0$. Thus we obtain

LEMMA 5.1. *If the horizontal lift X^H to $T(M)$ of a vector field X in M is a projective Killing vector field in $T(M)$, then it is necessarily an affine Killing vector field in $T(M)$.*

Taking account of theorem 4.5 and lemma 5.1, we have

THEOREM 5.2. *Let M be a compact Riemannian manifold. The horizontal lift X^H to $T(M)$ of a vector field X in M is a projective Killing vector field in $T(M)$ if and only if X is a parallel Killing vector field in M. Moreover, if M is irreducible, then a projective Killing vector field X^H other than zero does not exist in $T(M)$.*

Next, suppose that $\tilde{X}=X^V$ is a projective Killing vector field in $T(M)$. Contracting with respect to the indices $\bar{a}$ and $\bar{b}$ in (5.6), we have

$$\phi_c=\frac{1}{n}{}''\nabla_{\bar{a}}Z_{c}{}^{\bar{a}}=-\frac{1}{2n}y^{d}K_{dfc}{}^{e}\nabla_{e}X^{f}\,. \tag{5.12}$$

Substituting (1.22), (3.17) and (5.12) into (5.3), we have

$$\mathcal{L}_X\left\{ {a \atop c\ \ b} \right\}=0\,, \tag{5.13}$$

$$K_{dfb}{}^{a}\nabla_{c}X^{f}+K_{dfc}{}^{a}\nabla_{b}X^{f}=-\frac{1}{n}(\delta_{c}^{a}K_{dfb}{}^{e}\nabla_{e}X^{f}+\delta_{b}^{a}K_{dfc}{}^{e}\nabla_{e}X^{f})\,. \tag{5.14}$$

Contracting with respect to the indices a and c in (5.14), we have

$$K_{dfb}{}^{e}\nabla_{e}X^{f}=-\frac{n+1}{n}K_{dfb}{}^{e}\nabla_{e}X^{f},$$

from which we have

$$K_{dfb}{}^{e}\nabla_{e}X^{f}=0. \tag{5.15}$$

Substituting (5.15) into (5.12), we have

$$\phi_{c}=0. \tag{5.16}$$

From (5.8) and (5.16), we see that $\tilde{\phi}=0$. Thus we have

LEMMA 5.3. *If the vertical lift X^V to $T(M)$ of a vector field X in M is a projective Killing vector field in $T(M)$, then it is necessarily an affine Killing vector field in $T(M)$.*

Taking account of theorem 4.8 and lemma 5.3, we have

THEOREM 5.4. *Let M be a compact Riemannian manifold. The vertical lift X^V to $T(M)$ of a vector field X in M is a projective Killing vector field in $T(M)$ if and only if X is a parallel Killing vector field in M. Moreover, if M is irreducible, then a projective Killing vector field X^V other than zero does not exist in $T(M)$.*

In the last, suppose that $\tilde{X}=X^C$ is a projective Killing vector field in $T(M)$. Substituting (1.22) and (3.22) into (5.3), and contracting with respect to the indices a and b, we have

$$\mathcal{L}_X\left\{{a \atop a\;\; c}\right\}+\frac{1}{2}y^{d}y^{f}K_{dec}{}^{a}\mathcal{L}_X\left\{{e \atop f\;\; a}\right\}=(n+1)\phi_{c}. \tag{5.17}$$

On the other hand, taking account of (3.24) and contracting with respect to the indices $\bar{a}$ and $\bar{b}$ in (5.6), we have

$$\mathcal{L}_X\left\{{a \atop a\;\; c}\right\}-\frac{1}{2}y^{d}y^{f}K_{dec}{}^{a}\mathcal{L}_X\left\{{e \atop f\;\; a}\right\}=n\phi_{c}. \tag{5.18}$$

Adding (5.17) to (5.18), we have

$$\phi_{c}=\frac{1}{2n+1}\mathcal{L}_X\left\{{a \atop a\;\; c}\right\}. \tag{5.19}$$

On the other hand, eliminating (5.18) from (5.17), we also have

$$\phi_{c}=y^{d}y^{f}K_{dec}{}^{a}\mathcal{L}_X\left\{{e \atop f\;\; a}\right\}. \tag{5.20}$$

From (5.19) and (5.20), we conclude that

$$\phi_{c}=0. \tag{5.21}$$

From (5.8) and (5.21), we have $\tilde{\phi}=0$. Thus we have

LEMMA 5.5. *If the complete lift X^C to $T(M)$ of a vector field X in M is a projective Killing vector field in $T(M)$, then it is necessarily an affine Killing vector field in $T(M)$.*

Taking account of theorem 4.9 and lemma 5.5, we have

THEOREM 5.6. *The complete lift X^C to $T(M)$ of a vector field X in M is a projective Killing vector field in $T(M)$ if and only if X is an affine Killing vector field in M.*

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