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# THE SPECTRUM OF THE LAPLACE OPERATOR FOR A SPECIAL RIEMANNIAN MANIFOLD

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**1. Introduction.** Let $(M, g)$ be a compact orientable Riemannian manifold of dimension $n$. Let $\Lambda^q(M)$ be the vector space of exterior $q$-forms on $M$, where $q=0, 1, \cdots, n$. We denote by $S_p^q(M, g)$ the spectrum of $\Delta$ on $\Lambda^q(M)$.

It was the following open problem. Does $S_p^q(M, g)$ determine the geometry of the Riemannian manifold $(M, g)$? The answer to this problem in general case is negative. This is a consequence of the counter example which is given in ([3]). If the Riemannian manifold $(M, g)$ is a special one, then problem remains open.

It has been proved ([4]) that the three spectrums $S_p^0(S^n, g_0)$, $S_p^1(S^n, g_0)$ and $S_p^2(S^n, g_0)$ determine completely the geometry of the standard sphere $(S^n, g_0)$.

One of the results of the present paper is to prove that for each standard sphere $(S^n, g_0)$ there is at least one integer $q\in[0, n]$ such that the spectrum $S_p^q(S^n, g_0)$ determines completely the geometry on the sphere $(S^n, g_0)$.

In the second paragraph we give some known results for the spectrum of the Laplace operator $\Delta$ which acts on the vector space $\Lambda^q(M)$, where $q=0, 1, \cdots, n$.

The spectrum of the Laplace operator on the $\Lambda^q(M)$, when the Riemannian manifold $(M, g)$ has constant sectional curvature different from zero, is studied in the third paragraph.

**2.** We consider a compact, orientable Riemannian manifold $(M, g)$ of dimension $n$. Let $\Lambda^q(M)$ be the vector space of all exterior $q$-forms on $M$, where $q=0, 1, \cdots, n$. For $q=0$, we obtain the set $\Lambda^0(M)$ of all differentiable functions on $M$.

Let $\Delta=-(d\delta+\delta d)$ be the Laplace operator which acts on the exterior algebra of $M$

$$\Lambda(M)=\Lambda^0(M)\oplus\Lambda^1(M)\oplus\cdots\oplus\Lambda^n(M)=\bigoplus_{q=0}^{n}\Lambda^q(M)$$

as follows

$$\Delta: \Lambda(M)\longrightarrow\Lambda(M), \quad \Delta: \Lambda^q(M)\longrightarrow\Lambda^q(M),$$

$$\Delta: \alpha\longrightarrow\Delta(\alpha)=-(d\delta+\delta d)(\alpha)=-d\delta(\alpha)-\delta d(\alpha), \ \forall\alpha\in\Lambda^q(M).$$

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If the exterior $q$-form is such that $\Delta\alpha=\lambda\alpha$, where $\lambda\in\boldsymbol{R}$, then $\alpha$ is called a $q$-eigenform, (or simply a $q$-form), and $\lambda$ the eigenvalue associated with $\alpha$.

The set of eigenvalues associated with the exterior $q$-forms is called the spectrum of $\Delta$ on $\Lambda^q(M)$, and is denoted by $S_p^q(M, g)$. Thus

$$S_p^q(M, g)=\{0\geqq\lambda_{1,q}=\cdots=\lambda_{1,q}>\lambda_{2,q}=\cdots=\lambda_{2,q}>\lambda_{3,q}>\cdots>-\infty\},$$

where each eigenvalue is repeated as many times as its multiplicity, which is finite and the spectrum $S_p^q(M, g)$ is discrete, since $\Delta$ is an elliptic operator.

The spectrum $S_p^q(M, g)$ exerts an influence on the geometry of $(M, g)$. The aim of the present paper is to show that $S_p^q(M, g)$ determines the geometry on $(M, g)$, when $(M, g)$ is a special Riemannian manifold and $q$ has a special value which depends on the dimension of the manifold.

In order to study the influence of $S_p^q(M, g)$ on the geometry of $(M, g)$ we need the Minakshisundarum-Pleijel-Gaffney asymptotic expansion given by

$$\sum_{i=1}^{\infty} e^{\lambda_{i,q}t}\sim \sum_{\substack{t>0\\ t\to 0}} (4\pi t)^{-n/2}(\alpha_{0,q}+\alpha_{1,q}t+\cdots+\alpha_{m,q}t^m)+0(t^{m-n/2}),$$

where $\alpha_{0,q}$, $\alpha_{1,q}$, $\alpha_{2,q}\cdots$ are numbers which can be expressed by

$$\alpha_{i,q}=\int_M u_{i,q}dM, \qquad i=0, 1, 2, \cdots,$$

where $dM$ is the volume element of $M$ and

$$u_{i,q}\colon M\longrightarrow \boldsymbol{R}, \qquad i=0, 1, 2, \cdots$$

are functions which are local Riemannian invariants. These can be expressed by the curvature tensor, its associated tensors, and their covariant derivatives.

Some of these have been computed ([5])

$$\alpha_{0,q}=\binom{n}{q}\mathrm{Vol}(M), \tag{2.1}$$

$$\alpha_{1,q}=\int_M C(n, q)S\, dM, \tag{2.2}$$

$$\alpha_{2,q}=\int_M [C_1(n, q)S^2+C_2(n, q)|E|^2+C_3(n, q)|R|^2]dM, \tag{2.3}$$

where

$$C(n, q)=\frac{1}{6}\binom{n}{q}-\binom{n-2}{q-1}, \tag{2.4}$$

$$C_1(n, q)=\frac{1}{72}\binom{n}{q}-\frac{1}{6}\binom{n-2}{q-1}+\frac{1}{2}\binom{n-4}{q-2}, \tag{2.5}$$

$$C_2(n, q)=-\frac{1}{180}\binom{n}{q}+\frac{1}{2}\binom{n-2}{q-1}-2\binom{n-4}{q-2}, \tag{2.6}$$

$$C_3(n, q)=\frac{1}{180}\binom{n}{q}-\frac{1}{12}\binom{n-2}{q-1}+\frac{1}{2}\binom{n-4}{q-2}, \tag{2.7}$$

and $R$, $E$ and $S$ are the curvature tensor field, the Ricci curvature, and the scalar curvature of $(M, g)$, respectively, and $|R|$, $|E|$ are the norms of $R$, $E$ with respect to $g$.

*Problem* 2.1. *Let $(M, g)$, $(M', g')$ be two compact orientable Riemannian manifolds. If $S_p^q(M, g)=S_p^q(M', g')$, is $(M, g)$ isometric to $(M', g')$?*

The answer to this problem is negative. This is a consequence of the following counter example (J. Milnor [3]).

There exist two lattices $L$ and $L'$ in $\boldsymbol{R}^{16}$ such that

$$S_p^0(\boldsymbol{R}^{16}/L, g_0/L)=S_p^0(\boldsymbol{R}^{16}/L', g_0/L'), \tag{2.8}$$

where $g_0$ is the Euclidean metric in $\boldsymbol{R}^{16}$.

Relation (2.8) implies that

$$S_p^q(\boldsymbol{R}^{16}/L, g_0/L)=S_p^q(\boldsymbol{R}^{16}/L', g_0/L').$$

But $(\boldsymbol{R}^{16}/L, g_0/L)$ is not isometric to $(\boldsymbol{R}^{16}/L', g_0/L')$.

From the relation

$$S_p^q(M, g)=S_p^q(M', g'),$$

we conclude that

(i) $\dim(M)=\dim(M')$, (ii) $\mathrm{Vol}(M)=\mathrm{Vol}(M')$, (iii) $b_q(M)=b_q(M')$.

That is, the $q$ Betti numbers are equal, since $b_q(M)$ is the multiplicity of 0 in $S_p^q(M, g)$.

**3.** We consider two compact, orientable, Riemannian manifolds $(M, g)$ and $(M', g')$, for which we further assume that

$$S_p^q(M, g)=S_p^q(M', g'). \tag{3.1}$$

We study special conditions, which taken with (3.1), determine the geometry on $(M, g)$.

THEOREM 3.1. *Let $(M, g)$, $(M', g')$ be two compact, orientable Riemannian manifolds. If $n$ is given, then we can find at least one integer $q$ (one of them is $q=\left[\frac{n}{3}\right]$ if $n\geqq 8$, or $q=2$, if $n\in\{6, 7\}$ or $q=0$ if $n\in\{2, 3, 4, 5\}$ such that $Sp^q(M, g)=Sp^q(M', g')$ implies that $(M, g)$ has constant sectional curvature $k$, if and only if $(M', g')$ has constant sectional curvature $k'$, and $k=k'$.*

*Proof*. Let $C$, $G$ be the Weyl conformal curvature tensor field and the Einstein tensor field respectively, on $(M, g)$. The components $(C_{ijkl})$ and $(G_{ij})$ of $C_1$ and $G$, respectively, with respect to a local coordinate system $(x^1, \cdots, x^n)$ on the manifold $(M, g)$ are given by

$$C_{ijkl}=R_{ijkl}-\alpha(E_{jk}g_{il}-E_{jl}g_{ik}-g_{jk}E_{il}-g_{il}E_{ik})+\beta(g_{jk}g_{il}-g_{jl}g_{ik})S\,, \tag{3.2}$$

where $\alpha=1/(n-1)$, $\beta=1/(n-1)(n-2)$, and

$$G_{ij}=E_{ij}-\gamma g_{ij}S\,, \tag{3.3}$$

where $\gamma=1/n$.

From (3.2) and (3.3) we obtain

$$|C|^2=|R|^2-4|E|^2/(n-2)+2S^2/(n-1)(n-2)\,, \tag{3.4}$$

$$|G|^2=|E|^2-S^2/n\,. \tag{3.5}$$

The formula (2.3) by virtue of (3.4) and (3.5) becomes

$$\alpha_{2,q}=\int_M[A_1|C|^2+A_2|G|^2+A_3S^2]dM\,, \tag{3.6}$$

where

$$A_1=A_1(n, q)=\frac{1}{180n(n-1)(n-2)(n-3)}\binom{n}{q}\cdot P_1(q, n)\,, \tag{3.7}$$

$$A_2=A_2(n, q)=\frac{1}{180n(n-1)(n-2)^2}\binom{n}{q}\cdot P_2(n, q)\,, \tag{3.8}$$

$$A_3=A_3(n, q)=\frac{1}{360n^2(n-1)^2}\binom{n}{q}\cdot P_3(n, q)\,. \tag{3.9}$$

The expressions $P_1(n, q)$, $P_2(n, q)$ and $P_3(n, q)$ in the formulas (3.7), (3.8) and (3.9) are given by

$$P_1(n, q)=90q(q-1)(n-q)(n-q-1)-15q(n-q)(n-2)(n-3)$$
$$+n(n-1)(n-2)(n-3)\,, \tag{3.10}$$

$$P_2(n, q)=-360q(q-1)(n-q)(n-q-1)+30q(n-q)(n-2)(3n-8)$$
$$-n(n-1)(n-2)(n-6)\,, \tag{3.11}$$

$$P_3(n, q)=180q(q-1)(n-q)(n-q-1)-60q(n-q)(n-2)^2$$
$$+n(n-1)(5n^2-7n+6)\,. \tag{3.12}$$

By assumption, the Riemannian manifold $(M', g')$ has constant sectional curvature $k'$. Therefore for $(M', g')$ we have $C'=0$, $G'=0$, and formula (3.6) in this case takes the form

$$\alpha'_{2,q}=\int_{M'} A_3(S')^2 dM' . \tag{3.13}$$

From (3.1), (3.6) and (3.13) we have

$$\int_M [A_1|C|^2+A_2|G|^2+A_3S^2]dM=\int_{M'} A_3(S')^2 dM' . \tag{3.14}$$

If $q=[n/3]$, then we have

$$A_1>0\,,\quad A_2>0\,,\quad A_3\geqq 0\,,\qquad \text{if}\quad n\geqq 7\,. \tag{3.15}$$

From the relation $\alpha_{1,q}=\alpha'_{1,q}$ by virtue of (2.2) yields

$$\int_M SdM=\int_{M'} S'dM' , \tag{3.16}$$

which, since $S'$=constant, implies

$$\int_M S^2 dM\geqq\int_{M'} (S')^2 dM' . \tag{3.17}$$

From (3.14), (3.15) and (3.17) we obtain $|C|^2=|G|^2=0$, which gives $C=G=0$. Hence the Riemannian manifold $(M, g)$ has constant sectional curvature $k$. Finally, the relation (3.16) implies $k=k'$.

If the dimension of the manifold is between 2 and 5 we take as $q=0$, ([1]).

If the dimension of the manifold is 6 or 7, then we take $q=2$, ([7]).

This completes the proof of the theorem. More details of this will be published later.

A consequence of the theorem (3.1) is the following corollary

COROLLARY 3.2. *Let* $(S^n, q_0)$ *be the standard Euclidean sphere. If* $n\geqq 6$ *then the* $Sp^{[n/3]}(S^n, g_0)$ *determines completely the geometry on* $(S^n, g_0)$. *Finally if* $n\in[2, 5]$, *then the* $Sp^0(S^n, q_0)$ *determines completely the geometry on* $(S^n, g_0)$.

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