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A REMARK ON CONFORMALLY FLAT TOTALLY REAL SUBMANIFOLDS

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1. Introduction.

The following result is well-known.

THEOREM A. (K. Yano [7]). *Let M^n, $n \geqq 4$, be a totally umbilical submanifold of a conformally flat Riemannian manifold. Then M^n is conformally flat.*

Corresponding to Theorem A one has the following results for submanifolds of Kaehlerian manifolds.

THEOREM B (S. Yamagushi and S. Sato [6]). *Let M^n, $n \geqq 4$, be a totally geodesic complex submanifold of a Kaehlerian manifold M^p, $p \geqq 8$, with vanishing Bochner curvature tensor. Then M^n has vanishing Bochner curvature tensor.*

THEOREM C (D. E. Blair [1], K. Yano [9]). *Let M^n, $n \geqq 4$, be a totally umbilical totally real submanifold of a Kaehlerian manifold with vanishing Bochner curvature tensor. Then M^n is conformally flat.*

A generalization of Theorem A is given in the following.

THEOREM D (B. Y. Chen and K. Yano [4]). *Let M^n, $n \geqq 4$, be a totally quasiumbilical submanifold of a conformally flat Riemannian manifold. Then M^n is conformally flat.*

In this paper we show that correspondingly Theorem C may be somewhat improved as follows.

THEOREM. *Let M^n, $n \geqq 4$, be a totally quasiumbilical totally real submanifold of a Kaehlerian manifold with vanishing Bochner curvature tensor. Then M^n is conformally flat.*

A proof of this property will be given in a straightforward way using an expression of the equation of Gauss for a totally real submanifold N of a Kaehlerian manifold $\tilde{N}$ which contains the Weyl curvature tensor of N and the

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Bochner curvature tensor $\tilde{N}$. This equation was obtained by K. Yano [9] and will be stated here as a Lemma.

Remark 1. Totally quasiumbilical Kaehler submanifolds are totally geodesic.

Remark 2. For a proof of Theorem D, see [2].

Remark 3. With respect to Theorems B and C, see also [11].

2. Preliminaries.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, $(h, j, i, k, l, t, s \in \{1', 2', \cdots, n'\})$. Let g_{ji}, ∇_j, $K_{kji}{}^h$, K_{ji} and K be the Riemannian metric tensor, the covariant differentiation of the corresponding Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of M^n, respectively. Then the *Weyl curvature tensor* of M^n is defined by

$$C_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki}, \tag{1}$$

where

$$L_{ji} = -\frac{1}{n-2} K_{ji} + \frac{1}{2(n-1)(n-2)} K g_{ji}, \quad L_k{}^h = L_{kt} g^{th}$$

and g^{ts} are the contravariant components of g_{ji}. According to a Theorem of H. Weyl the vanishing of $C_{kji}{}^h$ characterizes the conformal flatness of M^n for $n \geqq 4$.

Let M^{2m} be a (real) $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{V; y^\alpha\}$, $(\alpha, \beta, \gamma, \lambda, \mu, \nu \in \{1, 2, \cdots, 2m\})$. Let $F_\alpha{}^\beta$, $g_{\alpha\beta}$, ∇_α, $R_{\nu\alpha\beta}{}^\mu$, $R_{\alpha\beta}$ and R be the complex structure tensor, the Kaehlerian metric tensor, the corresponding covariant differentiation operator, the Riemann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of M^{2m}, respectively. Then the *Bochner curvature tensor* of M^{2m} is defined by

$$\begin{aligned} B_{\nu\mu\lambda}{}^\alpha = R_{\nu\mu\lambda}{}^\alpha + \delta_\nu^\alpha A_{\mu\lambda} - \delta_\mu^\alpha A_{\nu\lambda} + A_\nu{}^\alpha g_{\mu\lambda} - A_\mu{}^\alpha g_{\nu\lambda} + F_\nu{}^\alpha \tilde{A}_{\mu\lambda} - F_\mu{}^\alpha \tilde{A}_{\nu\lambda} \\ + \tilde{A}_\nu{}^\alpha g_{\mu\lambda} - \tilde{A}_\mu{}^\alpha F_{\nu\lambda} - 2(F_{\nu\mu}\tilde{A}_\lambda{}^\alpha + \tilde{A}_{\nu\mu} F_\lambda{}^\alpha), \end{aligned} \tag{2}$$

where

$$A_{\mu\lambda} = -\frac{1}{2(m+2)} R_{\mu\lambda} + \frac{1}{8(m+1)(m+2)} R g_{\mu\lambda}, \quad R_\nu{}^\alpha = A_{\nu\gamma} g^{\gamma\alpha},$$

$$\tilde{A}_{\mu\lambda} = -A_{\mu\gamma} F_\lambda{}^\gamma, \quad \tilde{A}_\nu{}^\alpha = \tilde{A}_{\nu\gamma} g^{\gamma\alpha}, \quad F_{\mu\lambda} = F_\mu{}^\gamma g_{\gamma\lambda}$$

and $g^{\lambda\mu}$ are the contravariant components of $g_{\alpha\beta}$ [5], [8], [11].

Now, let M^n be a Riemannian manifold isometrically immersed in a Kaehlerian manifold M^{2m}. Let the immersion be represented by $y^\alpha = y^\alpha(x^h)$ and put

$B_i{}^\alpha = \dfrac{\partial y^\alpha}{\partial x^i}$. Let $C_y{}^\alpha$ be $2m-n$ mutually orthogonal unit normal vectors of M^n in M^{2m}, $(x, y, z \in \{(n+1)', (n+2)', \cdots, (2m)'\})$. Then

(3) $$g_{ji} = g_{\mu\lambda} B_{ji}^{\mu\lambda}, \quad B_{ji}^{\mu\lambda} = B_j{}^\mu B_i{}^\lambda$$

and the metric tensor induced in the normal bundle is given by

(4) $$g_{zy} = g_{\mu\lambda} C_{zy}^{\mu\lambda}, \quad C_{zy}^{\mu\lambda} = C_z{}^\mu C_y{}^\lambda .$$

The *second fundamental tensors* $H_{ji}{}^x$ of M^n with respect to the normals $C_x{}^\alpha$ are determined by the formulas of Gauss and Weingarten

(5) $$\nabla_j B_i{}^\alpha = H_{ji}{}^x C_x{}^\alpha ,$$

(6) $$\nabla_j C_y{}^\alpha = -H_j{}^i{}_y B_i{}^\alpha ,$$

where $$H_j{}^i{}_y = H_{jty} g^{ti}, \; H_{jty} = H_{jt}{}^z g_{zy}.$$

The equations of Gauss for the submanifold M^n of M^{2m} are given by

(7) $$K_{kji}{}^h = R_{\nu\mu\lambda}{}^\alpha B_{kji\alpha}^{\nu\mu\lambda h} + H_k{}^h{}_x H_{ji}{}^x - H_j{}^h{}_x H_{ki}{}^x$$

where $$B_{kji\alpha}^{\nu\mu\lambda h} = B_k{}^\nu B_j{}^\mu B_i{}^\lambda B^h{}_\alpha , \quad B^h{}_\alpha = B_i{}^\gamma g^{ih} g_{\gamma\alpha} .$$

M^n is said to be *totally real* or *anti-invariant submanifold* of M^{2m} if the complex structure F of M^{2m} maps every tangent vector of M^n to a vector which is normal to M^n [3], [10]. Thus a totally real submanifold satisfies equations of the form

(8) $$F_\lambda{}^\alpha B_i{}^\lambda = -f_i{}^x C_x{}^\alpha , \quad F_\lambda{}^\alpha C_y{}^\lambda = f_y{}^h B_h{}^\alpha + f_y{}^x C_x{}^\alpha .$$

3. Equations of Gauss for a totally real submanifold in terms of the curvature tensors of Weyl and Bochner [9]

Using the *conformal second fundamental tensors* $M_{ji}{}^x$ with respect to $C_x{}^\alpha$ [7],

(9) $$M_{ji}{}^x = H_{ji}{}^x - H^x g_{ji} , \quad H^x = \frac{1}{n} g^{ts} H_{ts}{}^x ,$$

the equations of Gauss for any submanifold M^n may be written as

(10) $$K_{kji}{}^h = R_{\nu\mu\lambda}{}^\alpha B_{kji\alpha}^{\nu\mu\lambda h} + M_k{}^h{}_x M_{ji}{}^x - M_j{}^h{}_x M_{ki}{}^x + \delta_k^h M_{ji}{}^x H_x - \delta_j^h M_{ki}{}^x H_x$$
$$+ M_k{}^h{}_x H^x g_{ji} - M_j{}^h{}_x H^x g_{ki} + H_x H^x (\delta_k^h g_{ji} - \delta_j^h g_{ki}) ,$$

where $$M_k{}^h{}_x = M_{ktx} g^{th} , \quad M_{ktz} = M_{kt}{}^y g_{yz} , \quad H_z = H^y g_{yz} .$$

Now, assume that M^n is totally real. Then from (8) it follows that

(11) $$F_\lambda{}^\alpha B_i{}^\lambda B^h{}_\alpha = 0 .$$

Using (11), transvection of (2) with $B^{\nu\mu\lambda h}_{kji\alpha}$ gives

$$R_{\nu\mu\lambda}{}^{\alpha}B^{\nu\mu\lambda h}_{kji}=B_{\nu\mu\lambda}{}^{\alpha}B^{\nu\mu\lambda h}_{kji}-\delta^h_k A_{\mu\lambda}B^{\mu\lambda}_{ji}+\delta^h_j A_{\mu\lambda}B^{\mu\lambda}_{ki} -A_{\mu\lambda}B^{\mu\lambda}_{kt}g^{th}g_{ji}+A_{\mu\lambda}B^{\mu\lambda}_{jt}g^{th}g_{ki}. \tag{12}$$

Substitution of (12) in (10) yields

$$\begin{aligned} K_{kji}{}^{h}=&B_{\nu\mu\lambda}{}^{\alpha}B^{\nu\mu\lambda h}_{kji\alpha}-\delta^h_k A_{\mu\lambda}B^{\mu\lambda}_{ji}+\delta^h_j A_{\mu\lambda}B^{\mu\lambda}_{ki}-A_{\mu\lambda}B^{\mu\lambda}_{kt}g^{th}g_{ji}\\ &+A_{\mu\lambda}B^{\mu\lambda}_{jt}g^{th}g_{ki}+M_k{}^h{}_x M_{ji}{}^x-M_j{}^h{}_x M_{ki}{}^x+\delta^h_k M_{ji}{}^x H_x\\ &-\delta^h_j M_{ki}{}^x H_x+M_k{}^h{}_x H^x g_{ji}-M_j{}^h{}_x H^x g_{ki}+H_x H^x(\delta^h_k g_{ji}-\delta^h_j g_{ki}). \end{aligned} \tag{13}$$

The Ricci tensor K_{ji} is obtained by contraction with respect to h and k:

$$\begin{aligned} K_{ji}=&B_{\nu\mu\lambda}{}^{\alpha}B^{\nu}_{\alpha}B^{\mu\lambda}_{ji}-(n-2)A_{\mu\lambda}B^{\mu\lambda}_{ji}-A_{\mu\lambda}B^{\mu\lambda}g_{ji}-M_j{}^t{}_x M_{ti}{}^x\\ &+(n-2)M_{ji}{}^x H_x+(n-1)H_x H^x g_{ji}, \end{aligned} \tag{14}$$

where
$$B^{\mu\lambda}=B^{\mu\lambda}_{ji}g^{ji},\quad B^{\nu}_{\alpha}=B_t{}^{\nu}B^t{}_{\alpha}.$$

Transvection with g^{ji} gives

$$A_{\mu\lambda}B^{\mu\lambda}=\frac{n}{2}H_x H^x+\frac{1}{2(n-1)}(B_{\nu\mu\lambda}{}^{\alpha}B^{\nu}_{\alpha}B^{\mu\lambda}-K-M_s{}^t{}_x M_t{}^{sx}), \tag{15}$$

where
$$M_t{}^{sx}=M_t{}^s{}_y g^{yx}.$$

Substitution of (15) in (14) yields

$$\begin{aligned} A_{\mu\lambda}B^{\mu\lambda}_{ji}=&L_{ji}+\frac{1}{n-2}B_{\nu\mu\lambda}{}^{\alpha}B^{\nu}_{\alpha}B^{\mu\lambda}_{ji}-\frac{1}{2(n-1)(n-2)}B_{\nu\mu\lambda}{}^{\alpha}B^{\nu}_{\alpha}B^{\mu\lambda}g_{ji}\\ &-\frac{1}{n-2}M_j{}^t{}_x M_{ti}{}^x+\frac{1}{2(n-1)(n-2)}M_s{}^t{}_x M_t{}^{sx}g_{ji}\\ &+M_{ji}{}^x H_x+\frac{1}{2}H_x H^x g_{ji}. \end{aligned} \tag{16}$$

Finally substitution of (16) in (13) gives the following.

LEMMA. *Let M^n be a totally real submanifold of a Kaehlerian manifold M^{2m}. Then*

$$\begin{aligned} C_{kji}{}^h=&B_{\nu\mu\lambda}{}^{\alpha}B^{\nu\mu\lambda h}_{kji\alpha}-\frac{1}{n-2}[\delta^h_k B_{\nu\mu\lambda}{}^{\alpha}B^{\nu}_{\alpha}B^{\mu\lambda}_{ji}-\delta^h_j B_{\nu\mu\lambda}{}^{\alpha}B^{\nu}_{\alpha}B^{\mu\lambda}_{ki}\\ &+B_{\nu\mu\lambda}{}^{\alpha}B^{\nu}_{\alpha}B^{\mu\lambda}_{kt}g^{th}g_{ji}-B_{\nu\mu\lambda}{}^{\alpha}B^{\nu}_{\alpha}B^{\mu\lambda}_{jt}g^{th}g_{ki}]\\ &+\frac{1}{(n-1)(n-2)}B_{\nu\mu\lambda}{}^{\alpha}B^{\nu}_{\alpha}B^{\mu\lambda}(\delta^h_k g_{ji}-\delta^h_j g_{ki})+M_k{}^h{}_x M_{ji}{}^x-M_j{}^h{}_x M_{ki}{}^x \end{aligned}$$

$$+\frac{1}{n-2}[\delta_k^h M_j{}^t{}_x M_{ti}{}^x-\delta_j^h M_k{}^t{}_x M_{ti}{}^x+M_k{}^t{}_x M_t{}^{hx} g_{ji}-M_j{}^t{}_x M_t{}^{hx} g_{ki}]$$

$$-\frac{1}{(n-1)(n-2)} M_s{}^t{}_x M_t{}^{sx}(\delta_k^h g_{ji}-\delta_j^h g_{ki}).$$

4. Proof of Theorem

If on U there exist two functions a^z, b^z and a unit vector field u^z such that

$$H_{ji}{}^z=a^z g_{ji}+b^z u_j^z u_i^z, \tag{17}$$

then M^n is said to be *quasiumbilical* with respect to the normal direction $C_z{}^\alpha$. This is equivalent to say that M^n has a principal curvature with multiplicity $\geqq n-1$ with respect to $C_z{}^\alpha$. If respectively identically $a^z=0$, $b^z=0$ or $a^z=b^z=0$ then M^n is said to be *cylindrical, umbilical* or *geodesic* with respect to $C_z{}^\alpha$. M^n is called a *totally* quasiumbilical, cylindrical, umbilical or geodesic submanifold of M^{2m} if M^n is quasiumbilical, cylindrical, umbilical or geodesic with respect to every normal direction $C_z{}^\alpha$.

From (9) and (17) it follows that the conformal second fundamental tensors $M_{ji}{}^z$ of a totally quasiumbilical submanifold are given by

$$M_{ji}{}^z=b^z\left(-\frac{1}{n}g_{ji}+u_j^z u_i^z\right). \tag{18}$$

In particular, M^n is totally umbilical if and only if $M_{ji}{}^z=0$.

The preceeding Lemma implies that a totally real submanifold M^n, $n\geqq 4$, of a Kaehlerian manifold M^{2m} with vanishing Bochner curvature tensor is conformally flat if and only if

$$\begin{aligned} M_k{}^h{}_x M_{ji}{}^x-M_j{}^h{}_x M_{ki}{}^x+\frac{1}{n-2}[\delta_k^h M_j{}^t{}_x M_{ti}{}^x-\delta_j^h M_k{}^t{}_x M_{ti}{}^x \\ +M_k{}^t{}_x M_t{}^{hx} g_{ji}-M_j{}^t{}_x M_t{}^{hx} g_{ki}] \\ -\frac{1}{(n-1)(n-2)} M_s{}^t{}_x M_t{}^{sx}(\delta_k^h g_{ji}-\delta_j^h g_{ki})=0. \end{aligned} \tag{19}$$

In particular, (19) is trivially satisfied when M^n is totally umbilical.

Now assume that M^n is totally quasiumbilical. Then using the fact that the $2m-n$ vectors $C_x{}^\alpha$ are orthonormal, it follows from (18) that

$$\begin{aligned} & M_k{}^h{}_x M_{ji}{}^x-M_j{}^h{}_x M_{ki}{}^x \\ &=\sum_x (b^x)^2\Big[\frac{1}{n^2}(\delta_k^h g_{ji}-\delta_j^h g_{ki})-\frac{1}{n}(\delta_k^h u_j^x u_i^x-\delta_j^h u_k^x u_i^x \\ &\qquad +g_{ji}g^{th}u_k^x u_t^x-g_{ki}g^{th}u_j^x u_t^x)\Big], \end{aligned} \tag{21}$$

$$
\begin{aligned}
(22)\qquad & \delta_k^h M_j{}^t{}_x M_{ti}{}^x - \delta_j^h M_k{}^t{}_x M_{ti}{}^x + M_k{}^t{}_x M_t{}^{hx} g_{gi} - M_j{}^t{}_x M_t{}^{hx} g_{ki} \\
& = \sum_x (b^x)^2 \Big[\frac{2}{n^2} (\delta_k^h g_{ji} - \delta_j^h g_{ki}) + \frac{n-2}{n} (\delta_k^h u_j^x u_i^x - \delta_j^h u_k^x u_i^x \\
& \qquad + g_{ji} g^{lh} u_k^x u_l^x - g_{ki} g^{lh} u_j^x u_l^x) \Big],
\end{aligned}
$$

$$
(23)\qquad M_s{}^t{}_x M_t{}^{sx} (\delta_k^h g_{ji} - \delta_j^h g_{ki}) = \frac{n-1}{n} (\delta_k^h g_{ji} - \delta_j^h g_{ki}) \sum_x (b^x)^2 .
$$

By substitutions with (21), (22) and (23) it is clear that (19) is indeed satisfied under the present assumption.

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