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# GENERIC SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF AN ODD-DIMENSIONAL SPHERE

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## § 0. Introduction.

Many authors have been studying the so-called generic (anti-holomorphic) submanifold of a Kaehlerian manifold by the method of Riemannian fibre bundles (see [6], [9], [10] and [12] etc.).

But, the present authors [17] studied a generic submanifold $M$ of an odd-dimensional unit sphere $S^{2m+1}(1)$ under the condition that the structure tensor $f$ induced on $M$ and the second fundamental tensor $h$ commute. Moreover, one of the present authors [4] gives characterizations of a generic minimal submanifold of $S^{2m+1}(1)$ that $h$ and $f$ anticommute.

The purpose of the present paper is devoted to generalize the notions of the previous facts and characterize a generic submanifold of $S^{2m+1}(1)$ tangent to the Sasakian structure vector field defined on $S^{2m+1}(1)$.

In § 1, we recall fundamental properties and structure equations for generic submanifolds immersed in a Sasakian manifold and define the structure tensor induced on the submanifold to be antinormal.

In § 2, we prepare a theorem on submanifolds of $S^{2m+1}(1)$ which is used later very usefully.

In § 3, we find some results of generic submanifolds of $S^{2m+1}(1)$ with $\xi^x \neq 0$, where $\xi^x$ is the normal part of the Sasakian structure vector $\xi$.

In § 4, we investigate $(m+1)$-dimensional generic submanifolds of $S^{2m+1}(1)$ with $\xi^x \neq 0$.

In § 5, we determine generic submanifolds with antinormal structure of $S^{2m+1}(1)$ with $\xi^x \neq 0$.

In the last § 6, we characterize generic submanifolds of $S^{2m+1}(1)$ tangent to the Sasakian structure vector field.

## § 1. Generic submanifolds of a Sasakian manifold

Let $M^{2m+1}$ be a $(2m+1)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U; y^h\}$ and $(\phi_j{}^h, g_{ji}, \xi^h)$ the set of structure

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tensors of $M^{2m+1}$, where here and in the sequel, the indices $h, i, j$ and $k$ run over the range $\{1, 2, \cdots, 2m+1\}$. We then have

$$\phi_j{}^h\phi_i{}^j = -\delta_i^h + \xi_i\xi^h\,,\quad \xi_j\phi_i{}^j = 0\,,\quad \phi_j{}^h\xi^j = 0\,, \tag{1.1}$$
$$\xi_j\xi^j = 1\,,\quad \xi_i = g_{ji}\xi^j\,,\quad \phi_j{}^k\phi_i{}^h g_{kh} = g_{ji} - \xi_j\xi_i$$

and

$$\nabla_j\xi^h = \phi_j{}^h\,,\quad \nabla_j\phi_i{}^h = -g_{ji}\xi^h + \delta_j^h\xi_i\,, \tag{1.2}$$

where $\nabla_j$ denotes the operator of covariant differentiation with respect to $g_{ji}$.

Let $M^n$ be an $n$-dimensional Riemannian manifold isometrically immersed in $M^{2m+1}$ by the immersion $i: M^n \to M^{2m+1}$ and identify $i(M^n)$ with $M^n$ itself and represent the immersion $i$ by $y^h = y^h(x^a)$ (throughout this paper the indices $a, b, c, d$ and $e$ run over the range $\{1, 2, \cdots, n\}$). If we put $B_b{}^h = \partial_b y^h$, $\partial_b = \partial/\partial x^b$, then $B_b{}^h$ are $n$ linearly independent vectors of $M^{2m+1}$ tangent to $M^n$. Denoting by $g_{cb}$ the fundamental metric tensor of $M^n$, we then have

$$g_{cb} = B_c{}^h B_b{}^k g_{hk} \tag{1.3}$$

because of the immersion isometric.

We now denote $C_x{}^h$ by $2m+1-n$ mutually orthogonal unit normals of $M^n$ (the indices $u, v, w, x, y$ and $z$ run over the range $\{n+1, \cdots, 2m+1\}$). Thus, denoting by $\nabla_c$ the operator of van der Waerden-Bortolotti covariant differentiation with respect to the Christoffel symbols $\{{}_c{}^a{}_b\}$ formed with $g_{cb}$, we obtain equations of Gauss and Weingarten

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h\,, \tag{1.4}$$

$$\nabla_c C_x{}^h = -h_c{}^a{}_x B_a{}^h \tag{1.5}$$

respectively, where $h_{cb}{}^x$ are the second fundamental tensors with respect to the normals $C_x{}^h$ and $h_c{}^a{}_x = h_{cb}{}^y g^{ab} g_{yx}$, $g_{yx}$ being the metric tensor of the normal bundle of $M^n$ given by $g_{yx} = g_{ji} C_y{}^j C_x{}^i$, and $(g^{cb}) = (g_{cb})^{-1}$.

A submanifold $M^n$ of a Sasakian manifold $M^{2m+1}$ is called a *generic* (an *anti-holomorphic*) submanifold if the normal space $N_P(M^n)$ of $M^n$ at any point $P \in M^n$ is always mapped into the tangent space $T_P(M^n)$ by the action of the structure tensor $\phi$ of the ambient manifold $M^{2m+1}$, that is, $\phi N_P(M^n) \subset T_P(M^n)$ for all $P \in M^n$ (see [4], [7] and [12]).

A submanifold $M^n$ of a Sasakian manifold $M^{2m+1}$ is said to be *anti-invariant* (*totally real*) if $\phi T_P(M^n) \subset N_P(M^n)$ for all $P \in M^n$ (see [11]).

From now on, we consider throughout this paper generic submanifolds immersed in a Sasakian manifold $M^{2m+1}$. Then we can put in each coordinate neighborhood

$$\phi_j{}^h B_c{}^j = f_c{}^a B_a{}^h - f_c{}^x C_x{}^h\,, \tag{1.6}$$

$$\phi_j{}^h C_x{}^j = f_x{}^a B_a{}^h\,, \tag{1.7}$$

$$\xi^h = \eta^a B_a{}^h + \xi^x C_x{}^h , \tag{1.8}$$

where $f_c{}^a$ is a tensor field of type (1, 1) defined on $M^n$, $f_c{}^x$ a local 1-form for each fixed index $x$, $\eta^a$ a vector field and $\xi^x$ a function for each fixed index $x$, and $f_x{}^a = f_c{}^y g^{ac} g_{yx}$.

Applying $\phi$ to (1.6) and (1.7) respectively and using (1.1) and these equations, we easily find ([4], [7])

$$\left\{\begin{array}{l} f_c{}^e f_e{}^a = -\delta_c^a + f_c{}^x f_x{}^a + \eta_c \eta^a , \\ f_c{}^e f_e{}^x = -\eta_c \xi^x , \quad f_x{}^e f_e{}^y = \delta_x^y - \xi_x \xi^y , \\ \eta^e f_e{}^a = -\xi^x f_x{}^a , \quad \eta^e f_e{}^x = 0 , \\ g_{de} f_c{}^d f_b{}^e = g_{cb} - f_c{}^x f_{xb} - \eta_c \eta_b , \quad \eta_a \eta^a + \xi_x \xi^x = 1 , \end{array}\right. \tag{1.9}$$

where $\eta_a = g_{ea}\eta^e$. But, the last relationship follows from (1.3), (1.8) and the fact that $\xi_j \xi^j = 1$.

Putting $f_{cb} = f_c{}^a g_{ab}$ and $f_{cx} = f_c{}^y g_{yx}$, then we easily verify from (1.9) that $f_{cb} = -f_{bc}$, $f_{cx} = f_{xc}$.

When the submanifold $M^n$ is a hypersurface of $M^{2m+1}$, (1.9) becomes the so-called $(f, g, u, v, \lambda)$-structure ([1], [2]), where we have put $f_c{}^x = u_c$, $\eta^a = v^a$, $\xi_x = \xi^x = \lambda$.

The aggregate $(f_c{}^a, g_{cb}, f_c{}^x, \eta^a, \xi^x)$ satisfying (1.9) is said to be *antinormal* ([4], [8]) if

$$h_c{}^{ex} f_e{}^a + f_c{}^e h_e{}^{ax} = 0 \tag{1.10}$$

holds, or equivalently

$$h_{ce}{}^x f_b{}^e = h_{be}{}^x f_c{}^e . \tag{1.11}$$

In characterizing a generic submanifold of an odd-dimensional sphere, we shall use the following theorem.

THEOREM A ([1], [8]). *Let $M^{2m}$ be a complete hypersurface with antinormal $(f, g, u, v, \lambda)$-structure of an odd-dimensional unit sphere $S^{2m+1}(1)$. If the function $\lambda$ does not vanish almost everywhere and the scalar curvature of $M^{2m}$ is a constant, then $M^{2m}$ is a great sphere $S^{2m}(1)$ or a product of two spheres $S^m(1/\sqrt{2}) \times S^m(1/\sqrt{2})$.*

Transvecting (1.11) with $f_a{}^b$ and using the first relation of (1.9), we find

$$h_{ce}{}^x(-\delta_a^e + f_a{}^z f_z{}^e + \eta_a \eta^e) = h_{be}{}^x f_c{}^e f_a{}^b ,$$

from which, taking the skew-symmetric part,

$$(h_{ce}{}^x f_z{}^e) f_b{}^z - (h_{be}{}^x f_z{}^e) f_c{}^z + (h_{ce}{}^x \eta^e)\eta_b - (h_{be}{}^x \eta^e)\eta_c = 0 . \tag{1.12}$$

Differentiating (1.6)~(1.8) covariantly along $M^n$ and using (1.1)~(1.5), we find respectively (see [4], [7])

$$\nabla_c f_b{}^a = -g_{cb}\eta^a + \delta_c^a \eta_b + h_{cb}{}^x f_x{}^a - h_c{}^a{}_x f_b{}^x, \tag{1.13}$$

$$\nabla_c f_b{}^x = g_{cb}\xi^x + h_{ce}{}^x f_b{}^e, \tag{1.14}$$

$$h_c{}^e{}_x f_e{}^y = h_c{}^{ey} f_{ex}, \tag{1.15}$$

$$\nabla_c \eta_b = f_{cb} + h_{cb}{}^x \xi_x, \tag{1.16}$$

$$\nabla_c \xi^x = -f_c{}^x - h_{ce}{}^x \eta^e \tag{1.17}$$

with the help of (1.6)~(1.8).

For an anti-invariant submanifold of an odd-dimensional unit sphere, Yano and Kon proved in Chapter 4, Theorem 6.5 of [11]

THEOREM B. *Let $M$ be an $(n+1)$-dimensional compact orientable anti-invariant submanifold with parallel mean curvature vector of $S^{2n+1}(1)$. If the normal connection of $M$ is flat, then we have $M=S^1(r_1)\times\cdots\times S^1(r_{n+1})$, $r_1^2+\cdots+r_{n+1}^2=1$.*

## § 2. Submanifolds of $S^{2m+1}(1)$

Let $M^n$ be an $n$-dimensional submanifold of an odd-dimensional unit sphere $S^{2m+1}(1)$, then the equations of Gauss, Codazzi and Ricci for $M^n$ are respectively given by

$$K_{dcb}{}^a = \delta_d^a g_{cb} - \delta_c^a g_{db} + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x, \tag{2.1}$$

$$\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x = 0, \tag{2.2}$$

$$K_{dcy}{}^x = h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y, \tag{2.3}$$

$K_{dcb}{}^a$ and $K_{dcy}{}^x$ being the curvature tensor of $M^n$ and that of the connection induced in the normal bundle respectively.

We now suppose that the connection induced in the normal bundle of $M^n$ is flat, that is, $K_{dcy}{}^x=0$. From the Ricci identity

$$\nabla_d\nabla_c h_{ba}{}^x - \nabla_c\nabla_d h_{ba}{}^x = -K_{dcb}{}^e h_{ae}{}^x - K_{dca}{}^e h_{be}{}^x,$$

we have

$$(g^{da}\nabla_d\nabla_a h_{cb}{}^x)h^{cb}{}_x - (\nabla_c\nabla_b h^x)h^{cb}{}_x = K_{ce}h_b{}^{ey}h^{cb}{}_y - K_{dcba}h^{day}h^{cb}{}_y \tag{2.4}$$

because of (2.2), where we have put $h^x = g^{cb}h_{cb}{}^x$, $K_{dcba} = K_{dcb}{}^e g_{ae}$, $K_{cb} = g^{da}K_{dcba}$.

We have from (2.1)

$$K_{cb} = (n-1)g_{cb} + h_x h_{cb}{}^x - h_c{}^e{}_x h_{be}{}^x \tag{2.5}$$

which implies

$$K=n(n-1)+h_x h^x - h_{cb}{}^x h^{cb}{}_x \,, \tag{2.6}$$

$K$ being the scalar curvature of $M^n$.

Moreover we have from (2.3)

$$h_{ce}{}^x h_b{}^e{}_y = h_{be}{}^x h_c{}^e{}_y \,. \tag{2.7}$$

Substituting (2.1) and (2.5) into (2.4) and taking account of the identity

$$\frac{1}{2}\Delta(h_{cb}{}^x h^{cb}{}_x)=(g^{da}\nabla_d\nabla_a h_{cb}{}^x)h^{cb}{}_x+\|\nabla_d h_{cb}{}^x\|^2 \,,$$

we have

$$\begin{aligned}\frac{1}{2}\Delta(h_{cb}{}^x h^{cb}{}_x)=&\, n h_{cb}{}^x h^{cb}{}_x - h_x h^x + h^x h_{cex} h_b{}^{ey} h^{cb}{}_y \\ &-(h_{cb}{}^x h^{cby})(h_{dax} h^{da}{}_y)+(\nabla_c\nabla_b h^x)h^{cb}{}_x+\|\nabla_d h_{cb}{}^x\|^2\end{aligned} \tag{2.8}$$

with the help of (2.7), where $\Delta=g^{da}\nabla_d\nabla_a$.

If the mean curvature vector of $M^n$ is parallel in the normal bundle, that is, $\nabla_c h^x=0$, then (2.8) implies

$$\begin{aligned}\frac{1}{2}\Delta(h_{cb}{}^x h^{cb}{}_x)=&\, n h_{cb}{}^x h^{cb}{}_x - h_x h^x + h_x h_{ce}{}^x h_b{}^{ey} h^{cb}{}_y \\ &-(h_{cb}{}^x h^{cby})(h_{dax} h^{da}{}_y)+\|\nabla_d h_{cb}{}^x\|^2 \,.\end{aligned} \tag{2.9}$$

For a submanifold of an $m$-dimensional sphere $S^m$, Yano and Kon [12] proved the following theorem:

THEOREM C. *Let $M$ be a complete n-dimensional submanifold of $S^m$ with flat normal connection. If the second fundamental form of $M$ is parallel, then $M$ is a small sphere, a great sphere or a pythagorean product of a certain number of spheres. Moreover, if $M$ is of essential codimension m-n, then $M$ is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)\,,\quad r_1^2+\cdots+r_N^2=1\,,\quad N=m-n+1\,,$$

*or a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_{N'}}(r_{N'})\subset S^{m-1}(r)\subset S^m\,,$$

$r_1^2+\cdots+r_{N'}^2=r^2<1$, $N'=m-n$.

## § 3. Generic submanifolds with $\xi_x\neq 0$ of $S^{2m+1}(1)$

In this section we consider a generic submanifold satisfying (1.11) of an odd-dimensional sphere $S^{2m+1}(1)$.

Transvecting (1.12) with $\eta^b$ and taking account of (1.9), we find

(3.1) $$-(h_{be}{}^{x}\eta^{b}f_{z}{}^{e})f_{c}{}^{z}+(1-\mu^{2})h_{ce}{}^{x}\eta^{e}-(h_{de}{}^{x}\eta^{d}\eta^{e})\eta_{c}=0\,,$$

where $\mu^2=\xi_x\xi^x$, from which, transvecting $f_y{}^c$ and using (1.9),

$$\mu^{2}h_{ce}{}^{x}\eta^{e}f_{y}{}^{c}=(h_{be}{}^{x}\eta^{b}f_{z}{}^{e}\xi^{z})\xi_{y}\,.$$

Thus (3.1) becomes

(3.2) $$\mu^{2}(1-\mu^{2})h_{ce}{}^{x}\eta^{e}=\mu^{2}(h_{de}{}^{x}\eta^{d}\eta^{e})\eta_{c}+(h_{be}{}^{x}\eta^{b}f_{z}{}^{e}\xi^{z})\xi_{y}f_{c}{}^{y}\,.$$

We now suppose that the function $\mu$ does not vanish almost everywhere and $n\neq m$, then so does $\mu(1-\mu^2)$. In fact, if $1-\mu^2$ vanishes identically, then we see from the last relation of (1.9) that $\eta_c=0$ and hence $f_{cb}=0$ because of (1.16). Thus, it follows that $0=f_{cb}f^{cb}=2(n-m)$ with the help of (1.9). Therefore $\mu(1-\mu^2)$ is nonzero almost everywhere.

Consequently (3.2) implies

(3.3) $$h_{ce}{}^{x}\eta^{e}=B^{x}\eta_{c}+A^{x}\xi_{z}f_{c}{}^{z}\,,$$

where we have put

$$A^{x}=(h_{de}{}^{x}\eta^{d}f_{z}{}^{e}\xi^{z})/\mu^{2}(1-\mu^{2})\,,\quad B^{x}=(h_{de}{}^{x}\eta^{d}\eta^{e})/(1-\mu^{2})\,.$$

Substituting (3.3) into (1.12), we find

$$(h_{ce}{}^{x}f_{z}{}^{e})f_{b}{}^{z}-(h_{be}{}^{x}f_{z}{}^{e})f_{c}{}^{z}+A^{x}(\xi_{z}f_{c}{}^{z}\eta_{b}-\xi_{z}f_{b}{}^{z}\eta_{c})=0\,,$$

from which, transvecting $f_y{}^b$ and making use of (1.9),

(3.4) $$h_{ce}{}^{x}f_{y}{}^{e}-(h_{ce}{}^{x}f_{z}{}^{e}\xi^{z})\xi_{y}-(h_{de}{}^{x}f_{z}{}^{e}f_{y}{}^{d})f_{c}{}^{z}-(1-\mu^{2})A^{x}\xi_{y}\eta_{c}=0\,.$$

Using (1.9), (1.11) and (3.3), we have

$$h_{ce}{}^{x}f_{z}{}^{e}\xi^{z}=-h_{ce}{}^{x}\eta^{a}f_{a}{}^{e}=-h_{ae}{}^{x}\eta^{a}f_{c}{}^{e}=-B^{x}\xi_{z}f_{c}{}^{z}+\mu^{2}A^{x}\eta_{c}\,.$$

Thus, (3.4) becomes

(3.5) $$h_{ce}{}^{x}f_{y}{}^{e}=P_{yz}{}^{x}f_{c}{}^{z}+A^{x}\xi_{y}\eta_{c}\,,$$

where we have put

$$P_{yz}{}^{x}=h_{de}{}^{x}f_{z}{}^{d}f_{y}{}^{e}-B^{x}\xi_{z}\xi_{y}\,,$$

which implies $P_{yz}{}^x=P_{zy}{}^x$.

Putting $P_{yzx}=P_{yz}{}^w g_{wx}$ and taking account of (1.15), we see from (3.5) that

(3.6) $$(P_{yzx}-P_{xzy})f_{c}{}^{z}+(A_{x}\xi_{y}-A_{y}\xi_{x})\eta_{c}=0\,,$$

where $A_x=g_{xy}A^y$. Transvection $\eta^c$ and $f_a{}^c$ gives respectively

(3.7) $$A_{x}\xi_{y}-A_{y}\xi_{x}=0\,,$$

(3.8) $$(P_{yzx}-P_{xzy})\xi^{z}=0\,,$$

because $1-\mu^2$ does not vanish almost everywhere. If we transvect (3.6) with $f_w{}^c$ and use (1.9) and (3.8), then we obtain $P_{yzx}=P_{xzy}$. Hence $P_{xyz}$ is symmetric for any index.

Transvecting (3.5) with $f_a{}^c$ and taking account of (1.9), we find

$$h_{ce}{}^x f_y{}^e f_a{}^c = -P_{yz}{}^x \xi^z \eta_a + A^x \xi_y (\xi_z f_a{}^z),$$

from which, using (1.9), (1.11) and (3.3),

$$P_{yz}{}^x \xi^z + B^x \xi_y = 0, \tag{3.9}$$

which implies

$$B_x \xi_y - B_y \xi_x = 0, \tag{3.10}$$

because $P_{xyz}$ is symmetric for all indices.

$\mu$ being nonzero almost everywhere, (3.7) and (3.10) give respectively

$$A^x = \beta \xi^x, \quad B^x = \alpha \xi^x, \tag{3.11}$$

where $\beta = A^x \xi_x / \mu^2$, $\alpha = B^x \xi_x / \mu^2$.

Thus (3.3), (3.5) and (3.9) reduce respectively to

$$h_{ce}{}^x \eta^e = \xi^x (\alpha \eta_c + \beta \xi_z f_c{}^z), \tag{3.12}$$

$$h_{ce}{}^x f_y{}^e = P_{yz}{}^x f_c{}^z + \beta \xi^x \xi_y \eta_c, \tag{3.13}$$

$$P_{yz}{}^x \xi^z = -\alpha \xi^x \xi_y. \tag{3.14}$$

Transvection (1.11) with $f^{cb}$ yields

$$\begin{aligned} 0 = h_{ce}{}^x(-g^{ce} + f^{cz} f_z{}^e + \eta^c \eta^e) &= -h^x + P_{yz}{}^x (g^{yz} - \xi^y \xi^z) + \alpha \xi^x (1-\mu^2) \\ &= -h^x + P^x + \alpha \xi^x \end{aligned}$$

with the help of (1.9), (3.12), (3.13) and (3.14), where $P^x = g^{yz} P_{yz}{}^x$. Hence, it follows that

$$h^x = P^x + \alpha \xi^x. \tag{3.15}$$

Transvecting (2.7) with $f_z{}^b$ and using (3.13), we get

$$h_{ce}{}^x (P_{wyz} f^{we} + \beta \xi_y \xi_z \eta^e) = h_c{}^e{}_y (P_{wz}{}^x f_e{}^w + \beta \xi^x \xi_z \eta_e),$$

from which, using (3.12)$\sim$(3.14),

$$P_{wyz} P_v{}^{wx} f_c{}^v = P_{wz}{}^x P_{vy}{}^w f_c{}^v. \tag{3.16}$$

If we transvect (3.16) with $f_a{}^c$ and $f_u{}^c$ and take account of (1.9), we get respectively

$$P_{wyz} P_v{}^{wx} \xi^v \eta_a = P_{wz}{}^x P_{vy}{}^w \xi^v \eta_a, \quad P_{wyz} P_v{}^{wx} (\delta_u^v - \xi_u \xi^v) = P_{wz}{}^x P_{vy}{}^w (\delta_u^v - \xi_u \xi^v).$$

The last two relationships give

$$P_{wyz}P_{vx}{}^{w}=P_{wzx}P_{vy}{}^{w} \tag{3.17}$$

because $1-\mu^2$ does not vanish almost everywhere, which implies

$$P_{xyz}P^{xyz}=P_xP^x\,, \tag{3.18}$$

where $P_x=P^z g_{zx}$.

LEMMA 3.1. *Let $M^n$ $(n\neq m)$ be an n-dimensional generic submanifold with flat normal connection of $S^{2m+1}(1)$. If the induced structure $(f_c{}^a,\ g_{cb},\ f_x{}^c,\ \eta^a,\ \xi^x)$ on $M^n$ is antinormal and the function $\xi_x\xi^x$ is nonzero almost everywhere. Then we have $\alpha(n-m-1)=0$.*

*Proof.* From (3.12) we have

$$h_{ce}{}^{x}\eta^e\xi_x=\alpha\mu^2\eta_c+\beta\mu^2(\xi_x f_c{}^x)\,.$$

Differentiating this covariantly and substituting (1.14), (1.16) and (1.17), we obtain

$$\begin{aligned}(\nabla_d h_{ce}{}^{x})\eta^e\xi_x&+h_c{}^{ex}\xi_x(f_{de}+h_{de}{}^{y}\xi_y)-h_{ce}{}^{x}\eta^e(f_{dx}+h_{dax}\eta^a)\\ =(\nabla_d(\alpha\mu^2))\eta_c&+(\nabla_d(\beta\mu^2))\xi_x f_c{}^x+\alpha\mu^2(f_{dc}+h_{dc}{}^{x}\xi_x)-\beta\mu^2 f_c{}^x(f_{dx}+h_{dex}\eta^e)\\ &+\beta\mu^2\xi_x(g_{dc}\xi^x+h_{de}{}^{x}f_c{}^e)\,,\end{aligned}$$

from which, taking the skew-symmetric part and using (1.11), (2.2) and (2.7),

$$\begin{aligned}(\nabla_d(\alpha\mu^2))\eta_c-(\nabla_c(\alpha\mu^2))\eta_d+(\nabla_d(\beta\mu^2))\xi_x f_c{}^x-(\nabla_c(\beta\mu^2))\xi_x f_d{}^x+2\alpha\mu^2 f_{dc}&\\ +\alpha(\beta\mu^2+1)(\xi_x f_d{}^x\eta_c-\xi_x f_c{}^x\eta_d)=0&\end{aligned} \tag{3.19}$$

with the help of (3.12).

If we transvect (3.19) with $\eta^c$ and take account of (1.9), then we get

$$(1-\mu^2)\nabla_d(\alpha\mu^2)=\eta^e(\nabla_e(\alpha\mu^2))\eta_d+\{\eta^e\nabla_e(\beta\mu^2)-2\alpha\mu^2-\alpha(\beta\mu^2+1)(1-\mu^2)\}\xi_x f_d{}^x\,. \tag{3.20}$$

Next, transvecting (3.20) with $\xi^z f_z{}^d$ and using (1.9), we get

$$\xi^z f_z{}^e\nabla_e(\alpha\mu^2)=\{\eta^e\nabla_e(\beta\mu^2)-2\alpha\mu^2-\alpha(\beta\mu^2+1)(1-\mu^2)\}\mu^2 \tag{3.21}$$

because $1-\mu^2$ does not vanish almost everywhere.

In the next step, transvect (3.19) with $f_z{}^c$ and use (1.9). Then we have

$$\begin{aligned}(1-\mu^2)\xi_z\{\nabla_d(\beta\mu^2)\}=f_z{}^e\{\nabla_e(\alpha\mu^2)\}\eta_d+f_z{}^e\{\nabla_e(\beta\mu^2)\}\xi_x f_d{}^x+2\alpha\mu^2\xi_z\eta_d&\\ +\alpha(\beta\mu^2+1)(1-\mu^2)\xi_z\eta_d\,.&\end{aligned}$$

If we transvect this with $\xi^z$, then we have

$$\mu^2(1-\mu^2)\{\nabla_d(\beta\mu^2)\}=\xi^z f_z{}^e\{\nabla_e(\alpha\mu^2)\}\eta_d+\xi^z f_z{}^e\{\nabla_e(\beta\mu^2)\}\xi_x f_d{}^x+2\alpha\mu^4\eta_d$$
$$+\alpha(\beta\mu^2+1)(1-\mu^2)\mu^2\eta_d\,.$$

Substituting (3.21) into this equation gives

$$\mu^2(1-\mu^2)\nabla_d(\beta\mu^2)=\mu^2\{\eta^e\nabla_e(\beta\mu^2)\}\eta_d+\{\xi^z f_z{}^e\nabla_e(\beta\mu^2)\}\xi_x f_d{}^x\,. \tag{3.22}$$

Substituting (3.20) and (3.22) into (3.19), we get

$$\alpha\{(1-\mu^2)f_{dc}-(\xi_x f_d{}^x\eta_c-\xi_x f_c{}^x\eta_d)\}=0\,,$$

because $\mu(1-\mu^2)$ does not vanish almost everywhere, from which, transvecting $f^{dc}$ and making use of (1.9), $2\alpha(1-\mu^2)(n-m-1)=0$, that is, $\alpha(n-m-1)=0$. This completes the proof of the lemma.

## §4. $(m+1)$-dimensional generic submanifolds with $\xi_x\neq 0$ of $S^{2m+1}(1)$

In this section we consider an $(m+1)$-dimensional generic submanifold of an odd-dimensional unit sphere $S^{2m+1}(1)$.

First of all, we prove

LEMMA 4.1 *Let $M^{m+1}$ be an $(m+1)$-dimensional generic submanifold with flat normal connection of $S^{2m+1}(1)$. If the induced structure on $M^{m+1}$ is antinormal and the function $\mu$ is nonzero almost everywhere. Then we have*

$$h_{cb}{}^x h^{cb}{}_y=P^z P_{yz}{}^x+(\alpha^2+2\beta^2\mu^2)\xi^x\xi_y\,. \tag{4.1}$$

*Proof.* We now compute

$$\|(1-\mu^2)f_{dc}-\eta_c\xi_x f_d{}^x+\eta_d\xi_x f_c{}^x\|^2=(1-\mu^2)^2 f_{dc}f^{dc}-2(1-\mu^2)(\xi_z f^{cz})(\xi_x f_c{}^x)$$
$$=(1-\mu^2)^2(f_{dc}f^{dc}-2\mu^2)=0$$

with the help of (1.9). Hence we have

$$(1-\mu^2)f_{dc}=\xi_x f_d{}^x\eta_c-\xi_x f_c{}^x\eta_d\,. \tag{4.2}$$

Using this, we have

$$(1-\mu^2)h_{ce}{}^x h_b{}^e{}_y f_d{}^b=h_{ce}{}^x h_b{}^e{}_y(\xi_z f_d{}^z\eta^b-\xi_z f^{bz}\eta_d)$$
$$=h_{ce}{}^x\xi_y(\alpha\eta^e+\beta\xi_w f^{ew})\xi_z f_d{}^z-h_c{}^{ex}\xi_z(P_{wy}{}^z f_e{}^w+\beta\xi^z\xi_y\eta_e)\eta_d$$

because of (3.12) and (3.13), from which, taking account of (3.12)~(3.14),

$$(1-\mu^2)h_{ce}{}^x h_b{}^e{}_y f_d{}^b=(\alpha^2+\beta^2\mu^2)\xi^x\xi_y(\xi_z f_d{}^z\eta_c-\xi_z f_c{}^z\eta_d)\,.$$

Thus, it follows that

$$h_{ce}{}^{x}h_{b}{}^{e}{}_{y}f_{d}{}^{b}=(\alpha^{2}+\beta^{2}\mu^{2})\xi^{x}\xi_{y}f_{dc} \tag{4.3}$$

because of (4.2) and the fact that $1-\mu^{2}$ does not vanish almost everywhere, which derived from $n=m+1$.

Transvecting (4.3) with $f^{dc}$ and making use of (1.9), we find

$$h_{ce}{}^{x}h_{b}{}^{e}{}_{y}(g^{cb}-f^{cz}f_{z}{}^{b}-\eta^{c}\eta^{b})=2\mu^{2}(\alpha^{2}+\beta^{2}\mu^{2})\xi^{x}\xi_{y}$$

because of $n=m+1$, from which, using (3.12)~(3.14),

$$\begin{aligned}h_{cb}{}^{x}h^{cb}{}_{y}-(P_{w}{}^{xz}f_{e}{}^{w}+\beta\xi^{x}\xi^{z}\eta_{e})(P_{yzv}f^{ev}+\beta\xi_{y}\xi_{z}\eta^{e})-\xi^{x}\xi_{y}(\alpha\eta_{e}+\beta\xi_{z}f_{e}{}^{z})(\alpha\eta^{e}+\beta\xi_{w}f^{ew})\\ =2\mu^{2}(\alpha^{2}+\beta^{2}\mu^{2})\xi^{x}\xi_{y}\,,\end{aligned}$$

or, taking account of (1.9), (3.14) and (3.17),

$$\begin{aligned}h_{cb}{}^{x}h^{cb}{}_{y}-P^{z}P_{yz}{}^{x}+\alpha^{2}\mu^{2}\xi^{x}\xi_{y}-\beta^{2}\mu^{2}(1-\mu^{2})\xi^{x}\xi_{y}-\alpha^{2}(1-\mu^{2})\xi^{x}\xi_{y}\\ -\xi^{x}\xi_{y}\beta^{2}\xi_{z}\xi_{w}(g^{zw}-\xi^{z}\xi^{w})\\ =2\mu^{2}(\alpha^{2}+\beta^{2}\mu^{2})\xi^{x}\xi_{y}\,.\end{aligned}$$

Hence, (4.1) is valid.

LEMMA 4.2. *Under the same the assumptions as those stated in Lemma 4.1, we have* $\alpha=\beta=0$ *if* $m>1$.

*Proof.* Applying the operator $\nabla^{c}$ to (1.11) and substituting (1.13), we find

$$\begin{aligned}(\nabla_{e}h^{x})f_{b}{}^{e}=-h_{e}{}^{cx}(-g_{cb}\eta^{e}+\delta_{c}^{e}\eta_{b}+h_{cb}{}^{z}f_{z}{}^{e}-h_{c}{}^{e}{}_{z}f_{b}{}^{z})\\ +h_{be}{}^{x}\{-(m+1)\eta^{e}+\eta^{e}+h^{z}f_{z}{}^{e}-h_{c}{}^{e}{}_{z}f^{cz}\}\end{aligned}$$

with the help of (2.2), from which, using (3.12), (3.13) and (4.1),

$$\begin{aligned}(\nabla_{e}h^{x})f_{b}{}^{e}=-(m-1)\xi^{x}(\alpha\eta_{b}+\beta\xi_{z}f_{b}{}^{z})-h^{x}\eta_{b}+h^{z}(P_{yz}{}^{x}f_{b}{}^{y}+\beta\xi^{x}\xi_{z}\eta_{b})\\ -2P_{yz}{}^{x}(P_{w}{}^{zy}f_{b}{}^{w}+\beta\xi^{z}\xi^{y}\eta_{b})-2\beta\mu^{2}\xi^{x}(\alpha\eta_{b}+\beta\xi_{z}f_{b}{}^{z})\\ +P^{z}P_{yz}{}^{x}f_{b}{}^{y}+(\alpha^{2}+2\beta^{2}\mu^{2})\xi^{x}\xi_{z}f_{b}{}^{z}\,,\end{aligned}$$

or, taking account of (3.14), (3.15) and (3.17),

$$(\nabla_{e}h^{x})f_{b}{}^{e}=-(m-1)\xi^{x}(\alpha\eta_{b}+\beta\xi_{z}f_{b}{}^{z})-h^{x}\eta_{b}+\beta(h^{z}\xi_{z})\xi^{x}\eta_{b}\,. \tag{4.4}$$

On the other hand, we have from (3.14) and (3.15)

$$h_{x}\xi^{x}=0\,. \tag{4.5}$$

If we differentiate (4.5) covariantly and substitute (1.17), we find

$$(\nabla_{d}h^{x})\xi_{x}-h^{x}(f_{dx}-h_{dex}\eta^{e})=0\,,$$

or, use (3.12) and (4.5), $\xi_x(\nabla_d h^x)=h^x f_{dx}$. Therefore, we have

$$\xi_x(\nabla_e h^x)f_b{}^e=h^x f_{ex}f_b{}^e=-h^x\xi_x\eta_b=0 \tag{4.6}$$

with the help of (1.9) and (4.5).

Transvecting (4.4) with $\xi_x$ and making use of (4.5) and (4.6), we get

$$(m-1)\mu^2(\alpha\eta_b+\beta\xi_z f_b{}^z)=0\,.$$

Thus, it follows that $\alpha=\beta=0$ because $\mu(1-\mu^2)$ does not vanish almost everywhere. Hence, Lemma 4.2 is proved.

Using Lemma 4.1 and Lemma 4.2, we now prove

THEOREM 4.3. *Let $M^{m+1}$ $(m>1)$ be an $(m+1)$-dimensional complete generic submanifold with flat normal connection of an odd-dimensional unit sphere $S^{2m+1}(1)$. If the mean curvature vector is parallel in the normal bundle, the induced structure on $M^{m+1}$ is antinormal and the function $\xi_x\xi^x$ does not vanish almost everywhere, then $M^{m+1}$ is a great sphere $S^{m+1}(1)$.*

*Proof.* From Lemma 4.1 and 4.2, we get

$$h_{cb}{}^x h^{cb}{}_x=h_x h^x \tag{4.7}$$

with the help of (3.15) with $\alpha=0$.

Since we see from (1.9), (3.13), (3.14), (3.15), (3.18) and Lemma 4.2 that

$$\|h_{cb}{}^x-P_{yz}{}^x f_c{}^y f_b{}^z\|^2=h_{cb}{}^x h^{cb}{}_x-P_{xyz}P^{xyz}=h_{cb}{}^x h^{cb}{}_x-h_x h^x\,,$$

the following relationship is valid:

$$h_{cb}{}^x=P_{yz}{}^x f_c{}^y f_b{}^z\,. \tag{4.8}$$

On the other hand, the mean curvature vector being parallel, (2.9) becomes

$$mh_x h^x+h^x h_{cex}h_b{}^{ey}h^{cb}{}_y-(h_{cb}{}^x h^{cby})(h_{dax}h^{da}{}_y)+\|\nabla_d h_{cb}{}^x\|^2=0$$

because of (4.7). Substituting (4.7) and (4.8) into this and taking account of (1.9), (3.13), (3.14), (3.18) and Lemma 4.2, we find

$$mh_x h^x+h^x P_{xyz}P^y P^z-P_{xyz}P^x P^y P^z+\|\nabla_d h_{cb}{}^x\|^2=0\,,$$

from which, using (3.15) with $\alpha=0$, $h^x=0$ and $\nabla_d h_{cb}{}^x=0$ and hence $h_{cb}{}^x=0$ by virtue of (4.7). Thus, by completeness, $M^{m+1}$ is a great sphere $S^{m+1}(1)$. This completes the proof of the theorem.

## § 5. Complete generic submanifolds with $\xi_x\neq 0$ of $S^{2m+1}(1)$

In this section, we consider that $M^n$ $(n\neq m)$ is an $n$-dimensional generic submanifold with flat normal connection of an odd-dimensional sphere $S^{2m+1}(1)$.

Moreover, we suppose that the induced structure on $M^n$ is antinormal and the function $\xi_x\xi^x$ does not vanish almost everywhere. Then we see from Lemma 3.1 and Lemma 4.2 that $\alpha=0$ on $M^n$. Thus, (3.12)~(3.15) reduce respectively to

$$h_{ce}{}^x\eta^e=\beta\xi^x\xi_z f_c{}^z, \tag{5.1}$$

$$h_{ce}{}^x f_y{}^e=P_{yz}{}^x f_c{}^z+\beta\xi^x\xi_y\eta_c, \tag{5.2}$$

$$P_{yz}{}^x\xi^z=0, \tag{5.3}$$

$$h^x=P^x. \tag{5.4}$$

From (5.2) and (5.4), we have

$$h_{ce}{}^x f_x{}^e=h_x f_c{}^x+\beta\mu^2\eta_c. \tag{5.5}$$

We first prove

LEMMA 5.1. *Let $M^n$ $(n\neq m,\ m>1)$ be an n-dimensional generic submanifold with flat normal connection of $S^{2m+1}(1)$. Suppose that the mean curvature vector is parallel, the induced structure on $M^n$ is antinormal and the function $\mu$ does not vanish almost everywhere. If the scalar curvature of $M^n$ is a constant, then we have $\beta=0$ or $\beta\mu^2=1$.*

*Proof.* Differentiating (5.5) covariantly and substituting (1.14) and (1.16), we find

$$\begin{aligned}(\nabla_d h_{ce}{}^x)f_x{}^e+h_c{}^{ex}(g_{de}\xi_x+h_{dax}f_e{}^a)\\ =h_x(g_{dc}\xi^x+h_{de}{}^x f_c{}^e)+\beta\mu^2(f_{dc}+h_{dcx}\xi^x)+(\nabla_d(\beta\mu^2))\eta_c,\end{aligned} \tag{5.6}$$

because the mean curvature vector is parallel, from which, taking the skew-symmetric part and using (1.11), (2.2) and (2.7),

$$2h_c{}^{ex}h_{dax}f_e{}^a=2\beta\mu^2 f_{dc}+(\nabla_d(\beta\mu^2))\eta_c-(\nabla_c(\beta\mu^2))\eta_d. \tag{5.7}$$

If we transvect (5.7) with $\eta^c$ and take account of (1.9) and (5.1), then we obtain

$$(1-\mu^2)\nabla_d(\beta\mu^2)=\{\eta^e\nabla_e(\beta\mu^2)\}\eta_d+2\beta\mu^2(\beta\mu^2-1)\xi_x f_d{}^x. \tag{5.8}$$

Substituting this into (5.7), we get

$$(1-\mu^2)h_c{}^{ex}h_{eax}f_d{}^a=\beta\mu^2(1-\mu^2)f_{dc}+\beta\mu^2(\beta\mu^2-1)(\xi_x f_d{}^x\eta_c-\xi_x f_c{}^x\eta_d) \tag{5.9}$$

because of (1.11).

On the other hand, we have

$$\begin{aligned}&h_c{}^{ex}h_{eax}f_d{}^a f^{dc}\\ &=h_c{}^{ex}h_{eax}(g^{ac}-f^{cz}f_z{}^a-\eta^c\eta^a)\end{aligned} \tag{5.10}$$

$$=h_{cb}{}^{x}h^{cb}{}_{x}-(P_{y}{}^{zx}f^{ey}+\beta\xi^{x}\xi^{z}\eta^{e})(P_{wzx}f_{e}{}^{w}+\beta\xi_{x}\xi_{z}\eta_{e})-(\beta\xi_{x}\xi_{z}f^{ez})(\beta\xi^{x}\xi_{y}f_{e}{}^{y})$$
$$=h_{cb}{}^{x}h^{cb}{}_{x}-P_{y}{}^{zx}P_{wzx}(g^{yw}-\xi^{y}\xi^{w})-\beta^{2}\mu^{4}(1-\mu^{2})-\beta^{2}\mu^{2}\xi_{z}\xi_{y}(g^{yz}-\xi^{y}\xi^{z})$$
$$=h_{cb}{}^{x}h^{cb}{}_{x}-h_{x}h^{x}-2\beta^{2}\mu^{4}(1-\mu^{2})$$

with the help of (1.9), (3.17) and (5.1)~(5.4).

Transvecting (5.9) with $f^{dc}$ and using (1.9) and (5.10), we find

$$(1-\mu^{2})\{h_{cb}{}^{x}h^{cb}{}_{x}-h_{x}h^{x}-2\beta^{2}\mu^{4}(1-\mu^{2})\}$$
$$=\beta\mu^{2}(1-\mu^{2})(2n-2m-2+2\mu^{2})+2\beta\mu^{4}(\beta\mu^{2}-1)(1-\mu^{2}),$$

from which,

$$h_{cb}{}^{x}h^{cb}{}_{x}=h_{x}h^{x}+2\beta\mu^{2}(n-m-1+\beta\mu^{2}) \tag{5.11}$$

because $1-\mu^2$ does not vanish almost everywhere. Thus, we see from (2.6) that the scalar curvature $K$ of $M^n$ is given by $K=n(n-1)-2\beta\mu^2(n-m-1+\beta\mu^2)$. Since $K$ is a constant, by differentiating we find $(n-m-1+2\beta\mu^2)\nabla_c(\beta\mu^2)=0$, which implies that $\beta=0$ or $\nabla_c(\beta\mu^2)=0$ because of $n-m-1+2\beta\mu^2\geqq 0$. Therefore, we see from (5.8) that $\beta\mu^2=1$ in the case of $\beta\neq 0$, that is, $\nabla_c(\beta\mu^2)=0$. This completes the proof of the lemma.

THEOREM 5.2. *Let $M^n$ ($n\neq m$, $m>1$) be an n-dimensional complete generic submanifold with flat normal connection of odd-dimensional unit sphere $S^{2m+1}(1)$. Suppose that the mean curvature vector is parallel in the normal bundle, the induced structure on $M^n$ is antinormal and the function $\xi_x\xi^x$ does not vanish almost everywhere. If the scalar curvature of $M^n$ is a constant, then $M^n$ is a great sphere $S^n(1)$ or a product of two spheres $S^m(1/\sqrt{2})\times S^m(1/\sqrt{2})$.*

*Proof*. By Lemma 5.1, we consider two cases that $\beta=0$ and $\beta\mu^2=1$. In the first case, we have from (5.11)

$$h_{cb}{}^{x}h^{cb}{}_{x}=h_{x}h^{x}. \tag{5.12}$$

Hence, as in the proof of Theorem 4.3, we see that $M^n$ is a great sphere $S^n(1)$.

In the next place, we consider the case in which $\beta\mu^2=1$. Differentiation covariantly yields

$$(\nabla_c\beta)\mu^{2}+2\beta\xi_{x}\nabla_{c}\xi^{x}=0, \tag{5.13}$$

from which, taking account of (1.17) and (5.1),

$$\mu^{2}(\nabla_{c}\beta)-2\beta(\xi_{x}f_{c}{}^{x}+\beta\mu^{2}\xi_{x}f_{c}{}^{x})=0$$

and consequently

$$\nabla_{c}\beta=4\beta^{2}\xi_{x}f_{c}{}^{x} \tag{5.14}$$

because of $\beta\mu^2=1$.

Differentiating (5.2) covariantly and substituting (1.14), (1.16), (1.17) and (5.14), we find

$$\begin{aligned}&(\nabla_d h_{cex})f_y{}^e+h_c{}^e{}_x(g_{de}\xi_y+h_{day}f_e{}^a)\\&=(\nabla_d P_{yzx})f_c{}^z+P_{yzx}(g_{dc}\xi^z+h_{de}{}^z f_c{}^e)+4\beta^2(\xi_z f_d{}^z)\eta_c\xi_x\xi_y\\&\quad-\beta(f_{dx}+\beta\xi_x\xi_z f_d{}^z)\xi_y\eta_c-\beta\xi_x\eta_c(f_{dy}+\beta\xi_y\xi_z f_d{}^z)+\beta\xi_x\xi_y(f_{dc}+h_{dcz}\xi^z)\end{aligned}$$

with the help of (5.1), from which, taking the skew-symmetric part and making use of (1.11) and (2.2),

$$\begin{aligned}(5.15)\qquad 2h_c{}^e{}_x h_{day}f_e{}^a=&(\nabla_d P_{yzx})f_c{}^z-(\nabla_c P_{yzx})f_d{}^z+2\beta^2\xi_x\xi_y(\xi_z f_d{}^z\eta_c-\xi_z f_c{}^z\eta_d)\\&+2\beta\xi_x\xi_y f_{dc}-\beta\{(\xi_x f_{dy}+\xi_y f_{dx})\eta_c-(\xi_x f_{cy}+\xi_y f_{cx})\eta_d\}.\end{aligned}$$

Transvecting (5.15) with $f_w{}^c\eta^d$ and taking account of (1.9) and (5.1)~(5.3), we get

$$\begin{aligned}&-2\beta^2\mu^2(1-\mu^2)\xi_x\xi_y\xi_w\\&=\eta^e\nabla_e P_{ywx}-(\eta^e\xi^z\nabla_e P_{zyx})\xi_w-2\beta^2(1-\mu^2)^2\xi_x\xi_y\xi_w\\&\quad-2\beta(1-\mu^2)\xi_x\xi_y\xi_w+\beta(1-\mu^2)\{\xi_x(g_{yw}-\xi_y\xi_w)+\xi_y(g_{wx}-\xi_w\xi_x)\}\end{aligned}$$

from which, using (5.3) and the fact that $\beta\mu^2=1$,

$$\eta^e\nabla_e P_{ywx}+P_{zyx}\eta^e(\nabla_e\xi^z)\xi_w-2\beta(\beta-1)\xi_x\xi_y\xi_w+(\beta-1)(\xi_x g_{yw}+\xi_y g_{xw})=0$$

or, taking account of (1.17), (5.1) and (5.3),

$$\eta^e\nabla_e P_{ywx}-2\beta(\beta-1)\xi_x\xi_y\xi_w+(\beta-1)(\xi_x g_{yw}+\xi_y g_{xw})=0\,.$$

If we transvect this with $g^{xw}$ and use (5.4), we obtain

$$\eta^e\nabla_e h^y-2(\beta-1)\xi^y+(\beta-1)(2m-n+2)\xi^y=0\,.$$

Since the mean curvature vector is parallel in the normal bundle, it follows that $n=2m$ because of $\beta\mu^2=1$. Hence $M^n$ is a hypersurface of $S^{2m+1}(1)$. According to Theorem A in §1, $M^n$ is a product of two spheres $S^m(1/\sqrt{2})\times S^m(1/\sqrt{2})$. Therefore Theorem 5.2 is proved.

## §6. Generic submanifolds with $\xi_x=0$ of $S^{2m+1}(1)$

In this section we suppose that a generic submanifold with $\xi^x=0$ and flat normal connection of $S^{2m+1}(1)$ satisfies (1.11). Then (1.9) reduces to

$$(6.1)\qquad\left\{\begin{aligned}&f_c{}^e f_e{}^a=-\delta_c^a+f_c{}^x f_x{}^a+\eta_c\eta^a\,,\\&f_c{}^e f_e{}^x=0\,,\quad \eta^e f_e{}^a=0\,,\quad \eta_e f^{ex}=0\,,\quad f_x{}^e f_e{}^y=\delta_x^y\,,\\&g_{ed}f_c{}^e f_b{}^d=g_{cb}-f_c{}^x f_{xb}-\eta_c\eta_b\,,\quad \eta_e\eta^e=1\end{aligned}\right.$$

and (1.14)~(1.17) to

$$\nabla_c f_b{}^x = h_{ce}{}^x f_b{}^e, \tag{6.2}$$

$$h_c{}^e{}_x f_e{}^y = h_c{}^{ey} f_{ex}, \tag{6.3}$$

$$\nabla_c \eta_b = f_{cb}, \tag{6.4}$$

$$h_{ce}{}^x \eta^e = -f_c{}^x. \tag{6.5}$$

Transvecting (1.11) with $f_y{}^b f_d{}^c$ and taking account of (6.1), we find

$$-h_{bd}{}^x f_y{}^b + (h_{be}{}^x \eta^e) f_y{}^b \eta_d + (h_{be}{}^x f_y{}^b f_z{}^e) f_d{}^z = 0,$$

from which, using (6.5),

$$h_{ce}{}^x f_y{}^e = P_{yz}{}^x f_c{}^z - \delta_y^x \eta_c, \tag{6.6}$$

where we have put $P_{yz}{}^x = h_{cb}{}^x f_y{}^c f_z{}^b$.

We put $P_{yzx} = P_{yz}{}^w g_{wx}$, then as in the proof of § 1, we see from (6.3) that $P_{yzx}$ is symmetric for all indices.

If we transvect (1.11) with $f^{cb}$ and make use of (6.1), then we get

$$h^x = h_{ce}{}^x f^{cz} f_z{}^e + h_{ce}{}^x \eta^c \eta^e,$$

or, use (6.5) and (6.6),

$$h^x = P^x \tag{6.7}$$

where we have put $P^x = g^{yz} P_{yz}{}^x$.

Since the normal connection of the submanifold is flat, by transvecting (2.7) with $f_z{}^b$ and taking account of (6.5) and (6.6), we get

$$P_{yz}{}^w (P_{wv}{}^x f_c{}^v - \delta_w^x \eta_c) + g_{yz} f_c{}^x = P_{wz}{}^x (P_{vy}{}^w f_c{}^v - \delta_y^w \eta_c) + \delta_z^x f_{cy},$$

from which, transvecting $f_u{}^c$ and using (6.1),

$$P_{yz}{}^w P_{wu}{}^x + g_{yz} \delta_u^x = P_{wz}{}^x P_{uy}{}^w + \delta_z^x g_{yu}. \tag{6.8}$$

Contraction with respect to $z$ and $x$ yields

$$P_{yzx} P_u{}^{xz} = P_x P_{yu}{}^x + (p-1) g_{yu}, \tag{6.9}$$

where $p = 2m+1-n$, and consequently

$$P_{xyz} P^{xyz} = h_x h^x + p(p-1) \tag{6.10}$$

with the help of (6.7).

Differentiating (6.6) covariantly and substituting (6.2) and (6.4), we find

$$(\nabla_d h_{ce}{}^x) f_y{}^e + h_c{}^{ex} h_{day} f_e{}^a = (\nabla_d P_{yz}{}^x) f_c{}^z + P_{yz}{}^x h_{de}{}^z f_c{}^e - \delta_y^x f_{dc},$$

from which, taking the skew-symmetric part with respect to $d$ and $c$, and using (1.11) and (2.2),

(6.11) $$2h_c{}^{ex}h_{eay}f_d{}^a=(\nabla_d P_{yz}{}^x)f_c{}^z-(\nabla_c P_{yz}{}^x)f_d{}^z-2\delta_y^x f_{dc}\,.$$

If we transvect (6.11) with $f_w{}^d$ and use (6.1), then we obtain

$$\nabla_c P_{yz}{}^x=(f_z{}^e\nabla_e P_{yw}{}^x)f_c{}^w\,.$$

Using $P_{yz}{}^x=P_{zy}{}^x$ and substituting this into (6.11), we have

$$h_{cex}h_a{}^e{}_y f_d{}^a=g_{yx}f_{cd}\,.$$

Transvection $f_b{}^d$ gives

$$h_{cex}h_a{}^e{}_y(-\delta_b^a+f_b{}^z f_z{}^a+\eta_b\eta^a)=g_{yx}(g_{cb}-f_c{}^z f_{zb}-\eta_c\eta_b)\,,$$

from which, using (6.5) and (6.6),

(6.12) $$h_{cex}h_b{}^e{}_y=P_{yz}{}^w P_{wvx}f_c{}^v f_b{}^z-P_{yzx}(f_b{}^z\eta_c+f_c{}^z\eta_b)$$
$$+2g_{yx}\eta_c\eta_b+f_{cx}f_{by}-g_{yx}(g_{cb}-f_c{}^z f_{zb})\,.$$

Transvecting (6.12) with $g^{cb}$ and taking account of (6.1) and (6.9), we get

$$h_{cbx}h^{cb}{}_y=P^z P_{zyx}+(2p+2-n)g_{yx}\,,$$

from which,

(6.13) $$h_{cb}{}^x h^{cb}{}_x=h_x h^x+p(2p+2-n)$$

and

(6.14) $$(h_{cb}{}^x h^{cby})(h_{dax}h^{da}{}_y)=P_{yzx}h^y h^z h^x+(p-1)h_x h^x$$
$$+2(2p+2-n)h_x h^x+p(2p+2-n)^2$$

with the help of (6.7) and (6.9).

Since we have from (6.9) and (6.12)

$$h_{ce}{}^x h_b{}^e{}_x=P^x P_{xyz}f_c{}^y f_b{}^z+2p(f_b{}^x f_{cx}+\eta_c\eta_b)-P^x(f_{bx}\eta_c+f_{xc}\eta_b)-pg_{cb}\,,$$

it follows that

(6.15) $$h^x h_{bax}h_c{}^a{}_y h^{cby}=P_{yzx}h^y h^z h^x+(2p+1)h_x h^x$$

with the help of (6.6)∼(6.9).

Substituting (6.13) and (6.14) into (2.8), we find

$$\frac{1}{2}\Delta(h_{cb}{}^x h^{cb}{}_x)=(n-p-1)\{3h_x h^x+2p(2p+2-n)\}+(\nabla_c\nabla_b h^x)h^{cb}{}_x+\|\nabla_d h_{cb}{}^x\|^2\,,$$

from which, using (6.13) and the fact that $p=2m+1-n$, (6.16) $$\frac{1}{2}\Delta(h_{cb}{}^{x}h^{cb}{}_{x})=2(n-m-1)\{2h_{cb}{}^{x}h^{cb}{}_{x}+h_{x}h^{x}\}+(\nabla_{c}\nabla_{b}h^{x})h^{cb}{}_{x}+\|\nabla_{d}h_{cb}{}^{x}\|^{2}.$$

Now, assuming the mean curvature vector is parallel in the normal bundle, that is, $\nabla_{c}h^{x}=0$, then we know that $h_{cb}{}^{x}h^{cb}{}_{x}$ is a constant because of (6.13). Thus, (6.16) implies

(6.17) $$(n-m-1)\{2h_{cb}{}^{x}h^{cb}{}_{x}+h_{x}h^{x}\}=0$$

and $\nabla_{d}h_{cb}{}^{x}=0$. Since we see from (6.1) that

(6.18) $$f_{cb}f^{cb}=2(n-m-1)\geqq 0.$$

If $2h_{cb}{}^{x}h^{cb}{}_{x}+h_{x}h^{x}=0$ and hence $h_{cb}{}^{x}=0$, then (6.5) means $P_{yz}{}^{x}f_{c}{}^{z}-\delta_{y}^{x}\eta_{c}=0$. Transvection $\eta^{c}$ gives $p=2m+1-n=0$. It contradicts the fact that the codimension $p\geqq 1$. Thus, (6.17) implies $n=m+1$. From (6.18) and the fact that the submanifold is $(m+1)$-dimensional, we have $f_{cb}=0$. Therefore, we see from (1.6) that the submanifold is anti-invariant. Moreover, if $M^{n}$ is compact orientable, according to Theorem B in § 1, then we have

THEOREM 6.1. *Let $M^{n}$ be an n-dimensional compact orientable generic submanifold with flat normal connection of an odd-dimensional unit sphere $S^{2m+1}(1)$. Suppose that the mean curvature vector is parallel in the normal bundle and the induced structure on $M^{n}$ is antinormal. If the Sasakian structure vector $\xi$ defined on $S^{2m+1}(1)$ is tangent to the submanifold, then $M^{n}$ is*

$$S^{1}(r_{1})\times\cdots\times S^{1}(r_{m+1}),\quad r_{1}^{2}+\cdots+r_{m+1}^{2}=1.$$

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