REMARKS ON THE EXISTENCE AND THE NON-EXISTENCE
OF SOME KIND OF PERIODIC ORBITS FOR
HAMILTONIAN SYSTEMS

By Haruki Yamada

§1. Introduction.

Let us consider a Hamiltonian system

$$(1) \qquad \dot{\xi}_k = H_{\eta_k}, \quad \dot{\eta}_k = -H_{\xi_k}, \qquad k = 1, 2, \cdots, n,$$

where we assume that the Hamiltonian $H(\xi, \eta)$ is real analytic near the origin $(\xi, \eta) = (0, 0)$ and $H_{\xi_k}(0, 0) = H_{\eta_k}(0, 0) = 0$. Hence H has the following expansion near the origin:

$$(2) \qquad H = \frac{1}{2} {}^t\zeta S\zeta + [\text{higher order terms with respect to } \zeta],$$

where $\zeta = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \boldsymbol{R}^{2n}$ and S is a $2n \times 2n$ real symmetric matrix. Then the Hamiltonian system (1) can be written in the following form:

$$(3) \qquad \dot{\zeta} = JS\zeta + [\text{higher order terms}]; \quad J = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix},$$

where E is the $n \times n$ unit matrix.

Recently, various researches have been done on the existence of periodic orbits near the equilibrium point $(0, 0)$. It is well known that if, among the eigenvalues $\pm\lambda_1, \cdots, \pm\lambda_n$ of JS, there is a pair of purely imaginary ones

$$(4) \qquad \pm\lambda_j \in \imath\boldsymbol{R}, \quad \lambda_j \neq 0$$

such that

$$(5) \qquad \lambda_k/\lambda_j \notin \boldsymbol{Z} \quad \text{for all } k = 1, \cdots, n; \ k \neq j,$$

then, there is a 1-parameter family of periodic orbits which emanate from the equilibrium and whose periods have the values near $2\pi/|\lambda_j|$. This is a theorem of Liapunov. (For precise statement and proof of this theorem see e.g. [8] §16). Though it is known that the condition (4) is essential for the existence of such

family of periodic orbits, it is often possible to remove the condition (5) or replace it by some other conditions. (see e. g. [1], [2], [3], [5], [9] and references cited there). On the contrary, it seems that [7] is the only known general result for the non-existence of (some kind of) periodic orbits. In [7] it is proved, for example, that if there is only one pair of purely imaginary eigenvalues $\pm\lambda_1 \in i\boldsymbol{R}$, $\lambda_1 \neq 0$, and Re $\lambda_j \neq 0$ for all $j=2, \cdots, n$, then in a sufficiently small neighbourhood of the origin, there are no periodic orbits other than those whose existence is guaranteed by the theorem of Liapunov.

The aim of this note is to present a somewhat systematic method of construction of examples for the non-existence of (some kind of) periodic orbits when all the eigenvalues of JS are purely imaginary. Before doing so, in § 2, we review on the normal forms and reality conditions for Hamiltonian systems. In § 3, we use our method to construct some examples in the case of two degrees of freedom. We also prove a simple existence theorem of infinitely many periodic orbits with arbitrary long periods when the Hamiltonian can be transformed into a normal form. In § 4, we indicate a generalization to the case of several degrees of freedom.

§ 2. Normal forms and reality conditions. (cf. [8], §§. 15, 30).

In what follows, for simplicity, we shall assume that all the eigenvalues of JS are non-zero purely imaginary numbers. Then we can take a linear canonical transformation

$$\zeta = Cw ; \quad w = \binom{u}{v} \in \boldsymbol{R}^{2n} ,$$

to transform the Hamiltonian (2) into the following form:

$$(6) \qquad H = H(w) = H(u, v) = \sum_{j=1}^{n} \lambda_j u_j v_j + [\text{higher order terms}] ,$$

where the reality condition is

$$(7) \qquad\qquad v_j = i\bar{u}_j , \qquad j = 1, \cdots, n.$$

In other words, if we find a solution $w(t) = \binom{u(t)}{v(t)}$ of

$$(8) \qquad\qquad \dot{u}_k = H_{v_k} , \quad \dot{v}_k = -H_{u_k} , \quad k = 1, \cdots, n$$

such that (7) is true for all t, then the function $\zeta(t)$ defined by

$$\zeta(t) = Cw(t)$$

is a real valued solution of (1). Further, if we take

(9) $$u_k = \frac{1}{\sqrt{2}}(p_k + iq_k), \quad v_k = \frac{1}{\sqrt{2}}(q_k + ip_k),$$

the transformation $\binom{u}{v} \rightarrow \binom{p}{q}$ is a canonical transformation and the reality condition (7) becomes

$$p_k \text{ and } q_k \text{ are real valued for all } k = 1, \cdots, n.$$

Then we have $u_k v_k = i(p_k^2 + q_k^2)/2$ and

$$H = \sum_{k=1}^{n} i\lambda_k (p_k^2 + q_k^2)/2 + \cdots.$$

If we take a further transformation

(10) $$x_k = \frac{1}{\sqrt{2}}(p_k - iq_k), \quad y_k = \frac{1}{\sqrt{2}}(p_k + iq_k) = \bar{x}_k,$$

(which is not a canonical transformation), the reality condition becomes

(11) $$x_k = \bar{y}_k, \qquad k = 1, \cdots, n,$$

and the Hamiltonian system (8) becomes

(12) $$\dot{x}_k = iH_{y_k}, \quad \dot{y}_k = -iH_{x_k}, \qquad k = 1, \cdots, n,$$

where

(13) $$H = \sum_{k=1}^{n} i\lambda_k x_k y_k + \cdots.$$

Note that under the reality condition (11), we have

$$x_k y_k = x_k \bar{x}_k = |x_k|^2 \geqq 0.$$

Further note that, though $\pm\lambda_k$ are eigenvalues of JS, the signs of $i\lambda_k$ in (13) are determined uniquely if we impose the reality condition (11). In the following we shall consider the Hamiltonian system in the form (12), (13) under the reality condition (11). Further, in what follows, for the simplicity, we shall say that a function $f = f(x, y)$ of x_k and y_k is *real valued* if it is real valued when we replace y_k by $\bar{x}_k$ (i.e. if the reality condition (11) is valid). Of course the Hamiltonian (13) is a real valued function.

Our method of construction of examples relies on a fact that if there is a real valued function $F(x, y)$ such that, for any solution $\binom{x(t)}{y(t)}$ of (12),

$$\frac{d}{dt} F(x, y) = \varphi(x, y) \geqq 0 \quad \text{for all } (x, y) \text{ with (11)},$$

then there are no periodic solutions other than those with $\varphi(x(t),\ y(t))\equiv 0$.

§3. The case of two degrees of freedom.

In this section we restrict ourselves to the case of $n=2$ in (1):

(14) $$\dot{\xi}_k=H_{\eta_k}\,,\quad \dot{\eta}_k=-H_{\xi_k}\,,\qquad k=1,\ 2.$$

We assume that the eigenvalues $\pm\lambda_1$, $\pm\lambda_2$ of JS are non-zero purely imaginary numbers. It is known that ([8], §30), if further

(15) $$\lambda_1 g_1+\lambda_2 g_2=0\,,\quad g_1,\ g_2\in \boldsymbol{Z}\Rightarrow g_1=g_2=0\,,$$

is valid, we can transform H by the composition of a *formal* canonical transformation (which is defined by a generating function expressed by a formal power series) and transformations (9) and (10) to the following form:

(16) $$H=i(\lambda_1 x_1 y_1+\lambda_2 x_2 y_2)+\frac{1}{2}R x_1^2 y_1^2+S x_1 y_1 x_2 y_2+\frac{1}{2}T x_2^2 y_2^2+h(\omega_1,\ \omega_2)\,,$$

where R, S and T are real constants and h is a real valued formal power series in products $\omega_1=x_1 y_1$, $\omega_2=x_2 y_2$ alone, the reality condition is the same as (11). Unfortunately, it is known that, in general, one has divergence (see [6]). But if it happens that it is convergent, we have the following result.

THEOREM 1. *Suppose that the Hamiltonian can be transformed into the form* (16) *by a convergent transformation.* (*We do not necessarily impose the condition* (15)). *Suppose further that*

(17) $$R\lambda_2\neq S\lambda_1\quad or\quad S\lambda_2\neq T\lambda_1$$

is valid. Then, for any small $\varepsilon>0$ and large $N>0$, there are infinitely many periodic orbits with primitive period larger than N in the ε-neighbourhood of the origin $0\in \boldsymbol{R}^4$.

Proof. If we write $\omega_1=x_1 y_1$, $\omega_2=x_2 y_2$, we have, by (12), the Hamiltonian system in the form

(18) $$\dot{x}_k=iH_{\omega_k}x_k\,,\quad \dot{y}_k=-iH_{\omega_k}y_k\,,$$

from which it follows that $\dot{\omega}_k\equiv 0$, $k=1,\ 2$. Hence the ω_k are (real valued) integrals of the system and hence $H_{\omega_k}(x,\ y)$ are independent of t if we take any solution $x_k=x_k(t)$, $y_k=y_k(t)$ of (18). Thus we can integrate (18) in the form

(19) $$x_k=c_k e^{iH_{\omega_k}t}\,,\quad y_k=d_k e^{-iH_{\omega_k}t}\,.$$

If we take $d_k=\bar{c}_k$ and use the fact that H, and hence H_{ω_k}, are real valued, we know that the solution (19) satisfies the reality conditions for all t. Note that,

in this case, $x_k y_k = \omega_k = c_k d_k = |c_k|^2$ (constant). The general solution of (14) can be written by a linear combination of $x_k(t)$, $y_k(t)$, $k=1, 2$, and $x_k(t)$, $y_k(t)$ are periodic functions with primitive period $2\pi/|H_{\omega_k}|$. Thus if we are able to take $(c_1, c_2) \in C^2$ in the ε-neighbourhood of the origin $(0, 0) \in C^2$ such that

$$2\pi/|H_{\omega_1}| : 2\pi/|H_{\omega_2}| = p : q ,$$

where p and q are relatively prime positive integers, we have a periodic orbit with primitive period

$$2q\pi/|H_{\omega_1}| = 2p\pi/|H_{\omega_2}| .$$

Now let us consider the ratio

$$\frac{H_{\omega_1}}{H_{\omega_2}} = \frac{\rho_1 + R\omega_1 + S\omega_2 + \cdots}{\rho_2 + S\omega_1 + T\omega_2 + \cdots} ,$$

where $\rho_1 = i\lambda_1$, $\rho_2 = i\lambda_2$ are non-zero real numbers and the dots $\cdots$ indicate the terms of order ≥ 2 with respect to ω_1, ω_2. If we take $\omega_1 = ks$, $\omega_2 = ls$ with $k, l > 0$, $k^2 + l^2 = 1$, $s > 0$, and put

$$f(s) = \frac{H_{\omega_1}(ks, ls)}{H_{\omega_2}(ks, ls)} = \frac{\rho_1 + (kR + lS)s + 0(s^2)}{\rho_2 + (kS + lT)s + 0(s^2)} ,$$

we have

$$f'(0) = \frac{1}{\rho_2^2} [(R\rho_2 - S\rho_1)k + (S\rho_2 - T\rho_1)l] .$$

Thus if (17) is valid, we have for almost all values k, l, $f'(0) \neq 0$ and hence $f(s)$ is a monotone function near $s = 0$. Thus there is a constant $\delta_0 > 0$ such that for any δ with $0 < \delta < \delta_0$, $\{f(s); 0 \leq s \leq \delta\}$ is a closed interval with inner points in R and it contains ρ_1/ρ_2 as a terminal point. It is easy to see that in this interval there are infinitely many rational numbers q/p with $(p, q) = 1$ and $|p|, |q| > M$ where M is an arbitrary fixed (large) number. If we take s such that $f(s) = q/p$ and take ω_1, ω_2 as $\omega_1 = ks$, $\omega_2 = ls$, we have a desired long periodic orbit

$$x_1 = \sqrt{\omega_1} e^{iH_{\omega_1} t} , \qquad x_2 = \sqrt{\omega_2} e^{iH_{\omega_2} t} ,$$

$$y_1 = \sqrt{\omega_1} e^{-iH_{\omega_1} t} , \qquad y_2 = \sqrt{\omega_2} e^{-iH_{\omega_2} t} ,$$

with period

$$\frac{2q\pi}{H_{\omega_1}} = \frac{2p\pi}{H_{\omega_2}} \left(\sim \frac{2q\pi}{\rho_1}, \frac{2p\pi}{\rho_2} \right) .$$

This proves the theorem.

Remark 1. It is clear that if

$$R\lambda_2^2 - 2S\lambda_1\lambda_2 + T\lambda_1^2 \neq 0 ,$$

that is, if the polynomial $R\xi^2 + 2S\xi\eta + T\eta^2$ can not be divisible by $\lambda_1\xi + \lambda_2\eta$, the

condition (17) is valid.

Now if λ_1/λ_2, $\lambda_2/\lambda_1 \notin \mathbf{Z}$, it always exist two 1-parameter family of periodic orbits with periods $\sim 2\pi/|\lambda_1|$ and $\sim 2\pi/|\lambda_2|$ which are guaranteed by the theorem of Liapunov. The above theorem says that if further H can be transformed into the normal form (16) and the condition (17) is valid, there are infinitely many long periodic orbits other than the above ones. On the other hand, for

$$(20) \qquad H = i(\lambda_1 x_1 y_1 + \lambda_2 x_2 y_2), \qquad \lambda_1/\lambda_2 \notin \mathbf{Q},$$

there are no periodic orbits at all other than those cited above. It is likely that under the conditions (15) and (17) (we do not impose the condition that the transformation into the normal form (16) is convergent), there always be infinitely many long periodic orbits near the origin. But we don't know whether this is true or not.

If $\lambda_1/\lambda_2 \in \mathbf{Q}$, on the other hand, all the solutions for (20) are periodic. But in general these long periodic orbits may disappear by some perturbations for H. To see this we start from somewhat general context.

Let us consider the Hamiltonian

$$(21) \qquad H = i(\lambda_1 x_1 y_1 + \lambda_2 x_2 y_2) + \frac{1}{2} R x_1^2 y_1^2 + S x_1 y_1 x_2 y_2 + \frac{1}{2} T x_2^2 y_2^2 + h(x, y),$$

where $h(x, y)$, which represents higher order terms, is not necessarily a function in ω_1 and ω_2 only. Then the Hamiltonian system is

$$(22) \qquad \begin{aligned} \dot{x}_1 &= i H_{y_1} = i(\rho_1 + R x_1 y_1 + S x_2 y_2) x_1 + i h_{y_1}, \\ \dot{x}_2 &= i H_{y_2} = i(\rho_2 + S x_1 y_1 + T x_2 y_2) x_2 + i h_{y_2}, \end{aligned}$$

where $\rho_1 = i\lambda_1$, $\rho_2 = i\lambda_2$. Note that other two equations can be given by taking the complex conjugates in (22) and replacing $\bar{x}_k$ and $\bar{y}_k$ by y_k and x_k respectively. Now take real valued functions $f(x, y)$ and $g(x, y)$ and put

$$(23) \qquad s = s(x, y) = \frac{g(x, y)}{f(x, y)}.$$

Then, if we take x_k, y_k as a solution of (22), we have

$$\dot{s} = \frac{1}{f^2}(\dot{g}f - \dot{f}g)$$

and

$$\dot{g}f - \dot{f}g = (g_{x_1}f - f_{x_1}g)\dot{x}_1 + (g_{x_2}f - f_{x_2}g)\dot{x}_2 + (g_{y_1}f - f_{y_1}g)\dot{y}_1 + (g_{y_2}f - f_{y_2}g)\dot{y}_2.$$

If we put

$$A = g_{x_1}f - f_{x_1}g, \qquad B = g_{y_1}f - f_{y_1}g,$$

$$C = g_{x_2}f - f_{x_2}g, \qquad D = g_{y_2}f - f_{y_2}g,$$

we have

$$\dot{g}f-\dot{f}g=i(x_1A-y_1B)(\rho_1+Rx_1y_1+Sx_2y_2)+i(x_2C-y_2D)(\rho_2+Sx_1y_1+Tx_2y_2)$$

$$+i(h_{y_1}A-h_{x_1}B+h_{y_2}C-h_{x_2}D).$$

Note that the condition $\lambda_1/\lambda_2=\rho_1/\rho_2\in\boldsymbol{Q}$ reflects the fact that we can take appropriate f and g to cancel out the term

$$i\rho_1(x_1A-y_1B)+i\rho_2(x_2C-y_2D).$$

Using the above formulations we shall prove the following fact.

THEOREM 2. *Suppose that* $\lambda_1/\lambda_2\in\boldsymbol{Q}$, λ_2/λ_1, $\lambda_1/\lambda_2\notin\boldsymbol{Z}$ *and* $\lambda_1/\lambda_2<0$. *Then there is a Hamiltonian* H *with quadratic term* $i\lambda_1x_1y_1+i\lambda_2x_2y_2$ *and* $R, S, T\neq0$ *such that there are no periodic orbits other than those two 1-parameter families whose existence is guaranteed by the theorem of Liapunov.*

Proof. By scaling the time variable t appropriately, it is enough to prove the theorem when

$$\lambda_1=-ip, \quad \lambda_2=iq \quad \text{(i. e. } \rho_1=p, \; \rho_2=-q),$$

where p and q are positive integers and $(p, q)=1$, $p\neq1$, $q\neq1$. Now consider the Hamiltonian in the form (21) with

$$h=x_1^qx_2^p+y_1^qy_2^p$$

and take $f=1$, $g=\dfrac{1}{i}(x_1^qx_2^p-y_1^qy_2^p)$ in (23). Then we have

$$\dot{s}=\dot{g}=q(x_1^qx_2^p+y_1^qy_2^p)(p+Rx_1y_1+Sx_2y_2)$$

$$+p(x_1^qx_2^p+y_1^qy_2^p)(-q+Sx_1y_1+Tx_2y_2)$$

$$+2(x_1y_1)^{q-1}(x_2y_2)^{p-1}(p^2x_1y_1+q^2x_2y_2)$$

$$=\{(qR+pS)x_1y_1+(qS+pT)x_2y_2\}(x_1^qx_2^p+y_1^qy_2^p)$$

$$+2(x_1y_1)^{q-1}(x_2y_2)^{p-1}(p^2x_1y_1+q^2x_2y_2).$$

Thus if we take R, S and T such that

(24) $$qR+pS=0, \quad qS+pT=0,$$

we have

(25) $$\dot{s}=2(x_1y_1)^{q-1}(x_2y_2)^{p-1}(p^2x_1y_1+q^2x_2y_2)\geqq0,$$

where the equality is valid identically in time t if and only if

$$x_1\equiv y_1\equiv0 \quad \text{or} \quad x_2\equiv y_2\equiv0.$$

But if $x_2 \equiv y_2 \equiv 0$ we have the Hamiltonian system

$$\dot{x}_1 = i(p + Rx_1y_1)x_1, \qquad \dot{x}_2 = 0,$$

$$\dot{y}_1 = -i(p + Rx_1y_1)y_1, \quad \dot{y}_2 = 0,$$

which has the 1-parameter family of periodic orbits

$$x_1 = \rho e^{i(p+R\rho^2)t}, \qquad x_2 \equiv 0,$$

$$y_1 = \rho e^{-i(p+R\rho^2)t}, \qquad y_2 \equiv 0,$$

where ρ is a non-negative real parameter. Similarly, if $x_1 \equiv y_1 \equiv 0$, we have another 1-parameter family of periodic orbits

$$x_1 \equiv 0, \qquad x_2 = \rho e^{i(-q+S\rho^2)t},$$

$$y_1 \equiv 0, \qquad y_2 = \rho e^{-i(-q+S\rho^2)t}.$$

These orbits have periods $2\pi/(p+R\rho^2)$ and $2\pi/(q-S\rho^2)$ respectively. (These orbits are those which are given by the theorem of Liapunov). By (25) all other orbits cannot be periodic. This proves the theorem.

Remark 2. The condition (24) can be written as

$$\lambda_2 R = \lambda_1 S, \quad \lambda_2 S = \lambda_1 T.$$

If one of these equality breaks down and if $h(x, y)$ is replaced by a function in ω_1 and ω_2, we have, by Theorem 1, there are infinitely many periodic orbits with arbitrary large primitive periods.

Remark 3. For the Hamiltonian which was given in the proof of Theorem 2, the result in Theorem 2 is valid *globally* in $\mathbf{R}^4$.

Remark 4. We do not know whether, if $\lambda_1/\lambda_2 > 0$, we can construct a similar example as in Theorem 2 or not. But if we consider non-Hamiltonian system, we are able to do so. In fact, e.g., consider the system

(26)

$$\dot{x}_1 = i(p + Rx_1y_1 + Sx_2y_2)x_1 - x_1^{q+1}y_2^p,$$

$$\dot{x}_2 = i(q + Sx_1y_1 + Tx_2y_2)x_2 + x_1^q y_2^{p-1},$$

$$\dot{y}_1 = -i(p + Rx_1y_1 + Sx_2y_2)y_1 - y_1^{q+1}x_2^p,$$

$$\dot{y}_2 = -i(q + Sx_1y_1 + Tx_2y_2)y_2 + y_1^q x_2^{p-1}.$$

Then if we take

$$s = \frac{g}{f} = \frac{y_1^q x_2^p + x_1^q y_2^p}{x_1^q y_1^q},$$

we have

$$\dot{s} = \imath q(x_1 y_1)^{-q}(x_1^q y_2^p - y_1^q x_2^p)(p + R x_1 y_1 + S x_2 y_2)$$

$$- \imath p(x_1 y_1)^{-q}(x_1^q y_2^p - y_1^q x_2^p)(q + S x_1 y_1 + T x_2 y_2) + 2(x_2 y_2)^{p-1}(q x_2 y_2 + p).$$

Accordingly, if we take R, S, and T as

(27) $$qR - pS = 0, \quad qS - pT = 0,$$

we have

$$\dot{s} = 2(x_2 y_2)^{p-1}(q x_2 y_2 + p) \geqq 0$$

and the equality is valid identically only if $x_2 \equiv y_2 \equiv 0$. Note that the above calculation is valid only if $x_1 \neq 0$, $y_1 \neq 0$. Thus this system has no periodic orbits other than two 1-parameter families of periodic orbits which correspond with $x_1 \equiv y_1 \equiv 0$ or $x_2 \equiv y_2 \equiv 0$. Note that the third and the fourth equations are consequences of the first and the second equations and the reality condition. Thus the system (26) has a meaning as the system in $\boldsymbol{R}^4$. Unfortunately (26) is not a Hamiltonian system.

Remark 5. In [2], it is shown that if we assume some conditions for the coefficients of higher order terms of the Hamiltonian, we have the existence of some kind of long periodic orbits other than those whose existence is guaranteed by the theorem of Liapunov. Of course our examples do not satisfy this condition.

In the case when $\lambda_1/\lambda_2 \in \boldsymbol{Z}$ we can prove easily the following theorem.

THEOREM 3. *Suppose that* $\lambda_1/\lambda_2 \in \boldsymbol{Z}$ *and* $\lambda_1/\lambda_2 < 0$. *Then there is a Hamiltonian with* R, S, $T \neq 0$, *for which there are no periodic orbits other than one 1-parameter family whose existence is guaranteed by the theorem of Liapunov.*

Proof. If $\lambda_1/\lambda_2 \neq -1$, we have only to take $q = 1$ in the proof of Theorem 2. Then if we take R, S, and T as

(28) $$R + pS = 0, \quad S + pT = 0,$$

we have

$$\dot{s} = 2(x_2 y_2)^{p-1}(p^2 x_1 y_1 + x_2 y_2) \geqq 0$$

and there must only be one 1-parameter family of periodic orbits corresponding $x_2 \equiv y_2 \equiv 0$. Note that the Hamiltonian can be written as

(29) $$H = p x_1 y_1 - x_2 y_2 + \frac{1}{2} R x_1^2 y_1^2 + S x_1 y_1 x_2 y_2 + \frac{1}{2} T x_2^2 y_2^2 + x_1 x_2^p + y_1 y_2^p.$$

Thus unless $p = 1$, the degree of $x_1 x_2^p + y_1 y_2^p$ is $p + 1 \geqq 3$. If $\lambda_1/\lambda_2 = -1$, we may take

$$H = x_1 y_1 - x_2 y_2 + (x_1 y_1 + x_2 y_2)(x_1 x_2 + y_1 y_2)$$

and $s=(x_1x_2-y_1y_2)/i$ (This is due to [4]). This proves the theorem.

Remark 6. In [5], it is proved that for the Hamiltonian (21), if $S+pT\neq0$, $p\geqq4$, there is a 1-parameter family of periodic orbits with primitive periods $\sim2\pi$ near the origin. (Note that by the theorem of Liapunov it always exists a 1-parameter family of periodic orbits with primitive periods $\sim2\pi/p$). Our example in the proof of Theorem 3 was constructed under the condition (28). We do not know whether we can construct an example of non-existence of long periodic orbits (with periods $\sim2\pi$) assuming only that $S+pT=0$ on the fourth order terms of the Hamiltonian.

Remark 7. In [9], it is proved that for the Hamiltonian $H=x_1y_1+px_2y_2+h(x,y)$, where h contains only terms of order $\geqq3$ and p is any positive real number, there are at least two periodic orbits on each energy surfaces $H=\varepsilon$ where $\varepsilon>0$ are sufficiently small real numbers. Thus the assumption $\lambda_1/\lambda_2<0$ in Theorem 3 is not so restrictive.

Remark 8. By investigation on the properties of $\dot{s}$ in the proof of Theorem 3, we have easily that, if $S+pT=0$, there are no periodic orbits for (29) in a domain near the plane $x_2=y_2=0$ in which there exist long periodic orbits when $S+pT\neq0$, $p\geqq4$.

§4. A generalization.

A part of our method in §3 can be extended to the case of several degrees of freedom. We shall mention only one example which is connected with a result of [9] (This is also a straight forward generalization of an example in [4]).

Consider the Hamiltonian of $2n$-degrees of freedom:

$$(30) \qquad H=\sum_{k=1}^{n}\rho_kx_ky_k-\sum_{k=1}^{n}\rho_kx_{n+k}y_{n+k}+\Big(\sum_{l=1}^{2n}x_ly_l\Big)\sum_{k=1}^{n}(x_kx_{n+k}+y_ky_{n+k}),$$

where ρ_k are non-zero real numbers. Then there are no periodic orbits at all for (30) other than the equilibrium. Let us prove this in the following manner:
The Hamiltonian system for (30) is

$$\dot{x}_j=iH_{y_j}=i\Big[\rho_jx_j+x_j\sum_{k=1}^{n}(x_kx_{n+k}+y_ky_{n+k})+y_{n+j}\Big(\sum_{l=1}^{2n}x_ly_l\Big)\Big],$$

$$(31)$$

$$\dot{x}_{n+j}=iH_{y_{n+j}}=i\Big[-\rho_jx_{n+j}+x_{n+j}\sum_{k=1}^{n}(x_kx_{n+k}+y_ky_{n+k})+y_j\Big(\sum_{l=1}^{2n}x_ly_l\Big)\Big],$$

$$j=1,\cdots,n.$$

Other $2n$ equations for y_j, y_{n+j}, $j=1, \cdots, n$, can be given by conjugating (31) and replacing $\bar{x}_l$ and $\bar{y}_l$, $l=1, \cdots, 2n$, by y_l and x_l respectively. If we take a real valued function

$$s = \frac{1}{i} \sum_{j=1}^{n} (x_j x_{n+j} - y_j y_{n+j}),$$

we have

$$\dot{s} = 2\left\{ \sum_{j=1}^{n} (x_j x_{n+j} + y_j y_{n+j}) \right\}^2 = 2\left(\sum_{l=1}^{2n} x_l y_l \right)^2 \geq 0.$$

The equality is valid identically only if $x_l \equiv y_l \equiv 0$ for all $l=1, \cdots, 2n$. This proves that the only periodic orbit for (31) is an equilibrium.

Remark 1. By a similar method, it is possible to make an example of Hamiltonian with the form

$$H = \sum_{k=1}^{n} \rho_k x_k y_k - \sum_{k=1}^{n+m} \rho_k x_{n+k} y_{n+k} + \cdots$$

with $\rho_k \in \mathbf{R}$, $\rho_k \neq 0$, $k=1, \cdots, n+m$, $n>0$, $m \geq 0$; $\rho_k / \rho_{n+l} \notin \mathbf{Z}$ for all $k=1, \cdots, n$, $l=1, \cdots, m$; $\rho_{n+l} / \rho_{n+s} \notin \mathbf{Z}$ for all $l, s=1, \cdots, m$, $l \neq s$, for which the Hamiltonian system has only m 1-parameter families of periodic orbits.

Remark 2. In [1], the following result is proved: Let

(32) $$H = \sum_{k=1}^{2n+m} \rho_k x_k y_k + h(x, y),$$

where $\rho_k > 0$ for $k=1, \cdots, n$, $\rho_k < 0$ for $k=n+1, \cdots, 2n+m$, and where h represents the terms of order ≥ 3. Suppose that there is a number $T \in \mathbf{R}$ such that

$$T/(2\pi/\rho_k) \in \mathbf{Z} \quad \text{for all } k=1, \cdots, 2n+m.$$

Then there is a left neighbourhood I_l and a right neighbourhood I_r of $T \in \mathbf{R}$ and a pair of integers $a, b \geq 0$ with $a+b=m$ such that, for all $\tau \in I_l$ (resp. I_r), the Hamiltonian system for (32) possesses at least a (resp. b) distinct non-trivial τ-periodic orbits.

Our example shows that the condition $m>0$ is essential for the existence of periodic orbits when we do not impose any additional conditions on the coefficients in $h(x, y)$ (i.e. terms of order ≥ 3).

REFERENCES

[1] E. R. FADELL AND P. H. RABINOWITZ, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems, Inv. Math., **45** (1978), 139-174.

[2] J. HENRARD, Periodic orbits emanating from a resonant equilibrium, Celes. Mech.,

1 (1970), 437–466.

[3] J. K. MOSER, Periodic orbits near an equilibrium and a theorem by Alan Wein-
stein, Comm. Pure Appl. Math., 29 (1976), 727–747.

[4] J. K. MOSER, Addendum to [3], Comm. Pure Appl. Math., 31 (1978), 529–530.

[5] D. S. SCHMIDT, Hopf bifurcation theorem and the center theorem of Liapunov
with resonance case, J. Math. Anal. Appl., 63 (1978), 354–370.

[6] C. L. SIEGEL, Über die Existenz einer Normalform analytischer Hamiltonscher
Differentialgleichungen in der Nähe einer Gleichgewichtslösung, Math. Ann.,
128 (1954), 144–170.

[7] C. L. SIEGEL, Periodische Lösungen von Differentialgleichungen, Nachr. Akad.
Wiss. Göttingen, Math-Phys. Kl. II, (1971), 261–283.

[8] C. L. SIEGEL AND J. K. MOSER. Lectures on Celestial Mechanics, Springer, 1971.

[9] A. WEINSTEIN, Normal modes for non-linear Hamiltonian systems, Inv. Math. 20
(1973), 47–57.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY